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Article

On Real Algebras Admitting Reflections which Commute

André Souleye Diabang[1,*], **Mankagna Albert Diompy**[2] **and Alhousseynou Ba**[2]

[1] Département de Mathématiques, UFR. Sciences et Technologies (SET), Université Iba Der Thiam, Thiés (UIDT), Sénégal

[2] Département de Mathématiques et Informatique, Faculté des Sciences et Techniques (FST), Université Cheikh Anta Diop, Dakar (UCAD), Sénégal

* **Correspondence:** andre.diabang@univ-thies.sn

Abstract: We study real algebras admitting reflections which commute. In dimension two, we show that two commuting reflections coincide. We specify the two and four-dimensional real algebras cases. We characterize real algebras of division of two-dimensional to third power-associative having a reflection. Finally We give a characterization in four-dimensional, the unitary real algebras of division at third power-associative having two reflections that commute. In eight-dimensional, we give an example of algebra so the group of automorphisms contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Keywords: Division algebra, Algebra isotopy, Derivation and reflexion

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1. Introduction

The classification of non-associative division algebras over a commutative field $\mathbb{K}$ with a characteristic different from 2 is a pastionning and topical problem, whose origins date back to the discovery of quaternions ($\mathbb{H}$, Hamilton 1843) and octonions ($\mathbb{O}$, Graves 1843, Cayley 1845). Fundamental results appeared, Hopf proved that the dimension of a real algebra of division of finite dimension n is a power of 2 and cannot exceed 2 in the commutative case.

Bott and Milnor [1] refined the result of Hopf by reducing the power of 2 to $n \in \{1, 2, 4, 8\}$. It is trivial to show that in dimension one the real algebra $\mathbb{R}$ is unique. In two-dimensional, the classification of these algebras is recently completed. The problem remains open in dimension 4 and 8. The study is quite interesting when there is a sufficient number of distinct reflections which commute for a finite-dimensional division algebra since the product of this last translates a certain symmetry and elegance.

In this paper, we give a description of real algebras of division in two-dimensional having one reflection, and in four-dimensional having two distinct reflections which commute. We recall that the subgroup generated by the latter is isomorphic to the Klein's group $\mathbb{Z}_2 \times \mathbb{Z}_2$ with $\mathbb{Z}_2 := \dfrac{\mathbb{Z}}{2\mathbb{Z}}$.

In eight-dimensional, we give a class of algebra whose group of automorphisms contains the Klein's group without giving the necessary and sufficient condition of the division.

2. Notations and Preliminary Results

Let A be an arbitrary non-associative real algebra. We define, $I(A) = \{x \in A, x^2 = x\}$ and let x, $y \in A$, $[x, y] = xy - yx$ and $(x, y, z) = (xy)z - x(yz)$.

- A is said to be
  - - division if the operations $L_x : A \to A$, $y \mapsto xy$ and $R_x : A \to A$, $y \mapsto yx$ are bijective, for all $x \in A$, $x \neq 0$.
  - -at third power-associative if $(x, x, x) = 0$ for all $x \in A$.
  - -at 121 power-associative if $(x, x^2, x) = 0$ for all $x \in A$.
- A linear map $\partial : A \to A$ is said to be a derivation of A if for all $x, y \in A$, we have $\partial(x.y) = \partial(x).y + x.\partial(y)$. The derivations of A form a vector subspace of the endomorphisms of A ($End_{\mathbb{K}}(A)$) which is a Lie algebra, for the bracket of Lie $[f, g] = f \circ g - g \circ f$. Such an algebra is called the Lie algebra of the derivations of A denoted by $Der(A)$.
- A linear map $f : A \to A$ is said to be an automorphism of A if f is bijective and for all x, $y \in A$, $f(x.y) = f(x).f(y)$. The automorphisms of A constitute a group $Aut_{\mathbb{K}}(A)$ for the usual law. $f \in Aut(A)$ is said to be a reflection of A, if f is involutive ($f \circ f = id_A$) not identical ($f \neq id_A$). Let f be an automorphism of A and $\lambda \in \mathbb{R}$, we denote by $E_\lambda(f)$ the kernel of $f - \lambda id_A$, id_A being the identity operator of A.
- Let $f, g : A \to A$ be linear bijections of A. Recall that the (f, g)-Albert isotope of A denoted by $A_{f,g}$ is a vector space A with the product $x \odot y = f(x)g(y)$.

Remark 1. *A linearization of* $(x, x, x) = 0$*, gives* $[x^2, y] + [xy + yx, x] = 0$ *for all* $x, y \in A$*. So by taking* $y = x^2$ *we get* $[x.x^2 + x^2.x, x] = 0 \Rightarrow 2[x.x^2, x] = 0 \Rightarrow (x.x^2)x - x(x.x^2) = 0 \Rightarrow (x.x^2)x - x(x^2.x) = 0 \Rightarrow (x, x^2, x) = 0$*. We can therefore affirm that if A is at third power-associative then it is at 121 power-associative. But the converse is not true in the case where the algebra is not of division. For example the algebra A having a basis* $\{e_1, e_2, e_3, e_4\}$ *whose product of the elements in this base is given by,* ${e_1}^2 = e_1$*,* $e_1e_2 = e_3$*,* $e_2e_1 = e_4$ *and other null products, is at 121 power-associative and it is not at third power-associative.*

In [2], we have the following result;

Lemma 1. *Let A be a real algebra of division of finite 2n-dimensional with* $n \in \{1, 2, 4\}$*. We suppose that there exists an automorphism* f *of A such that* $sp(f) = \{-1, 1\}$*. Then the following inclusions between subalgebras of A are strict,*

$$\{0\} \subset E_1(f) \subset E_1(f) + E_{-1}(f).$$

In that case $dimE_1(f) = n \geq 1$*, and* $dim(E_1(f) + E_{-1}(f)) = 2n$*.*

In addition, the following statements are equivalent,

(a) f *is a reflection of* A*.*
(b) $A = E_1(f) \oplus E_{-1}(f)$*.*

And the following equalities hold,

$$E_1(f)E_{-1}(f) = E_{-1}(f)E_1(f) = E_{-1}(f),$$

$$E_{-1}(f).E_{-1}(f) = E_1(f).$$

Definition 1. *Let* $\alpha, \beta, \gamma, \alpha', \beta'$ *and* γ' *be real numbers. We define algebra A having a base* $\{e, e_1, e_2, e_3\}$

for which the product is given as

$$\begin{array}{c|c|c|c|c} \odot & e & e_1 & e_2 & e_3 \\ \hline e & e & e_1 & e_2 & e_3 \\ \hline e_1 & e_1 & -e & \gamma e_3 & \beta' e_2 \\ \hline e_2 & e_2 & \gamma' e_3 & -e & \alpha e_1 \\ \hline e_3 & e_3 & \beta e_2 & \alpha' e_1 & -e \end{array} \tag{1}$$

We denote this algebra A by $\mathbb{H}(\alpha,\beta,\gamma,\alpha',\beta',\gamma')$.

In [3], the author gives the following results:

Theorem 1. *A necessary condition that the algebra* $\mathbb{H}(\alpha,\beta,\gamma,\alpha',\beta',\gamma')$ *should be a division algebra, is that the six nonzero constants* α, β, γ, $-\alpha'$, $-\beta'$, $-\gamma'$ *should have the same sign. This sign may be taken to be positive without loss of generality. If the six constants are positive, a sufficient (but not necessary) condition for a division algebra is that they satisfy the relation* $f(\alpha,\beta,\gamma) = f(-\alpha',-\beta',-\gamma')$ *where* f *is defined by*

$$f(x,y,z) = x + y + z - xyz.$$

Theorem 2. *A necessary and sufficient condition that the division algebra* $\mathbb{H}(\alpha,\beta,\gamma,\alpha',\beta',\gamma')$ *should have two sided rank 2 is that the equations* $\alpha' = \alpha$, $\beta' = \beta$ *and* $\gamma' = \gamma$ *hold true. In the contrary case, the algebra has two sided rank 4.*

Proposition 1. *Let A be a finite dimensional real algebra. The following proposals are equivalent,*

1. $Aut(A)$ *contains two distinct reflections which commute.*
2. $Aut(A)$ *contains a subgroup isomorphic to the Klein's group* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. (1) $\Longrightarrow$ (2) Let A be a real algebra such that $Aut(A)$ contains two reflections which commute f and g. Then $h = f \circ g \in Aut(A)$ and is a different reflection of f and g. The subgroup of $Aut(A)$ generated by f and g is isomorphic to the klein's group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(2) $\Longrightarrow$ (1) Obvious because the elements of the subgroup of $Aut(A)$ which is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$ different from the identity are distinct reflections that commute. □

3. Two-dimensional Division Algebra Case

Proposition 2. *Let A be a two-dimensional division algebra and let* f, g *be two commuting reflections of A. Then* $f = g$

Proof. The endomorphisms f and g of A are diagonalizable which commute, consequently there exists a common basis $\{e, e_1\}$ of A, formed of eigenvectors associated with eigenvalues 1 and -1. According to Lemma 1. The eigpaces $E_1(f)$, $E_{-1}(f)$, $E_1(g)$, $E_{-1}(g)$ of f and g are one-dimensional. By setting $E_1(f) = \mathbb{R}e$ and $E_{-1}(f) = \mathbb{R}e_1$.

Suppose that $E_1(g) = \mathbb{R}e_1$ and $E_{-1}(g) = \mathbb{R}e$, let $x = ae + be_1 \in A$, we have

$$f(x) = af(e) + bf(e_1) = ae - be_1 = -ag(e) - bg(e_1) = -g(ae + be_1) = -g(x),$$

thus $f = -g$ (which is not an automophism of A), absurd. So $E_1(g) = \mathbb{R}e$ and $E_{-1}(g) = \mathbb{R}e_1$, therefore f and g coincide on $A = E_1(f) \oplus E_{-1}(f)$. □

Theorem 3. *Let A be a two-dimensional division algebra and let* f *be a reflections of A. Then there exists a basis* $B = \{e, e_1\}$ *of A such that the product of A in this basis is given as,*

$$\begin{array}{c|c|c} \odot & e & e_1 \\ \hline e & e & \alpha e_1 \\ \hline e_1 & \beta e_1 & \gamma e \end{array} \tag{2}$$

with α, β, γ *are non-zero real numbers and* $\alpha\beta\gamma < 0$. *We will denote this algebra* $A(\alpha,\beta,\gamma)$.

Proof. The eigenspaces $E_1(f)$, $E_{-1}(f)$ of f are one-dimensional and $A = E_1(f) \oplus E_{-1}(f)$. There exists $e \in E_1(f)$ and $e_1 \in E_{-1}(f)$, such that $\{e, e_1\}$ is a basis of A. Taking into account the equalities $(*)$ of Lemma 1, we obtain the product of the elements of this basis of (2). □

For the division, the Theorem 3 of [4] gives the result.

Corollary 1. *Let A be a two-dimensional division algebra, then the following propositions are equivalent,*

1. *$Aut(A)$ contains a reflection.*
2. *A is isomorphic to $A(\alpha,\beta,\gamma)$ with $\alpha\beta\gamma < 0$.*

Proof. (1) ⇒ (2) The Theorem 3 gives the result.

(2) ⇒ (1) The endomorphism f of $A(\alpha,\beta,\gamma)$ defined by $f(x_0e + x_1e_1) = x_0e - x_1e_1$ is a reflection of $A(\alpha,\beta,\gamma)$. □

Theorem 4. *Let A be a two-dimensional division algebra and let f be a reflection of A, then the following propositions are equivalent,*

1. *A is commutative.*
2. *A is isomorphic to $A(\alpha,\beta,\gamma)$ with $\beta = \alpha$ and $\gamma < 0$.*
3. *A is at third power-associative.*
4. *A is at 121 power-associative.*

Proof. (1) ⇒ (2) Since f be a reflection of A, the Corollary 1 asserts that A is isomorphic to $A(\alpha,\beta,\gamma)$ with $\alpha\beta\gamma < 0$. Since A is commutative, $e_1e = ee_1 \Longrightarrow \alpha = \beta$, therefore we get the result.

(2) ⇒ (3) It is easy to show that $A(\alpha,\beta,\gamma)$, with $\beta = \alpha$ and $\gamma < 0$, is at third power-associative.

(3) ⇒ (4) The Remark 1 gives the result.

(4) ⇒ (1) A is isomorphic to $A(\alpha,\beta,\gamma)$ with $\alpha\beta\gamma < 0$ and it is at 121 power-associative, We have $(e_1, {e_1}^2, e_1) = 0 \Rightarrow \gamma^2(\alpha-\beta)e = 0 \Rightarrow \alpha = \beta$. and $\gamma < 0$. Then A is commutative. □

Theorem 5. *Let A be a two-dimensional division algebra and let f be a reflection of A. We are obtained the idempotents and the derivations of A.*

	$I(A(\alpha,\beta,\gamma))$	$Der(A(\alpha,\beta,\gamma))$
$\alpha+\beta \neq 0$ *et* $\gamma(\alpha+\beta-1) > 0$	$\{e, \lambda_0 e - \lambda_1 e_1, \lambda_0 e + \lambda_1 e_1\}$	$\{0\}$
otherwise	$\{e\}$	$\{0\}$

with $\lambda_0 = \frac{1}{\alpha+\beta}$, ${\lambda_1}^2 = \frac{\alpha+\beta-1}{\gamma(\alpha+\beta)^2}$.

Proof. A is isomorphic to $A(\alpha,\beta,\gamma)$ with $\alpha\beta\gamma < 0$.

- Let $x = \lambda_0 e + \lambda_1 e_1 \in I(A)$, we have

$$x^2 = x \Leftrightarrow \begin{cases} {\lambda_0}^2 + \gamma{\lambda_1}^2 = \lambda_0 \\ (\alpha+\beta)\lambda_0\lambda_1 = \lambda_1 \end{cases}$$

  We obtain $I(A)$ by resolving the system and discussing on $\alpha+\beta$ and $\dfrac{\alpha+\beta-1}{\gamma}$.
- $A(\alpha,\beta,\gamma)$ with $\alpha\beta\gamma < 0$ be a two-dimensional division algebra, then $Der(A) = \{0\}$.

□

Proposition 3. *Let A be a two-dimensional division algebra and f be a reflection of A. Then the following propositions are equivalent,*

1. *$Aut(A)$ is isomorphic to S_3.*

2. *A is isomorphic to McClay* $\overset{*}{\mathbb{C}} := (\mathbb{C}, \odot)$ *with* $x \odot y = \overline{x}\,\overline{y}$ *for* $x, y \in \mathbb{C}$.

Here, $\overline{x}$ *and* $\overline{y}$*, are the respective conjugates of* x *and* y.

Proof. (1) $\Longrightarrow$ (2) By hypothesis A is isomorphic to $A(\alpha,\beta,\gamma)$ with $\alpha\beta\gamma < 0$ and $Aut(A) \simeq S_3$, then there exists $f \in Aut(A)$ not identical of order 3.

We suppose that $\alpha + \beta = 0$ where $\gamma(\alpha + \beta - 1) \leq 0$. Since $f(e) \in I(A) = \{e\}$ then $f(e) = e$ and $f(e_1) = \pm e_1$ therefore $f^2 = id_A$, absurd because f is of order 3. We now assume that $\alpha + \beta \neq 0$ and $\gamma(\alpha + \beta - 1) > 0$. Since $f(e) \in I(A) = \{e, \lambda_0 e_0 + \lambda_1 e_1, \lambda_0 e_0 - \lambda_1 e_1\}$. If $f(e) = e$ we have $f(e_1) = \pm e_1$, contradiction because f is of order 3. There is only one case left $f(e) = \lambda_0 e \pm \lambda_1 e_1$. Let $f(e_1) = x_0 e_0 + x_1 e_1$, with x_0, x_1 non-zero real numbers.

Looking at the components of e of the equations $f(e)f(e_1) = \alpha f(e_1)$ et $f(e_1)f(e) = \beta f(e_1)$ we have $\alpha x_0 = \beta x_o \Rightarrow \alpha = \beta$. Therefore A is commutative so it is reflexive. In conclusion A is reflexive and $Aut(A) \simeq S_3$, hence the result follows.

(2) $\Longrightarrow$ (1) By simple calculation we have $Aut(\overset{*}{\mathbb{C}}) \simeq S_3$. □

4. Four-dimensional Division Algebra Case

Proposition 4. *Let A be a four-dimensional division algebra, let f and g are reflections of A. Then there exists a basis* $B = \{e, e_1, e_2, e_3\}$ *of A where* $e^2 = e$ *and*

$$E_1(f) = \mathbb{R}e + \mathbb{R}e_1,$$

$$E_{-1}(f) = \mathbb{R}e_2 + \mathbb{R}e_3,$$

$$E_1(g) = \mathbb{R}e + \mathbb{R}e_2,$$

and

$$E_{-1}(g) = \mathbb{R}e_1 + \mathbb{R}e_3.$$

Proof. Since f and g are diagonalizable and commute, so there is a common basis $\{e, e_1, e_2, e_3\}$ formed by eigenvectors associated to eigenvalues 1 or -1. The Lemma 1 shows that $E_1(f)$, $E_{-1}(f)$, $E_1(g)$ and $E_{-1}(g)$ are vector spaces of two-dimensional. If $E_1(f) = \mathbb{R}e + \mathbb{R}e_1$, $E_{-1}(f) = \mathbb{R}e_2 + \mathbb{R}e_3$, then one of the eigenvectors e, e_1 (and only one) belongs to $E_1(g)$, otherwise f and g coincide in $A = E_1(f) \oplus E_{-1}(f)$.

We can therefore set $E_1(g) = \mathbb{R}e + \mathbb{R}e_2$ and $E_{-1}(g) = \mathbb{R}e_1 + \mathbb{R}e_2$. We have

$$e_1 \in E_1(f) \Rightarrow {e_1}^2 \in E_1(f).E_1(f) = E_1(f),$$

and

$$e_1 \in E_{-1}(g) \Rightarrow {e_1}^2 \in E_{-1}(g).E_{-1}(g) = E_1(g).$$

Therefore $e_1 \in E_1(f) \cap E_1(g) = \mathbb{R}e$. Thus by analogy, the elements ${e_i}^2 \in E_1(f) \cap E_1(g) = \mathbb{R}e$ for all $i \in \{2, 3\}$. Since $E_1(f) \cap E_1(g)$ is a subalgebra of A, the element e is a scalar multiple of an idempotent and can be assumed to be idempotent. □

Proposition 5. *Let A be a real division algebra of unit four-dimensional of unit e. Then the following propositions are equivalent,*

1. $Aut(A)$ *contains two distinct reflections which commute.*
2. $Aut(A)$ *contains a subgroup isomorphic to the Klein's group* $\mathbb{Z}_2 \times \mathbb{Z}_2$.
3. *A is isomorphic to the algebra* $\mathbb{H}(\alpha,\beta,\gamma,\alpha',\beta',\gamma')$.

Proof. (1) $\Longleftrightarrow$ (2) This is true according to Proposition 1.

(2) $\Longleftrightarrow$ (3) We have two distinct reflections which commute. Proposition 2 ensures the existence of a basis $\{e, e_1, e_2, e_3\}$ of A, and the subalgebras $\mathbb{R}e + \mathbb{R}e_i$, for all $i \in \{1, 2, 3\}$, are isomorphic to the algebra $\mathbb{C}$ since they are real algebras of unit division of two-dimensional. We can now set

$$E_1(f) = \mathbb{R}e + \mathbb{R}e_1,$$

$$E_{-1}(f) = \mathbb{R}e_2 + \mathbb{R}e_3,$$

$$E_1(g) = \mathbb{R}e + \mathbb{R}e_2$$

and

$$E_{-1}(g) = \mathbb{R}e_1 + \mathbb{R}e_3.$$

We have $e_1 \in E_1(f)$ *et* $e_2 \in E_{-1}(f) \Rightarrow e_1e_2,\ \ e_2e_1 \in E_1(f).E_{-1}(f) = E_{-1}(f)$, $e_1 \in E_{-1}(g)$ and $e_2 \in E_1(g) \Rightarrow e_1e_2,\ \ e_2e_1 \in E_{-1}(g).E_1(g) = E_{-1}(g)$.

As a result e_1e_2, and $e_2e_1 \in E_{-1}(f) \cap E_{-1}(g) = \mathbb{R}e_3$. In the same way we have

$$e_1e_3, \qquad e_3e_1 \in E_{-1}(f) \cap E_1(g) = \mathbb{R}e_2,$$
$$e_2e_3, \qquad e_3e_2 \in E_1(f) \cap E_{-1}(g) = \mathbb{R}e_1.$$

This gives the multiplication table for (1).

(3) $\Longrightarrow$ (1) $f : A \longrightarrow A$; $\lambda_0 e + \sum_{i=1}^{3} \lambda_i e_i \longmapsto \lambda_0 e + \lambda_1 e_1 - \sum_{i=2}^{3} \lambda_i e_i$ and $g : A \longrightarrow A$; $\lambda_0 e + \sum_{i=1}^{3} \lambda_i e_i \longmapsto \lambda_0 e - \lambda_1 e_1 + \lambda_2 e_2 - \lambda_3 e_3$ are distinct reflections that commute. □

Corollary 2. *If A is a real division algebra of four-dimensional with two commute distinct reflections then A is isotope in the sens of Albert to* $\mathbb{H}(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$ *with* $e \in I(A)$.

Proof. Let A be a real division algebra of four-dimensional having two distinct reflections that commute f and g. According to Proposition 2 there is a basis $\{e, e_1, e_2, e_3\}$ where $e \in I(A)$ and $f(e) = g(e) = e$. Let $x \in A$, we have

$$f \circ R_e(x) = f(R_e(x)) = f(xe) = f(x)f(e) = f(x)e = R_e(f(x)) = R_e \circ f(x),$$

so

$$f \circ R_e = R_e \circ f \Leftrightarrow f \circ {R_e}^{-1} = {R_e}^{-1} \circ f.$$

As well as $f \circ {L_e}^{-1} = {L_e}^{-1} \circ f$, $(A, \odot)$ where $x \odot y = {R_e}^{-1}(x).{L_e}^{-1}(y)$ for all $x, y \in A$, is isotope to A, and is unitary of unit e. We have,

$$f(x \odot y) = f({R_e}^{-1}(x).{L_e}^{-1}(y)) = f({R_e}^{-1}(x)).f({L_e}^{-1}(y)) = {R_e}^{-1}(f(x)).{L_e}^{-1}(f(y)) = f(x) \odot f(y),$$

then $f \in Aut((A, \odot))$. We also show by analogy that $g \in Aut((A, \odot))$. Hence $(A, \odot)$ verifies the assumptions of Proposition 5 therefore it is isomorphic to $\mathbb{H}(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$. □

Theorem 6. *Let A be a real division algebra of unit four-dimensional having two reflections that commute. Then the following statements are equivalent,*

1. *A is isomorphic to* $\mathbb{H}(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$*, with* $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \gamma$.
2. *A is at third power-associative.*
3. *A is at 121 power-associative.*

Proof. (1) $\Longrightarrow$ (2) For all $x = \lambda_0 e + \sum_{i=1}^{3} \lambda_i e_i \in \mathbb{H}(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$, with $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \gamma$, we have $x^2 = -N(x)e + 2\lambda_0 x$ with $N(x) = \sum_{i=0}^{3} {\lambda_i}^2$ and it is easy to check that $(x, x, x) = 0$

(2) $\Longrightarrow$ (3) It is clear from Remark 1

(3) $\Longrightarrow$ (1) A simple calculation show that $\alpha = \alpha'$, $\beta = \beta'$ et $\gamma = \gamma'$. □

5. A Note on Eight-dimensional Division Algebras having a Reflection

Proposition 6. *Let A be a division algebra of eight-dimensional, the following statements are equivalent,*

1. *$Aut(A)$ contains two distinct reflections which commute.*
2. *$Aut(A)$ contains a subgroup isomorphic to the Klein's group $\mathbb{Z}_2 \times \mathbb{Z}_2$.*
3. *There are vector subspaces X, Y, Z and T of two-dimensional, for which the multiplication of A is given by,*

	X	Y	Z	T
X	X	Y	Z	T
Y	Y	X	T	Z
Z	Z	T	X	Y
T	T	Z	Y	X

(3)

Proof. (1) $\Longrightarrow$ (2) This is true according to Proposition 5.

(2) $\Longrightarrow$ (3) The group $Aut(A)$ contains two distinct reflections f and g which commute. Then there is a basis $\{e, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ consisting of eigenvectors common to f and g where $E_1(f) = Lin\{e, e_1, e_2, e_3\}$ and $E_{-1}(f) = Lin\{e_4, e_5, e_6, e_7\}$. As $f \neq g$ the subalgebras $E_1(f)$, $E_1(g)$ of dimension 4, cannot coincide. Consequently the subalgebra $X := E_1(f) \cap E_1(g)$, is of dimension ≤ 2.

Moreover the subalgebra X cannot be reduced to $\mathbb{R}e$. Otherwise the vector subspace $Lin\{e_1, e_2, e_3\} := E$ of $E_1(f)$, would be contained in $E_{-1}(g)$ and we would have $E^2 \subset E_1(f)^2 \cap E_{-1}(g)^2 = E_1(f) \cap E_1(g) = \mathbb{R}e$ nonsense because, e_1, e_2 $e_3 \in E$ then $e_1e_2, e_1e_3 \in E^2 \subset \mathbb{R}e$ so there exist $\alpha, \beta \in \mathbb{R}$ nonzero, such that $e_1.e_2 = \alpha e$ and $e_1e_3 = \beta e$. We have $e_1.(\beta e_2 - \alpha e_3) = 0 \Longrightarrow \beta e_2 - \alpha e_3 = 0$, as A is of division so $\beta e_2 = \alpha e_3$ contradicting the fact that e_2 and e_3 are linearly independent. So X is of dimension 2.

If for example $X = Lin\{e, e_1\}$ we can state that $E_1(g) = Lin\{e, e_1, e_4, e_5\}$ and $E_{-1}(g) = Lin\{e_2, e_3, e_6, e_7\}$, we then obtain the following sub-vector spaces of dimension 2.

$$Y := E_1(f) \cap E_{-1}(g) = Lin\{e_2, e_3\},$$
$$Z := E_1(g) \cap E_{-1}(f) = Lin\{e_4, e_5\},$$
$$T := E_{-1}(f) \cap E_{-1}(g) = Lin\{e_6, e_7\}.$$

It is easy to show that the multiplication of A is done according to (3).

(3) $\Longrightarrow$ (1) The vector space A decomposes into a direct sum of the subspaces vector espaces X, Y, Z, T and the two endomorphisms $f, g : A := X \oplus Y \oplus Z \oplus T \to X \oplus Y \oplus Z \oplus T$ defined by, for all $u = x + y + z + t \in X \oplus Y \oplus Z \oplus T$ we have $f(u) = x + y - z - t$ and $g(u) = x - y + z - t$. They are distinct reflections, which commute. Thus the subgroup of $Aut(A)$ generated by f and g is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

Example 1. *Let A be a division algebra of eight-dimensional whose product in the base $B = \{e, u_1, \ldots, u_7\}$ is given by,*

.	e	u_1	u_2	u_3	u_4	u_5	u_6	u_7
e	e	u_1	u_2	u_3	u_4	u_5	u_6	u_7
u_1	u_1	$-e$	γu_3	$-\beta' u_2$	δu_5	$-\eta' u_4$	λu_7	$-\mu' u_6$
u_2	u_2	$-\gamma' u_3$	$-e$	αu_1	σu_6	ηu_7	$-\rho' u_4$	$-\xi' u_5$
u_3	u_3	βu_2	$-\alpha' u_1$	$-e$	πu_7	τu_6	$-\varepsilon' u_5$	$\kappa' u_4$
u_4	u_4	$-\delta' u_5$	$\sigma' u_6$	$-\pi' u_7$	$-e$	θu_1	ωu_2	ιu_3
u_5	u_5	ηu_4	$-\eta' u_7$	$-\pi' u_6$	$-\theta' u_1$	$-e$	χu_3	ζu_2
u_6	u_6	$-\lambda' u_7$	ρu_4	εu_5	$-\omega' u_2$	$-\chi' u_3$	$-e$	νu_1
u_7	u_7	μu_6	ξu_5	κu_4	$-\iota' u_3$	$-\zeta' u_2$	$-\nu' u_1$	$-e$

Let $x = x_0 e + \sum_{i=1}^{7} x_i u_i \in A$*, the endomorphisms* $f : A \longrightarrow A$ *and* $g : A \longrightarrow A$ *defined by* $f(x) = x_0 e + x_1 u_1 - x_2 u_2 - x_3 u_3 + x_4 u_4 + x_5 u_5 - x_6 u_6 - x_7 u_7$ *and* $g(x) = x_0 e + x_1 u_1 + x_2 u_2 + x_3 u_3 - x_4 u_4 - x_5 u_5 - x_6 u_6 - x_7 u_7$ *are automorphisms of A which commute. Thus* $Aut(A)$ *contains a subgroup isomorphic to the Klein's group* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

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Conflict of interest

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