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Article

On Some Exact Formulas for 2-Color Off-diagonal Rado Numbers

Jin Jing[1] **and Y. M. Mei**[2,*]

[1] College of Taizhou, Nanjing Normal University, Taizhou, 225300, Jiangsu Province, P. R. China
[2] School of Mathematical Sciences, Suzhou University of Science and Technology, Suzhou, 215009, Jiangsu Province, P. R. China

* **Correspondence:** mei@mail.usts.edu.cn

Abstract: Let ε_0, ε_1 be two linear homogenous equations, each with at least three variables and coefficients not all the same sign. Define the 2-color off-diagonal Rado number $R_2(\varepsilon_0, \varepsilon_1)$ to be the smallest N such that for any 2-coloring of $[1, N]$, it must admit a monochromatic solution to ε_0 of the first color or a monochromatic solution to ε_1 of the second color. Mayers and Robertson gave the exact 2-color off-diagonal Rado numbers $R_2(x + qy = z, x + sy = z)$. Xia and Yao established the formulas for $R_2(3x+3y = z, 3x+qy = z)$ and $R_2(2x+3y = z, 2x+2qy = z)$. In this paper, we determine the exact numbers $R_2(2x + qy = 2z, 2x + sy = 2z)$, where q, s are odd integers with $q > s \geq 1$.

Keywords: Schur number, Ramsey theory, Off-diagonal Rado number

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1. Introduction

Let $[a, b]$ denote the set $\{x \in \mathbb{Z} | a \leq x \leq b\}$. A function Δ: $[1, n] \to [0, k - 1]$ is called a k-coloring of the set $[1, n]$. If ε is a system of equations in m variables, then we say that a solution $x_1, x_2, \ldots, x_m$ to ε is monochromatic if and only if

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m).$$

In 1916, Schur [1] proved that for every integer $k \geq 2$, there exists a least integer $n = S(k)$ such that for every k-coloring of the set $[1, n]$, there exists a monochromatic solution to $x + y = z$, the integer S_k is called Schur number. Rado [2, 3] generalized the work of Schur to arbitrary system of linear equations and found necessary and sufficient conditions to determine if an arbitrary system of linear equations admits a monochromatic solution for every coloring of the natural numbers with a finite number of colors. For a given system of linear equations ε, the least integer n, provided that it exists, such that for every coloring of the set $[1, n]$ with k colors, there exists a monochromatic solution to ε, is called k-color Rado number. If such an integer n does not exist, then the k-color Rado number for the system ε is infinite. In recent years there has been considerable interest in finding the exact Rado numbers for particular linear equations and in several other closely related problems, see for example [4–12].

For positive integer $k \geq 2$ and equations ε_i, where $i = 0, 1, \ldots, k-1$, the k-color off-diagonal Rado number is the least integer N provided that it exists for which any k-coloring of $[1, N]$ must admit a monochromatic solution of color i to ε_i for some $i \in [0, k - 1]$. Note that if $\varepsilon_i = \varepsilon_j$ for all i, j

$(1 \le i, j \le k)$, then the k-color off-diagonal Rado number is the k-color Rado number. Robertson and Schaal [13] gave some 2-color off-diagonal Rado numbers for particular linear equations. In [14], Mayers and Robertson gave the exact 2-color off-diagonal Rado numbers when the two equations are of the form $x + qy = z$ and $x + sy = z$. They showed that

$$R_2(x + qy = z, x + y = z) = 2q + 2\left\lfloor \frac{q+1}{2} \right\rfloor + 1$$

and for $s \ge 2$,

$$R_2(x + qy = z, x + sy = z) = qs + q + 2s + 1.$$

Motivated by Myers and Robertson's work, Xia and Yao [15] proved the following results:

$$R_2(3x + 3y = z, 3x + 3qy = z) = 54q + 57, \qquad q \ge 2$$

and

$$R_2(2x + 3y = z, 2x + 2qy = z) = 20q + 26, \qquad q \ge 2.$$

In this paper, we are also interested in precise values of 2-color off-diagonal Rado numbers and let $R_2(\varepsilon_0, \varepsilon_1)$ denote it. Let blue and red be the two colors and denoted by 0 and 1, respectively. We determine the exact numbers $R_2(2x+qy = 2z, 2x+sy = 2z)$, where q, s are odd integers and $q > s \ge 1$. The main results of this paper can be stated as follows.

Theorem 1. *For odd integers q, s and $q > s > 2$, we have*

$$R_2(2x + qy = 2z, 2x + sy = 2z) = \min\{qs + q + 2s + 1, qs + 2q + 1\}.$$

Theorem 2. *For odd integer $q \ge 3$, we have*

$$R_2(2x + qy = 2z, 2x + y = 2z) = 2q + 4.$$

2. Lower Bounds

Lemma 1. *For odd integers q, s and $q > s > 2$, we have*

$$R_2(2x + qy = 2z, 2x + sy = 2z) \ge \min\{qs + q + 2s + 1, qs + 2q + 1\}.$$

Proof. Let $N = \min\{qs + q + 2s + 1, qs + 2q + 1\}$, consider the 2-coloring of $[1, N - 1]$ defined by coloring

$$R = \{2i | i = 1, 2, \dots, s\} \cup \{j | qs + q + 1 \le j \le N - 1 \text{ and } j \text{ is an odd integer}\}$$

red and its complement blue. Now, we show that there is no suitable red solution to $2x + sy = 2z$ and no suitable blue solution to $2x + qy = 2z$. Assume that (x_0, y_0, z_0) is a red solution to $2x + sy = 2z$. Note that

$$qs + q + 1 > s(s + 2)$$

since $q > s$. If $z_0 < qs + q + 1$, then there exist integers a and b such that $x_0 = 2a$, $z_0 = 2b$ and $1 \le a < b < s$. Thus, $sy_0 = 4(b - a)$. The fact that s is an odd integer implies that $4 | y_0$ and $y_0 \ge 4$. However,

$$z_0 = \frac{2x_0 + sy_0}{2} \ge \frac{4 + 4s}{2} = 2s + 2,$$

which is a contradiction and therefore $z_0 \ge qs + q + 1$. If $y_0 \ge qs + q + 1$, then

$$z_0 = \frac{2x_0 + sy_0}{2} \ge \frac{4 + s(qs + q + 1)}{2} > N - 1,$$

which is a contradiction and hence, $y_0 \in \{2i | i = 1, 2, \ldots, s\}$. If $x_0 \in \{2i | i = 1, 2, \ldots, s\}$, we see that

$$z_0 = \frac{2x_0 + sy_0}{2} \leq 2s + s^2 < qs + q + 1,$$

which is a contradiction and thus, $x_0 \geq qs+q+1$. Note that z_0 and x_0 are all odd, therefore, $4|(2z_0-2x_0)$, i.e., $4|sy_0$. Recall that s is odd, therefore, $4|y_0$ and $y_0 \geq 4$. However,

$$z_0 = \frac{2x_0 + sy_0}{2} \geq \frac{2(qs + q + 1) + 4s}{2} = qs + q + 2s + 1 > N - 1,$$

which is a contradiction, and there is no suitable red solution to $2x + sy = 2z$.

Assume that (x_0, y_0, z_0) is a blue solution to $2x + qy = 2z$. Note that y_0 must be even this is because q is an odd integer. Therefore $y_0 \geq 2s + 2$ and

$$z_0 = \frac{2x_0 + qy_0}{2} \geq \frac{2 + q(2s + 2)}{2} = qs + q + 1,$$

which implies that z_0 must be even. If $y_0 = 2s+2$, then $2x_0 = 2z_0 - q(2s+2)$, that is, $x_0 = z_0 - q(s+1)$. Therefore, $2|x_0$ and $x_0 \geq 2s + 2$. However,

$$z_0 = \frac{2x_0 + qy_0}{2} \geq \frac{2(2s + 2) + q(2s + 2)}{2} = qs + q + 2s + 2 > N - 1,$$

which is a contradiction. If $y_0 \geq 2s + 4$, then

$$z_0 = \frac{2x_0 + qy_0}{2} \geq \frac{2 \times 1 + q(2s + 4)}{2} = qs + 2q + 1 > N - 1,$$

which is a contradiction. Thus, there is no suitable blue solution to $2x + qy = 2z$. The proof is complete. □

Lemma 2. *For odd integer $q > 2$, we have*

$$R_2(2x + qy = 2z, 2x + y = 2z) \geq 2q + 4. \tag{1}$$

Proof. Let $\{1\} \bigcup [3, 2q] \bigcup \{2q + 2\}$ be colored blue and $\{2, 2q + 1, 2q + 3\}$ be colored red. It is easy to verify that there is no suitable blue solutions to $2x + qy = 2z$ and no suitable red solution to $2x + y = 2z$. □

3. Proof of Theorem 1

Proof. It follows from Lemma 1 that

$$R_2(2x + qy = 2z, 2x + sy = 2z) \geq \min\{qs + q + 2s + 1, qs + 2q + 1\}, \tag{2}$$

where q, s are odd integers and $q > s \geq 3$. In order to prove this theorem, it suffices to show that

$$R_2(2x + qy = 2z, 2x + sy = 2z) \leq \min\{qs + q + 2s + 1, qs + 2q + 1\},$$

where q, s are odd integers and $q > s \geq 3$. Let Δ be a 2-coloring of $[1, \min\{qs+q+2s+1, qs+2q+1\}]$ using the colors red and blue. Without loss of generality, we assume, for contradiction, that there is no blue solution to $2x + qy = 2z$ and no red solution to $2x + sy = 2z$. We break the argument into 4 cases;

Case 1: $\Delta(2) = 0$ and $\Delta(s) = 0$.

$\Delta(2) = 0$ and $\Delta(s) = 0$ imply that $\Delta(q+s) = 1$, otherwise $(s, 2, q+s)$ is a blue solution to $2x+qy = 2z$. Since $\Delta(q+s) = 1$, we must have $\Delta(q+s+\frac{(q+s)s}{2}) = 0$ or else $(q+s, q+s, q+s+\frac{(q+s)s}{2})$ is a red solution to $2x+sy = 2z$. Now, $\Delta(q+s+\frac{(q+s)s}{2}) = 0$ and $\Delta(2) = 0$, so $\Delta(2q+s+\frac{(q+s)s}{2}) = 1$, otherwise $(q+s+\frac{(q+s)s}{2}, 2, 2q+s+\frac{(q+s)s}{2})$ is a blue solution to $2x+qy = 2z$. Note that

$$2q + s + \frac{(q+s)s}{2} \leq \min\{qs + q + 2s + 1, qs + 2q + 1\}$$

since $q - s \geq 2$. $\Delta(2q+s+\frac{(q+s)s}{2}) = 1$ and $\Delta(q+s) = 1$ imply that $\Delta(2q+s) = 0$, otherwise $(2q+s, s+q, 2q+s+\frac{(q+s)s}{2})$ is a red solution to $2x+sy = 2z$. Since $\Delta(2q+s) = 0$ and $\Delta(s) = 0$, we must have $\Delta(4) = 1$ or else $(s, 4, 2q+s)$ is a blue solution to $2x+qy = 2z$. If $\Delta(q+3s) = 1$, then $(q+s, 4, q+3s)$ is a red solution to $2x+sy = 2z$, so we may assume that $\Delta(q+3s) = 0$. If $\Delta(3s) = 0$, then $(3s, 2, q+3s)$ is a blue solution to $2x+qy = 2z$, so we may assume that $\Delta(3s) = 1$. Now, $\Delta(3s) = 1$ and $\Delta(4) = 1$, we must have $\Delta(5s) = 0$ or else $(3s, 4, 5s)$ is a red solution to $2x+sy = 2z$. $\Delta(5s) = 0$ and $\Delta(2) = 0$, imply that $\Delta(q+5s) = 1$, otherwise $(5s, 2, q+5s)$ is a blue solution to $2x+qy = 2z$. Since $\Delta(q+5s) = 1$ and $\Delta(q+s) = 1$, we must have $\Delta(8) = 0$ or else $(q+s, 8, q+5s)$ is a red solution to $2x+sy = 2z$. If $\Delta(4q+s) = 0$, then $(s, 8, 4q+s)$ is a blue solution to $2x+qy = 2z$, so we may assume that $\Delta(4q+s) = 1$. If $\Delta(4q-s) = 1$, then $(4q-s, 4, 4q+s)$ is a red solution to $2x+sy = 2z$, so we may assume that $\Delta(4q-s) = 0$. Since $\Delta(4q-s) = 0$ and $\Delta(2) = 0$, we must have $\Delta(3q-s) = 1$ or else $(3q-s, 2, 4q-s)$ is a blue solution to $2x+qy = 2z$. $\Delta(3q-s) = 1$ and $\Delta(4) = 1$ imply that $\Delta(3q+s) = 0$, otherwise $(3q-s, 4, 3q+s)$ is a red solution to $2x+sy = 2z$. Now we have $\Delta(3q+s) = 0$, $\Delta(2) = 0$ and $\Delta(2q+s) = 0$, and then $(2q+s, 2, 3q+s)$ is a blue solution to $2x+qy = 2z$, which is a contradiction.

Case 2: $\Delta(2) = 0$ and $\Delta(s) = 1$.

If $\Delta(2) = \Delta(4) = \cdots = \Delta(2q+2) = 0$ and then $(2, 4, 2q+2)$ is a blue solution to $2x+qy = 2z$, which is a contradiction. So we can assume that k $(1 \leq k < q+1)$ is the least number such that $\Delta(2) = \cdots = \Delta(2k) = 0$ and $\Delta(2k+2) = 1$.

Subcase 1: $k \leq \min\{\frac{(s-1)q+1}{s}, q-2\}$.

$\Delta(2k+2) = 1$ and $\Delta(s) = 1$ implies that $\Delta(ks+2s) = 0$, otherwise $(s, 2k+2, ks+2s)$ is a red solution to $2x+sy = 2z$. Since $\Delta(ks+2s) = 0$ and $\Delta(2) = 0$, we must have $\Delta(q+ks+2s) = 1$ or else $(ks+2s, 2, q+ks+2s)$ is a blue solution to $2x+qy = 2z$. $\Delta(q+ks+2s) = 1$ and $\Delta(2k+2) = 1$ imply that $\Delta(q+s) = 0$, otherwise $(q+s, 2k+2, q+ks+2s)$ is a red solution to $2x+sy = 2z$. If $\Delta(2q+s) = 0$, then $(q+s, 2, 2q+s)$ is a blue solution to $2x+qy = 2z$, so we may assume that $\Delta(2q+s) = 1$. If $\Delta(2q+ks+2s) = 1$ then $(2q+s, 2k+2, 2q+ks+2s)$ is a red solution to $2x+sy = 2z$, so we may assume that $\Delta(2q+ks+2s) = 0$. Note that

$$2q + ks + 2s \leq \min\{qs + q + 2s + 1, qs + 2q + 1\}$$

since $k \leq \min\{\frac{(s-1)q+1}{s}, q-2\}$. Since $\Delta(2q+ks+2s) = 0$ and $\Delta(ks+2s) = 0$, we must have $\Delta(4) = 1$, otherwise $(ks+2s, 4, ks+2s+2q)$ is a blue solution to $2x+qy = 2z$. $\Delta(4) = 1$ and $\Delta(2q+s) = 1$ imply that $\Delta(2q-s) = 0$ or else $(2q-s, 4, 2q+s)$ is a red solution to $2x+sy = 2z$. If $\Delta(3q-s) = 0$, then $(2q-s, 2, 3q-s)$ is a blue solution to $2x+qy = 2z$, so we may assume that $\Delta(3q-s) = 1$. Since $\Delta(3q-s) = 1$ and $\Delta(4) = 1$, we must have $\Delta(3q+s) = 0$ or else $(3q-s, 4, 3q+s)$ is a red solution to $2x+sy = 2z$. If $\Delta(4q+s) = 0$, then $(3q+s, 2, 4q+s)$ is a blue solution to $2x+qy = 2z$, so we may assume that $\Delta(4q+s) = 1$. If $\Delta(4q-s) = 1$, then $(4q-s, 4, 4q+s)$ is a red solution to $2x+sy = 2z$, so we may assume that $\Delta(4q-s) = 0$. $\Delta(s) = 1$ and $\Delta(4) = 1$ imply that $\Delta(3s) = 0$ or else $(s, 4, 3s)$ is a red solution to $2x+sy = 2z$. Since $\Delta(3s) = 0$ and $\Delta(2) = 0$, we must have $\Delta(q+3s) = 1$ or else $(3s, 2, q+3s)$ is a blue solution to $2x+qy = 2z$. $\Delta(4) = 1$ and $\Delta(q+3s) = 1$ imply that $\Delta(q+5s) = 0$, otherwise $(q+3s, 4, q+5s)$

is a red solution to $2x + sy = 2z$. $\Delta(q + 5s) = 0$ and $\Delta(2) = 0$ imply that $\Delta(5s) = 1$, otherwise $(5s, 2, q + 5s)$ is a blue solution to $2x + qy = 2z$. Since $\Delta(5s) = 1$ and $\Delta(s) = 1$, we must have $\Delta(8) = 0$ or else $(s, 8, 5s)$ is a red solution to $2x + sy = 2z$. When $s = 3$, by the above proof, we have $\Delta(4q + s) = \Delta(4q + 3) = 1$ and $\Delta(q + 3s) = \Delta(q + 9) = 1$, then $\Delta(2q - 4) = 0$, otherwise $(q + 9, 2q - 4, 4q + 3)$ is a red solution to $2x + 3y = 2z$. $\Delta(2) = 0$ implies that $\Delta(q + 2) = 1$ or else $(2, 2, q + 2)$ is a blue solution to $2x + qy = 2z$. Since $\Delta(q + 2) = 1$ and $\Delta(4) = 1$, we must have $\Delta(q - 4) = 0$, otherwise $(q - 4, 4, q + 2)$ is a red solution to $2x + 3y = 2z$. Now we have $\Delta(2q - 4) = 0$, $\Delta(q - 4) = 0$ and $\Delta(2) = 0$, and then $(q - 4, 2, 2q - 4)$ is a blue solution to $2x + qy = 2z$, which is a contradiction. When $s \geq 5$, since $\Delta(8) = 0$ and $\Delta(q + s) = 0$, we must have $\Delta(5q + s) = 1$, otherwise $(q + s, 8, 5q + s)$ is a blue solution to $2x + qy = 2z$. Note that

$$5q + s \leq \min\{qs + q + 2s + 1, qs + 2q + 1\},$$

since $s \geq 5$. $\Delta(5q + s) = 1$ and $\Delta(4) = 1$ imply that $\Delta(5q - s) = 0$ or else $(5q - s, 4, 5q + s)$ is a red solution to $2x + sy = 2z$. Now we have $\Delta(5q - s) = 0$, $\Delta(2) = 0$ and $\Delta(4q - s) = 0$, and then $(4q - s, 2, 5q - s)$ is a blue solution to $2z + qy = 2z$, which is a contradiction.

Subcase 2: $\min\{\frac{(s-1)q+1}{s}, q - 2\} < k \leq q$.
It is easy to verify that

$$q + 1 \leq 2 \min\left\{\frac{(s-1)q + 1}{s}, q - 2\right\}$$

and

$$s + 1 < 2s \leq 2 \min\left\{\frac{(s-1)q + 1}{s}, q - 2\right\}.$$

Thus, $q + 1 \leq 2k$ and $s + 1 < 2s \leq 2k$, and we have $\Delta(s + 1) = \Delta(2s) = \Delta(q + 1) = 0$. Obviously, $k \geq 2$ and $\Delta(2) = \Delta(4) = 0$, then $\Delta(2q + 2) = 1$, otherwise $(2, 4, 2q + 2)$ is a blue solution to $2x + qy = 2z$. Since $\Delta(q + 1) = 0$ and $\Delta(2) = 0$, we must have $\Delta(1) = 1$ or else $(1, 2, q + 1)$ is a blue solution to $2x + qy = 2z$. $\Delta(1) = 1$ and $\Delta(2q + 2) = 1$ imply that $\Delta(qs + s + 1) = 0$, otherwise $(1, 2q + 2, qs + s + 1)$ is a red solution to $2x + sy = 2z$. Now we have $\Delta(qs + s + 1) = 0$, $\Delta(2s) = 0$ and $\Delta(s + 1) = 0$, and then $(s + 1, 2s, qs + s + 1)$ is a blue solution to $2x + qy = 2z$, which is a contradiction.

Case 3: $\Delta(2) = 1$ and $\Delta(s) = 0$.
If $\Delta(2) = \Delta(4) = \cdots = \Delta(2s + 2) = 1$, then $(2, 4, 2s + 2)$ is a red solution to $2x + sy = 2z$, which is a contradiction. So we can assume that l $(1 \leq l < s + 1)$ is the least integer such that $\Delta(2) = \cdots = \Delta(2l) = 1$ and $\Delta(2l + 2) = 0$.

Subcase 1: $l \leq s - 1$.
$\Delta(2) = 1$ implies that $\Delta(s + 2) = 0$, otherwise $(2, 2, s + 2)$ is a red solution to $2x + sy = 2z$. If $\Delta((l+1)q+s+2) = 0$, then $(s+2, 2l+2, (l+1)q+s+2)$ is a blue solution to $2x+qy = 2z$, so we may assume that $\Delta((l+1)q+s+2) = 1$. If $\Delta((l+1)q+2s+2) = 1$, then $((l+1)q+s+2, 2, (l+1)q+2s+2)$ is a red solution to $2x + sy = 2z$, so we may assume that $\Delta((l + 1)q + 2s + 2) = 0$. Note that

$$(l + 1)q + 2s + 2 \leq \min\{qs + q + 2s + 1, qs + 2q + 1\}$$

since $l \leq s - 1$. $\Delta((l + 1)q + 2s + 2) = 0$ and $\Delta(2l + 2) = 0$ imply that $\Delta(2s + 2) = 1$, otherwise $(2s + 2, 2l + 2, (l + 1)q + 2s + 2)$ is a blue solution to $2x + qy = 2z$. Since $\Delta(2s + 2) = 1$ and $\Delta(2) = 1$, we must have $\Delta(4) = 0$ or else $(2, 4, 2s + 2)$ is a red solution to $2x + sy = 2z$. $\Delta(s) = 0$ and $\Delta(4) = 0$ imply that $\Delta(2q + s) = 1$, otherwise $(s, 4, 2q + s)$ is a blue solution to $2x + qy = 2z$. If $\Delta(2q+2s) = 1$, then $(2q+s, 2, 2q+2s)$ is a red solution to $2x+sy = 2z$, so we may assume that $\Delta(2q + 2s) = 0$. If $\Delta(2s) = 0$, then $(2s, 4, 2q + 2s)$ is a blue solution to $2x + qy = 2z$, so we may assume that $\Delta(2s) = 1$. Now, $\Delta(2s) = 1$ and $\Delta(2) = 1$, we must have $\Delta(3s) = 0$ or else $(2s, 2, 3s)$ is a red solution to $2x + sy = 2z$. If $\Delta(3s + 2q) = 0$, then $(3s, 4, 3s + 2q)$ is a blue solution to

$2x + qy = 2z$, so we may assume that $\Delta(3s + 2q) = 1$. Since $\Delta(3s + 2q) = 1$ and $\Delta(2) = 1$, we must have $\Delta(2q + 4s) = 0$ or else $(3s + 2q, 2, 4s + 2q)$ is a red solution to $2x + sy = 2z$. Since $\Delta(2q + 4s) = 0$ and $\Delta(4) = 0$, we must have $\Delta(4s) = 1$, otherwise $(4s, 4, 2q + 4s)$ is a blue solution to $2x + qy = 2z$. $\Delta(4s) = 1$ and $\Delta(2) = 1$ imply that $\Delta(5s) = 0$, otherwise $(4s, 2, 5s)$ is a red solution to $2x + sy = 2z$. If $\Delta(5s + 2q) = 0$, then $(5s, 4, 5s + 2q)$ is a blue solution to $2x + qy = 2z$, so we may assume that $\Delta(5s + 2q) = 1$. When $q = 5$, then $s = 3$, by the above proof, we have $\Delta(2s + 2) = \Delta(8) = 1$ and $\Delta(2) = 1$, then $\Delta(14) = 0$, otherwise $(2, 8, 14)$ is a red solution to $2x + 3y = 2z$. Now we have $\Delta(14) = 0$ and $\Delta(4) = 0$, so $(4, 4, 14)$ is a blue solution to $2x + 5y = 2z$, which is a contradiction. When $q > 5$ and $s = 3$, $\Delta(4) = 0$ implies that $\Delta(2q+4) = 1$, otherwise $(4, 4, 2q+4)$ is a blue solution to $2x+qy = 2z$. Since $\Delta(2q+4) = 1$ and $\Delta(2) = 1$, we must have $\Delta(2q + 1) = 0$ or else $(2q + 1, 2, 2q + 4)$ is a red solution to $2x + 3y = 2z$. If $\Delta(4q + 1) = 0$, then $(2q + 1, 4, 4q + 1)$ is a blue solution to $2x + qy = 2z$, so we may assume that $\Delta(4q + 1) = 1$. If $\Delta(4q + 4) = 1$, then $(4q + 1, 2, 4q + 4)$ is a red solution to $2x + 3y = 2z$, so we may assume that $\Delta(4q + 4) = 0$. Now we have $\Delta(8) = 0$, $\Delta(4) = 0$ and $\Delta(4q + 4) = 0$, so $(4, 8, 4q + 4)$ is a blue solution to $2x + qy = 2z$, which is a contradiction. When $q > s \geq 5$, we see that

$$\max\{2q + 6s, 6q + s\} \leq \min\{qs + q + 2s + 1, qs + 2q + 1\}.$$

Since $\Delta(5s + 2q) = 1$ and $\Delta(2) = 1$, then $\Delta(2q + 6s) = 0$ or else $(2q + 5s, 2, 2q + 6s)$ is a red solution to $2x + sy = 2z$. If $\Delta(6s) = 1$, then $(6s, 4, 2q + 6s)$ is a blue solution to $2x + qy = 2z$. Since $\Delta(6s) = 1$ and $\Delta(2s) = 1$, we must have $\Delta(8) = 0$, otherwise $(2s, 8, 6s)$ is a red solution to $2x + sy = 2z$. $\Delta(2q + s) = 1$ and $\Delta(2) = 1$ imply that $\Delta(2q) = 0$ or else $(2q, 2, 2q + s)$ is a red solution to $2x + sy = 2z$. Since $\Delta(2q) = 0$ and $\Delta(4) = 0$, then $\Delta(4q) = 1$, otherwise $(2q, 4, 4q)$ is a blue solution to $2x + qy = 2z$. If $\Delta(4q + s) = 1$, then $(4q, 2, 4q + s)$ is a red solution to $2x + sy = 2z$, so we may assume that $\Delta(4q + s) = 0$. If $\Delta(6q + s) = 0$, then $(4q + s, 4, 6q + s)$ is a blue solution to $2x + qy = 2z$. Since $\Delta(6q + s) = 1$ and $\Delta(2) = 1$, we must have $\Delta(6q) = 0$, otherwise $(6q, 2, 6q + s)$ is a red solution to $2x + sy = 2z$. Now we have $\Delta(2q) = 0$, $\Delta(8) = 0$ and $\Delta(6q) = 0$, so $(2q, 8, 6q)$ is a blue solution to $2x + qy = 2z$, which is a contradiction.

Subcase 2: $l = s$.

We have $\Delta(2) = \Delta(4) = \cdots = \Delta(2s) = 1$ and $\Delta(2s + 2) = 0$. Note that $2|(s + 1)$ and $s + 1 < 2s$, thus, $\Delta(s + 1) = 1$. Since $\Delta(2) = 1$ and $\Delta(s + 1) = 1$, we must have $\Delta(2s + 1) = 0$ and $\Delta(1) = 0$ or else $(s + 1, 2, 2s + 1)$ and $(1, 2, s + 1)$ are red solutions to $2x + sy = 2z$. When $2s \leq q$, since $\Delta(2s+1) = 0$ and $\Delta(2s+2) = 0$, then $\Delta(qs+q+2s+1) = 1$, otherwise $(2s+1, 2s+2, qs+q+2s+1)$ is a blue solution to $2x+qy = 2z$. If $\Delta(qs+q+1) = 0$, then $(1, 2s+2, qs+q+1)$ is a blue solution to $2x + qy = 2z$, so we may assume that $\Delta(qs + q + 1) = 1$. Now we have $\Delta(qs + q + 1) = 1$, $\Delta(4) = 1$ and $\Delta(qs + q + 2s + 1) = 1$, thus $(qs + q + 1, 4, qs + q + 2s + 1)$ is a red solution to $2x + sy = 2z$, which is a contradiction. When $2s \geq q + 1$, note that $2|(q + 1)$, so $\Delta(q + 1) = 1$. If $\Delta(q + 1 - s) = 1$, then $(q + 1 - s, 2, q + 1)$ is a red solution to $2x + sy = 2z$, so we may assume that $\Delta(q + 1 - s) = 0$. Since $\Delta(q + 1 - s) = 0$ and $\Delta(2s + 2) = 0$, we must have $\Delta(qs + 2q + 1 - s) = 1$ or else $(q + 1 - s, 2s + 2, qs + 2q + 1 - s)$ is a blue solution to $2x + qy = 2z$. $\Delta(4) = 1$ implies that $\Delta(2s + 4) = 0$, otherwise $(4, 4, 2s + 4)$ is a red solution to $2x + sy = 2z$. If $\Delta(qs+2q+1) = 0$, then $(1, 2s+4, qs+2q+1)$ is a blue solution to $2x+qy = qz$, so we may assume that $\Delta(qs + 2q + 1) = 1$. Now we have $\Delta(qs + 2q + 1) = 1$, $\Delta(2) = 1$ and $\Delta(qs + 2q + 1 - s) = 1$, so $(qs + 2q + 1 - s, 2, qs + 2q + 1)$ is a red solution to $2x + sy = 2z$, which is a contradiction.

Case 4: $\Delta(2) = 1$ and $\Delta(s) = 1$.

Since $\Delta(2) = \Delta(s) = 1$, we must have $\Delta(s+2) = 0$ and $\Delta(2s) = 0$ or else $(2, 2, s+2)$ and $(s, 2, 2s)$ are red solutions to $2x + sy = 2z$. If $\Delta(qs + s + 2) = 0$, then $(s + 2, 2s, qs + s + 2)$ is a blue solution to $2x + qy = 2z$, so we may assume that $\Delta(qs + s + 2) = 1$. If $\Delta(qs + 2s + 2) = 1$, then $(qs + s + 2, 2, qs + 2s + 2)$ is a red solution to $2x + sy = 2z$, so we may assume that $\Delta(qs + 2s + 2) = 0$. Since $\Delta(qs + 2s + 2) = 0$ and $\Delta(2s) = 0$, we must have $\Delta(2s + 2) = 1$, otherwise $(2s+2, 2s, qs+2s+2)$ is a blue solution to $2x+qy=2z$. $\Delta(2s+2)=1$ and $\Delta(2)=1$ imply that $\Delta(4)=0$, otherwise $(2,4,2s+2)$ is a red solution to $2x+sy=2z$. $\Delta(4)=0$ implies that $\Delta(2q+4)=1$, otherwise $(4,4,2q+4)$ is a blue solution to $2x+qy=2z$. If $\Delta(2q+4-s)=1$, then $(2q+4-s, 2, 2q+4)$ is a red solution to $2x+sy=2z$, so we may assume that $\Delta(2q+4-s)=0$. If $\Delta(4q+4-s)=0$, then $(2q+4-s, 4, 4q+4-s)$ is a blue solution to $2x+qy=2z$, so we may assume that $\Delta(4q+4-s)=1$. If $\Delta(4q+4)=1$, then $(4q+4-s, 2, 4q+4)$ is a red solution to $2x+sy=2z$, so we may assume that $\Delta(4q+4)=0$. Since $\Delta(4q+4)=0$ and $\Delta(4)=0$, we must have $\Delta(8)=1$, otherwise $(4,8,4q+4)$ is a blue solution to $2x+qy=2z$. $\Delta(8)=1$ and $\Delta(s)=1$ imply that $\Delta(5s)=0$ or else $(s,8,5s)$ is a red solution to $2x+sy=2z$. Since $\Delta(4)=0$ and $\Delta(5s)=0$, we must have $\Delta(2q+5s)=1$ or else $(5s, 4, 2q+5s)$ is a blue solution to $2x+qy=2z$. Note that

$$\min\{qs+q+2s+1, qs+2q+1\} \geq 2q+5s$$

since $s \geq 3$ and $q \geq 5$. $\Delta(2q+5s)=1$ and $\Delta(2)=1$ imply that $\Delta(2q+4s)=0$, otherwise $(2q+4s, 2, 2q+5s)$ is a red solution to $2x+sy=2z$. If $\Delta(4s)=0$, then $(4s, 4, 2q+4s)$ is a blue solution to $2x+qy=2z$, so we may assume that $\Delta(4s)=1$. If $\Delta(3s)=1$, then $(3s,2,4s)$ is a red solution to $2x+sy=2z$, so we may assume that $\Delta(3s)=0$. If $\Delta(2q+3s)=0$, then $(3s,4,2q+3s)$ is a blue solution to $2x+qy=2z$, so we may assume that $\Delta(2q+3s)=1$. If $\Delta(2q+2s)=1$, then $(2q+2s, 2, 2q+3s)$ is a red solution to $2x+sy=2z$, so we may assume that $\Delta(2q+2s)=0$. Now we have $\Delta(2s)=0$, $\Delta(4)=0$ and $\Delta(2q+2s)=0$, so $(2s, 4, 2q+2s)$ is a blue solution to $2x+qy=2z$, which is a contradiction and the proof is complete.

□

4. Proof of Theorem 2

Proof. It follows from Lemma 2 that when $q \geq 3$ is an odd integer,

$$R_2(2x+qy=2z, 2x+y=2z) \geq 2q+4.$$

In order to prove this theorem, it suffices to show that when $q \geq 3$ is an odd integer,

$$R_2(2x+qy=2z, 2x+y=2z) \leq 2q+4.$$

Let Δ be a 2-coloring of $[1, 2q+4]$ using the colors red and blue. Without loss of generality, we assume, for contradiction, that there is no blue solution to $2x+qy=2z$ and no red solution to $2x+y=2z$. We break our proof into 3 cases.

Case 1: $\Delta(2)=1$.
Since $\Delta(2)=1$, then the solution $(2,4,4)$ to $2x+y=2z$ forces $\Delta(4)=0$ and the solution $(2,2,3)$ to $2x+y=2z$ forces $\Delta(3)=0$. Since $\Delta(4)=0$, then the solution $(4,4,2q+4)$ to $2x+qy=2z$ forces $\Delta(2q+4)=1$. Since $\Delta(2)=1$ and $\Delta(2q+4)=1$, then the solution $(2q+3, 2, 2q+4)$ to $2x+y=2z$ forces $\Delta(2q+3)=0$. Now we have $\Delta(3)=\Delta(4)=\Delta(2q+3)=0$, so $(3,4,2q+3)$ is a blue solution to $2x+qy=2z$, which is a contradiction.

Case 2: $\Delta(2)=0$ and $\Delta(4)=1$.
$\Delta(4)=1$ implies that $\Delta(6)=0$ or else $(4,4,6)$ is a red solution to $2x+y=2z$. Since $\Delta(6)=0$ and $\Delta(2)=0$, we must have $\Delta(q+6)=1$, otherwise $(6,2,q+6)$ is a blue solution to $2x+qy=2z$. If $\Delta(2q+4)=1$, then $(4, 2q+4, q+6)$ is a red solution to $2x+y=2z$, so we may assume that $\Delta(2q+4)=0$. If $\Delta(q+4)=0$, then $(q+4, 2, 2q+4)$ is a blue solution to $2x+qy=2z$, so we may assume that $\Delta(q+4)=1$. If $\Delta(q+2)=1$, then $(q+2, 4, q+4)$ is a red solution to $2x+y=2z$, so we have $\Delta(q+2)=0$. Now we have $\Delta(2)=0$ and $\Delta(q+2)=0$, so $(2,2,q+2)$ is a blue solution to $2x+qy=2z$, which is a contradiction.

Case 3: $\Delta(2) = \Delta(4) = 0$.

$\Delta(2) = 0$ implies that $\Delta(q + 2) = 1$, otherwise $(2, 2, q + 2)$ is a blue solution to $2x + qy = 2z$. If $\Delta(2q + 2) = 0$, then $(2, 4, 2q + 2)$ is a blue solution to $2x + qy = 2z$, so we may assume that $\Delta(2q + 2) = 1$. Since $\Delta(q + 2) = 1$ and $\Delta(2q + 2) = 1$, we must have $\Delta(1) = 0$, otherwise $(1, 2q + 2, q + 2)$ is a red solution to $2x + y = 2z$. $\Delta(1) = 0$ and $\Delta(2) = 0$ imply that $\Delta(q + 1) = 1$, otherwise $(1, 2, q + 1)$ is a blue solution to $2x + qy = 2z$. Now we have $\Delta(2q + 2) = 1$ and $\Delta(q + 1) = 1$, so $(q + 1, 2q + 2, 2q + 2)$ is a red solution to $2x + y = 2z$, which is a contradiction and this completes the proof.

□

Conflict of interest

The authors declare no conflict of interest.

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