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Article

Order Structure of Good Sets in Hypercube

N. V. Shinde[1] **S. A. Tapadia**[2] **and B. N. Waphare**[3*]

[1] Department of Mathematics, COEP Technological University, Pune-411005, India.
[2] Department of Engineering Sciences, Vishwakarma University, Pune-411048, India.
[3] Center for Advanced Studies in Mathematics, Department of Mathematics, Savitribai Phule Pune University, Pune-411007, India.

* **Correspondence:** waphare@yahoo.com.

Abstract: A good set on k vertices is a vertex induced subgraph of the hypercube Q_n that has the maximum number of edges. The long-lasting problem of characterizing graphs that are cover graphs of lattices is NP-complete. This paper constructs and studies lattice theoretic properties of a class of lattices whose cover graphs are isomorphic to good sets.

Keywords: hypercube, partial cube, lattice.

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1. Introduction

The n-dimensional hypercube, Q_n, is a graph in which a binary n-tuple is assigned to every vertex, and any two vertices are adjacent if they differ in exactly one coordinate. Q_n is an n-regular, n-connected, bipartite, and vertex transitive graph on 2^n vertices. Moreover, Q_n can be viewed as a lattice. It is the direct product of 2-element chain n times. Its cover graph is the same as Q_n.

Hart [1] defined *good set* as the vertex induced subgraph of hypercube having the maximum number of edges on the given set of vertices. It is well known that hypercube is one of the most popular topologies for the interconnection of computing nodes in multiprocessor systems. Nevertheless, as the number of nodes in such systems must be a power of 2, there are significant gaps in the sizes of the systems that can be built. To overcome this restriction, Katseff [2] proposed incomplete hypercube that can be viewed as a hypercube that operates in a degraded manner after some nodes become faulty. Interestingly, incomplete hypercubes are good sets only. Compact hypercubes [3], composite hypercubes [4] are some more synonyms of good sets. Tzeng [5] studied some structural properties, while Sen et al. [6] proved some topological properties of good sets. Tapadia and Waphare proved graph theoretic properties like good sets are partial cubes, Hamiltonian, bipancyclic, etc. [7]. One can refer for more properties of good sets in [8–12].

We look at the lattice theoretic aspect of a good set in view of the following classic problem in lattice theory.

Open Problem: Which graphs are the cover graphs of finite posets/lattices?

This problem was first posed by Ore [13] for partially ordered sets and then restated by Rival [14] for ordered sets. Cover graphs of finite posets have already been characterised. However, the problem is NP-complete for lattices [15]. In this paper, we construct a lattice called *upper good lattice* in Q_n, whose underlying cover graph is isomorphic to the good set. We also study lattice theoretic properties of the upper good lattice.

2. Preliminaries

A partially ordered set(poset) $(L; \leq)$ consists of a nonempty set L and a binary relation $\leq$ on L, which is reflexive, antisymmetric and transitive. If a covers b, then we use notation $b \prec a$. A poset $(L; \leq)$ is a lattice if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in L$. Also, if $(L; \leq)$ is a lattice, so is its dual $(L; \geq)$. For a lattice $(L; \leq)$ and $M \subseteq L$, $(M; \leq)$ is a sublattice of $(L; \leq)$ if $a, b \in M$ implies that $\sup\{a, b\}$ and $\inf\{a, b\}$ lies in M. A poset $(L; \leq)$ is a join-semilattice if $\sup\{a, b\}$ exists for all $a, b \in L$. If $(L; \leq)$ is a poset and $M \subseteq L$ such that $a, b \in M$ implies that $sup\{a, b\} \in M$, then M is called a join-subsemilattice. In the similar manner, we define meet-semilattice and meet-subsemilattice. Two lattices $(L_1; \leq_1)$ and $(L_2; \leq_2)$ are said to be isomorphic if there exists a bijection $\phi : L_1 \to L_2$ such that $a \leq_1 b$ in L_1 implies that $\phi(a) \leq_2 \phi(b)$ in L_2. A lattice that is isomorphic to its dual is called a self dual lattice. For $a, b \in L$, we use notations $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$ and we call $\vee$, the join and $\wedge$, the meet. In a bounded lattice $(L; \leq)$, a is a complement of b if and only if $a \wedge b = 0$ and $a \vee b = 1$.

If the vertices of Q_n are denoted as $0, 1, \cdots, 2^n - 1$ in the usual way, that is, two vertices are adjacent if and only if their corresponding numbers in the binary representation differ in exactly one digit, then $\{0, 1, \cdots, k-1\}$ is a good set of k vertices, denoted by G_k. For every vertex i, the corresponding binary n-tuple $(i_{n-1}, i_{n-2}, \ldots, i_1, i_0)$ is such that $i = \sum_{j=0}^{n-1} i_j\, 2^j$, where $0 \leq i \leq 2^n - 1$.

Q_n can be viewed as a set of all binary n-tuples equipped with three operations *meet* $(\wedge)$, *join* $(\vee)$ and *complement* $(^c)$ as follows.

For two vertices $a = (a_{n-1}, a_{n-2}, \cdots, a_0)$, $b = (b_{n-1}, b_{n-2}, \cdots, b_0)$ of Q_n we define $a \wedge b = (a_{n-1} \wedge b_{n-1}, a_{n-2} \wedge b_{n-2}, \cdots, a_0 \wedge b_0)$, $a \vee b = (a_{n-1} \vee b_{n-1}, a_{n-2} \vee b_{n-2}, \cdots, a_0 \vee b_0)$, and $a^c = (a^c_{n-1}, a^c_{n-2}, \cdots, a^c_0)$, where $0^c = 1$, $1^c = 0$ and meet-table, join-table are described as follows:

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

For an element $a = (a_{n-1}, a_{n-2}, \cdots, a_0) \in Q_n$, we define support of a as $supp(a) = \{j : a_j = 1\}$. Then, $a = \sum_{j \in supp(a)} 2^j$. So for $a, b \in V(Q_n)$,

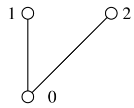
(i) $a \leq b$ if and only if $supp(a) \subseteq supp(b)$.
(ii) Similarly, $a \nleq b$ if and only if $supp(a) \cap supp(b) = \emptyset$.
(iii) If $supp(a) \cap supp(b) \neq \emptyset$, $a \wedge b = \sum_{j \in supp(a) \cap supp(b)} 2^j$
(iv) $a \vee b = \sum_{j \in supp(a) \cup supp(b)} 2^j$ for $a, b \in V(Q_n)$

For $0 < m \leq 2^n$, $n \geq 0$, we denote by S_m the subposet with m elements $\{0, 1, \cdots, m-1\}$ in Q_n and $S^c_m = \{0^c, 1^c, \cdots, (m-1)^c\}$, the set of complements in Q_n of the elements in S_m. It can be easily observed that S^c_m is a subposet of Q_n.

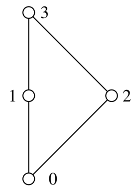
In the following figure, we have drawn S_m in Q_3.

$S_1 = \{0\}$

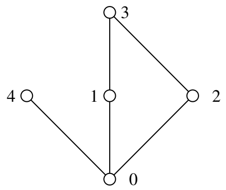
$S_2 = \{0, 1\}$



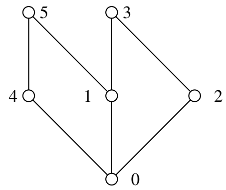
$S_3 = \{0, 1, 2\}$



$S_4 = \{0, 1, 2, 3\}$



$S_5 = \{0, 1, 2, 3, 4\}$



$S_6 = \{0, 1, 2, 3, 4, 5\}$

3. Properties of the poset S_m, $m \leq 2^n$

Now, we give some useful lemmas.

Lemma 1. *The underlying graph of S_m is isomorphic to a good set G_m in Q_n.*

Proof. Let H be the underlying graph of an upper good lattice S_m. Let $a \in H$. Define a map $\phi : H \to G_m$ which maps $a \in H$ to $\phi(a) = a$. Clearly, ϕ is a bijection. It is easy to observe that a is adjacent to b in H if and only if $\phi(a)$ is adjacent to $\phi(b)$ in G_m. This completes the proof. $\square$

Lemma 2. *All lower bounds of every element of S_m in Q_n exist in S_m.*

Proof. If $m = 2^t$ for $1 \leq t \leq n$, then $S_m = Q_t$ which is a sublattice of Q_n. Therefore, the result is true when m is a power of 2. So, suppose that m is not a power of 2. Note that S_m is a set of m consecutive nonnegative integers. So, for any $a \in S_m$, all previous nonnegative integers $0, 1, \cdots, a-1$ lie in S_m. In other words, for any $a \in S_m$, all elements b with $supp(b) \subseteq supp(a)$ lie in S_m. This proves the result. $\square$

In the similar manner, the following lemma can be proved.

Lemma 3. *All upper bounds of every element of S^c_m in Q_n exist in S^c_m.*

Lemma 4. *S_m is a cover preserving meet subsemilattice of Q_n.*

Proof. Let $a, b \in S_m$. Then, $a \wedge b \leq a$ and $a \wedge b \leq b$. By using Lemma 2, all lower bounds of a and b in Q_n exist in S_m also. So, $a \wedge b \in S_m$. Therefore, S_m is a meet subsemilattice.
Suppose $b \prec a$ in S_m. Assume that there exists $d \in Q_n$ such that $b \prec d \prec a$. Then, $d \in S_m$, by Lemma 2. It implies that $b \nprec a$ in S_m, a contradiction. Therefore, S_m is a cover preserving meet subsemilattice of Q_n. $\square$

Corollary 1. *S^c_m is a cover preserving join subsemilattice of Q_n.*

Proof. Since Q_n is a bounded distributive lattice, an element in Q_n can have only one complement. Let $a, b \in S^c_m$. Then, a^c, b^c lie in S_m. By using Lemma 4, $a^c \wedge b^c$ lies in S_m. But, $a^c \wedge b^c = (a \vee b)^c$ by using De Morgan's identities. Therefore, $a \vee b$ lies in S^c_m.
Since $a^c \wedge b^c$ in S_m is same as $a^c \wedge b^c$ in Q_n, $a \vee b$ in S^c_m is same as $a \vee b$ in Q_n. Therefore, S^c_m is join subsemilattice of Q_n.
S^c_m is cover preserving by the similar arguments the proof of the above lemma. $\square$

Here, we give a necessary and sufficient condition for S_m to be a lattice.

Lemma 5. *S_m is a lattice if and only if $m = 2^t$ for some $0 \le t \le n$.*

Proof. Let $m = 2^t$ for some $0 \le t \le n$. Since, S_m is cover preserving and $S_m = Q_t$, it is a lattice. Conversely, assume that m is not a power of 2, that is, $m = 2^t + l$ for some $0 \le t \le n$ and $0 < l < 2^t$. Consider the binary representation of $l - 1$, say $(l_{n-1}, l_{n-2}, \cdots, l_0)$. Since, $l - 1 < 2^t - 1$, $l_i = 0$ for all $t \le i \le n-1$ and there is at least one $0 \le j \le t-1$ such that $l_j = 0$. Let the binary representation of $m - 1$ be $(m_{n-1}, m_{n-2}, \cdots, m_0)$. Since $m - 1 = 2^t + (l-1)$, $m_t = 1$ and $m_i = l_i$ for all $i \neq t, 0 \le i \le n-1$. Let p be an element with the binary representation $(p_{n-1}, p_{n-2}, \cdots, p_0)$ such that $p_i = l_i$ for all $i \neq j, 0 \le i \le n-1$ and $p_j = 1$. Since, $p < 2^t$, p lies in S_m. Clearly, $p = 2^j + (l-1)$. Now, $(m-1) \vee p = (m_{n-1} \vee p_{n-1}, m_{n-2} \vee p_{n-2}, \cdots, m_0 \vee p_0)$, where $m_i \vee p_i = 0$ for all $t+1 \le i \le n-1$, $m_t \vee p_t = 1$, $m_j \vee p_j = 1$ and $m_i \vee p_i = l_i$ for all $i \neq j, 0 \le i \le t-1$. Thus, $(m-1) \vee p = 2^t + 2^j + (l-1) > m-1$. It implies that for $m-1, p \in S_m$, $(m-1) \vee p$ doesnot lie in S_m. Thus, if m is not a power of 2, then S_m is not a lattice. □

4. An upper good lattice and its properties

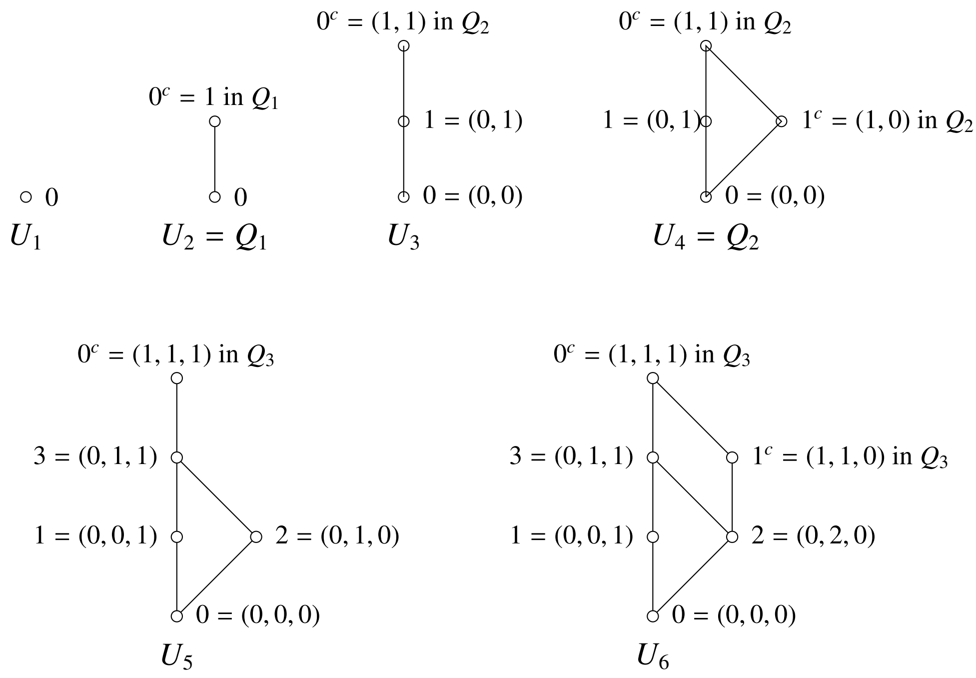
Now, we define an upper good lattice in Q_n.

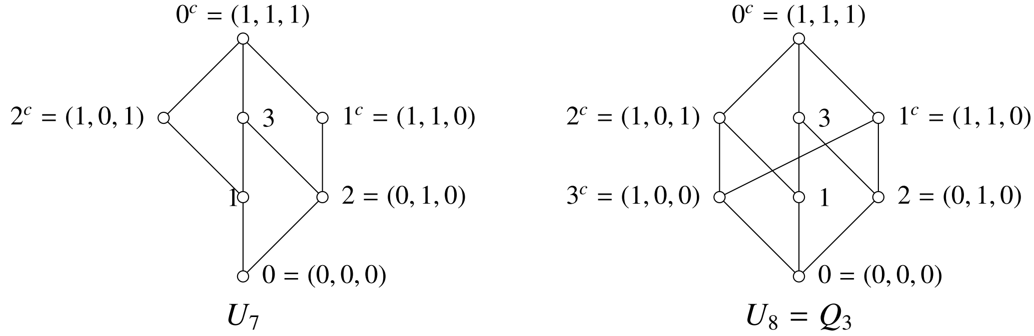
Definition 1. *The set U_k with k elements is an upper good lattice if (i) $k = 1$ or (ii) Suppose that $2^{n-1} < k \le 2^n$, $n > 0$. Then, consider the hypercube Q_n as union of two copies of $n-1$ dimensional cubes Q^0_{n-1}, Q^1_{n-1}, such that $V(Q^0_{n-1}) = \{0\} \oplus V(Q_{n-1})$ and $V(Q^1_{n-1}) = \{1\} \oplus V(Q_{n-1})$. Then, U_k is a subposet of Q_n consisting of elements in $Q^0_{n-1} \bigcup S^c_{k-2^{n-1}}$.*

Observation 1. *For $2^{n-1} < k \le 2^n$, $n > 0$, $U_k = U_{k-1} \bigcup (k - 2^{n-1} - 1)^c$. Then, the binary representation of $(k - 2^{n-1} - 1)^c$ has the form $(1, d_{n-2}, \cdots, d_0)$. It implies that $(k - 2^{n-1} - 1)^c$ covers exactly one element $(0, d_{n-2}, \cdots, d_0)$ of Q^0_{n-1}. Moreover, at least one element of U_{k-1} which is not in Q^0_{n-1} covers $(k - 2^{n-1} - 1)^c$.*

Dually, a lower good lattice L_k can be defined.

In the following figure, we have drawn upper good lattices in Q_3. Corresponding binary representations of the elements are given in the round brackets.





Here after, unless we mention specifically, U_k is an upper good lattice on k elements, where $2^{n-1} < k \leq 2^n$, for some $n > 0$.

Proposition 1. *The underlying graph of an upper good lattice U_k is isomorphic to a good set G_k.*

Proof. Let H be the underlying graph of an upper good lattice U_k. Let $a \in H$ with $a = (a_{n-1}, a_{n-2}, \cdots, a_0)$. Define a map $\phi : H \to G_k$ which maps $a \in H$ to $\phi(a) = (a_{n-1}, a^c_{n-2}, \cdots, a^c_0)$. Clearly, ϕ is a bijection. It is easy to observe that a is adjacent to b in H if and only if $\phi(a)$ is adjacent to $\phi(b)$ in G_k. This completes the proof. □

Lemma 6. *An upper good lattice U_k is cover preserving.*

Proof. Let $a, b \in U_k$ with the binary representations $(a_n, a_{n-1}, \cdots, a_0)$, $(b_n, b_{n-1}, \cdots, b_0)$, respectively. If both a and b lie in either Q^0_{n-1} or $S^c_{k-2^{n-1}}$, then a is covered by b in U_k if and only if a is covered by b in Q_n. So, let $a \in Q^0_{n-1}$ and $b \in S^c_{k-2^{n-1}}$. Then, $a_n = 0$ and $b_n = 1$. Suppose that a is covered by b in Q_n. Since a is covered by b in Q_n, $a_i = b_i$ for $0 \leq i \leq n-1$. Thus, a is covered by b in U_k also.
Conversely, suppose that a is covered by b in U_k and not in Q_n means there exists $d \in Q_n - U_k$ such that $a \prec d \prec b$. Now, consider b' whose binary representation is $(0, b_{n-1}, \cdots, b_0)$. Then, $supp(b') \subseteq supp(b)$ and $|supp(b')| + 1 = |supp(b)|$. Since $a \in Q^0_{n-1}$ and $a \prec b$ in U_k, $supp(a) \subseteq supp(b')$. Because of our assumption $a = b'$. It implies that b' is not covered by b in Q_n, a contradiction. Thus, a is covered by b in Q_n also. □

Lemma 7. *An upper good lattice U_k is a join subsemilattice of Q_n.*

Proof. Let $a, b, c \in U_k$ such that $a \leq c$ and $b \leq c$. To prove that $a \vee b \leq c$.
Case 1: If $a, b \in S^c_{k-2^{n-1}}$, then by using Lemma 1, $a \vee b \leq c$.
Case 2: If $a, b \in Q^0_{n-1}$, then $a \vee b \leq c$.
Case 3: Let $a \in Q^0_{n-1}$ and $b \in S^c_{k-2^{n-1}}$. By using Lemma 2, $a \vee b \leq c$. This completes the proof. □

Proposition 2. *An upper good lattice U_k is a lattice.*

Proof. By using Lemma 7, U_k is a join semilattice. Moreover, the zero element lies in U_k. Thus, it is a lattice. □

Corollary 2. *An upper good lattice U_k is an upper semimodular lattice.*

Proof. Since Q_n is modular, it satisfies the Upper and the Lower Covering Conditions. By using Lemma 6 and Lemma 7, U_k is a cover preserving join subsemilattice of Q_n. Therefore, U_k satisfies the Upper Covering Condition. This proves the result. □

We know that $U_{2^t} = Q_t$ for $t < n$ is a sublattice of Q_n. Now, we give values of k other than power two, for which U_k is a sublattice of Q_n.

Proposition 3. *Let $2^{n-1} < k < 2^n$, for some $n > 0$. An upper good lattice U_k is a sublattice of Q_n if and only if either k is a power of 2 or $k = 2^{n-1} + 2^m$ for some $0 \leq m \leq n-1$.*

Proof. If k is a power of 2, it can be proved easily proved that U_k is sublattice. Suppose that $k = 2^{n-1} + 2^m$ for some $0 \leq m < n-1$. In this case, $S^c_{k-2^{n-1}}$ is Q^c_m. By using Lemma 7, U_k is a join subsemilattice of Q_n. It is enough to prove that U_k is a meet subsemilattice of Q_n. Let $a, b \in U_k$. If both a and b lie in either Q^0_{n-1} or Q^c_m, then $a \wedge b$ exists in U_k which is same as $a \wedge b$ in Q_n. Now, suppose that $a \in Q^0_{n-1}$ and $b \in Q^c_m$ with their binary representations $(0, a_{n-1}, \cdots, a_0)$, $(1, b_{n-1}, \cdots, b_0)$, respectively. Let $b' \in Q^0_{n-1}$ be the corresponding element such that $b' \prec b$. Then, the binary representation of b' is $(0, b_{n-1}, \cdots, b_0)$. Since a and b' lie in Q^0_{n-1}, $a \wedge b'$ exists in Q^0_{n-1} with the binary representation $(0, a_{n-1} \wedge b_{n-1}, \cdots, a_0 \wedge b_0)$. Then, the binary representation of $a \wedge b$ is $(0 \wedge 1, a_{n-1} \wedge b_{n-1}, \cdots, a_0 \wedge b_0)$ =$(0, a_{n-1} \wedge b_{n-1}, \cdots, a_0 \wedge b_0)$ which is an element of Q^0_{n-1}. Therefore, $a \wedge b$ exists in U_k and is same as $a \wedge b$ in Q_n. Conversely, assume that $k \neq 2^{n-1} + 2^m$ for some $0 \leq m < n-1$, that is, $k = 2^{n-1} + l$ for some $l \neq 2^m$ for $0 \leq m \leq n-1$. It means that $k - 2^{n-1}$ is not a power of 2. By using Lemma 5, $S_{k-2^{n-1}}$ is not a sublattice of Q_n. We can find two elements in $S_{k-2^{n-1}}$, say p, q, such that $p \vee q$ doesnot exist in $S_{k-2^{n-1}}$. Since $p, q \in S_{k-2^{n-1}}$, p^c, q^c lie in $S^c_{k-2^{n-1}} \subseteq U_k$. But, $p^c \wedge q^c = (p \vee q)^c$. Since $p \vee q$ doesnot exist in $S_{k-2^{n-1}}$, $(p \vee q)^c$ doesnot exist in $S^c_{k-2^{n-1}}$. Thus, U_k is not a sublattice of Q_n if $k \neq 2^{n-1} + 2^m$ for some $0 \leq m < n-1$. □

Corollary 3. *Let $2^{n-1} < k \leq 2^n$, for some $n > 0$. An upper good lattice U_k is distributive if and only if either k is a power of 2 or $k = 2^{n-1} + 2^m$ for some $0 \leq m \leq n-1$.*

Proof. Note that Q_n is distributive and a sublattice of a distributive lattice is distributive. By using Proposition 3, if k is a power of 2 or $k = 2^{n-1} + 2^m$ for some $0 \leq m \leq n-1$, then U_k is a sublattice of Q_n and hence, distributive.
Conversely, assume that U_k is distributive. So, it is modular. Since U_k is a finite lattice, it is modular if and only if it satisfies the upper covering condition (UCC) and the lower covering condition(LCC). By using Corollary 2, U_k satisfies UCC. By using Proposition 3, U_k satisfies both UCC and LCC if and only if $k = 2^{n-1} + 2^m$ for some $0 \leq m \leq n-1$. This completes the proof. □

By the similar arguments, one can easily prove the following corollary.

Corollary 4. *An upper good lattice U_k is distributive if and only if it is modular.*

We know that Q_n is self dual for all values of n. Now, we find k different from power of 2, for which U_k is self dual.

Proposition 4. *Let $2^{n-1} < k < 2^n$, for some $n > 1$. An upper good lattice U_k is a selfdual lattice if and only if either k is a power of 2 or $k = 2^{n-1} + 2^{n-2}$.*

Proof. If k is a power of 2, clearly U_k is self-dual. Let $k = 2^{n-1} + 2^{n-2}$ for $n \geq 2$. In this case, $S_{k-2^{n-1}} = S_{2^{n-2}} = Q_{n-2}$. Let $a \in U_k$ with $a = (a_{n-1}, a_{n-2}, \cdots, a_0)$. Define a map $\phi : (U_k; \leq) \to (U_k; \geq)$ which maps $a \in S_{2^{n-2}} \bigcup S^c_{2^{n-2}}$ to $\phi(a) = (a^c_{n-1}, a^c_{n-2}, \cdots, a^c_0)$ and $a \in Q^0_{n-1} - S_{2^{n-2}}$ to $\phi(a) = (a_{n-1}, a_{n-2}, a^c_{n-3}, \cdots, a^c_0)$. Clearly, ϕ is a bijection. Now, we prove that ϕ is a homomorphism. Let $a = (a_{n-1}, a_{n-2}, \cdots, a_0)$, $b = (b_{n-1}, b_{n-2}, \cdots, b_0) \in U_k$. Clearly, $a_i \leq b_i$ if and only if $a^c_i \geq b^c_i$ for all $0 \leq i \leq n-1$.
Case 1: Let $a, b \in Q^0_{n-1} - S_{2^{n-2}}$. Since $a_i \leq b_i$ if and only if $a^c_i \geq b^c_i$ for all $0 \leq i \leq n-3$, $a \leq b$ if and only if $\phi(a) \geq \phi(b)$.
Case 2: Let $a, b \in S_{2^{n-2}} \bigcup S^c_{2^{n-2}}$. Then, $a \leq b$ if and only if $a^c \geq b^c$.
Case 3: Let $a \in Q^0_{n-1} - S_{2^{n-2}}$ and $b \in S_{2^{n-2}}$ be such that $b \leq a$. Then, $\phi(a) = (a_{n-1}, a_{n-2}, a^c_{n-3}, \cdots, a^c_0)$, $\phi(b) = (b^c_{n-1}, b^c_{n-2}, \cdots, b^c_0)$ and $b_i \leq a_i$ for all $0 \leq i \leq n-1$. Since $a \in Q^0_{n-1} - S_{2^{n-2}}$, $2^{n-2} \leq a \leq 2^{n-1} - 1$ which implies that $a_{n-1} = 0$ and $a_{n-2} = 1$. In the similar manner, $b \in S_{2^{n-2}}$ implies that $b_{n-1} = 0$ and $b_{n-2} = 0$. Thus, $\phi(a) = (0, 1, a^c_{n-3}, \cdots, a^c_0)$, $\phi(b) = (1, 1, b^c_{n-3}, \cdots, b^c_0)$ and $b^c_i \geq a^c_i$ for all $0 \leq i \leq n-3$. Therefore, $\phi(b) \geq \phi(a)$.
Case 4: Let $a \in Q^0_{n-1} - S_{2^{n-2}}$ and $b \in S^c_{2^{n-2}}$ be such that $a \leq b$. Then, $\phi(a) = (a_{n-1}, a_{n-2}, a^c_{n-3}, \cdots, a^c_0)$, $\phi(b) = (b^c_{n-1}, b^c_{n-2}, \cdots, b^c_0)$ and $a_i \leq b_i$ for all $0 \leq i \leq n-1$. Since $a \in Q^0_{n-1} - S_{2^{n-2}}$, $2^{n-2} \leq a \leq 2^{n-1} - 1$

which implies that $a_{n-1} = 0$ and $a_{n-2} = 1$. In the similar manner, $b \in S^c_{2^{n-2}}$ implies that $b_{n-1} = 1$ and $b_{n-2} = 1$. Thus, $\phi(a) = (0, 1, a^c_{n-3}, \cdots, a^c_0)$, $\phi(b) = (0, 0, b^c_{n-3}, \cdots, b^c_0)$ and $a^c_i \geq b^c_i$ for all $0 \leq i \leq n-3$. Therefore, $\phi(a) \geq \phi(b)$.

This proves that if $k = 2^{n-1} + 2^{n-2}$ for $n \geq 2$, then U_k is a selfdual lattice. Conversely, assume that U_k is a self-dual lattice. By Proposition 7, U_k is a join subsemilattice. Since U_k is self-dual, it is meet subsemilattice also. So, U_k is a sublattice. By Proposition 3, $k = 2^{n-1} + 2^m$ for some $0 \leq m \leq n-1$. Note that there are $n-1$ atoms in U_k. In $S_{2^m} = Q_m$, there are m atoms and the complement in Q_n of an atom in S_{2^m} is the dual atom in U_k. So, the dual atoms in U_k are $2^{n-1} - 1 = (0, 1, 1, \cdots, 1)$ and complements of m atoms in S_{2^m}. Thus, there are $m+1$ dual atoms in U_k if $k = 2^{n-1} + 2^m$ for some $0 \leq m \leq n-1$. Since U_k is self dual, the number of atoms in U_k is same as the number of dual atoms in it. Therefore, $n-1 = m+1$ which implies that $m = n-2$. This completes the proof. □

5. Concluding remarks

In this paper, we studied lattice theoretic properties of upper good lattices. These properties follow dually for lower good lattices. We strongly feel that the upper and lower good lattices are the only lattices whose cover graphs are good sets. However, the following problem is open.

Problem 1. *How many lattices are there whose cover graphs are good sets?*

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