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Order Structure of Good Sets in Hypercube

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Abstract: A good set on k vertices is a vertex induced subgraph of the hypercube \( Q_n \) that has the maximum number of edges. The long-lasting problem of characterizing graphs that are cover graphs of lattices is NP-complete. This paper constructs and studies lattice theoretic properties of a class of lattices whose cover graphs are isomorphic to good sets.

Keywords: hypercube, partial cube, lattice.

1. Introduction

The \( n \)-dimensional hypercube, \( Q_n \), is a graph in which a binary \( n \)-tuple is assigned to every vertex, and any two vertices are adjacent if they differ in exactly one coordinate. \( Q_n \) is an \( n \)-regular, \( n \)-connected, bipartite, and vertex transitive graph on \( 2^n \) vertices. Moreover, \( Q_n \) can be viewed as a lattice. It is the direct product of 2-element chain \( n \) times. Its cover graph is the same as \( Q_n \).

Hart [1] defined good set as the vertex induced subgraph of hypercube having the maximum number of edges on the given set of vertices. It is well known that hypercube is one of the most popular topologies for the interconnection of computing nodes in multiprocessor systems. Nevertheless, as the number of nodes in such systems must be a power of 2, there are significant gaps in the sizes of the systems that can be built. To overcome this restriction, Katseff [2] proposed incomplete hypercube that can be viewed as a hypercube that operates in a degraded manner after some nodes become faulty. Interestingly, incomplete hypercubes are good sets only. Compact hypercubes [3], composite hypercubes [4] are some more synonyms of good sets. Tzeng [5] studied some structural properties, while Sen et al. [6] proved some topological properties of good sets. Tapadia and Waphare proved graph theoretic properties like good sets are partial cubes, Hamiltonian, bipancyclic, etc. [7]. One can refer for more properties of good sets in [8–12].

We look at the lattice theoretic aspect of a good set in view of the following classic problem in lattice theory.

Open Problem: Which graphs are the cover graphs of finite posets/lattices?

This problem was first posed by Ore [13] for partially ordered sets and then restated by Rival [14] for ordered sets. Cover graphs of finite posets have already been characterised. However, the problem is NP-complete for lattices [15]. In this paper, we construct a lattice called upper good lattice in \( Q_n \), whose underlying cover graph is isomorphic to the good set. We also study lattice theoretic properties
2. Preliminaries

A partially ordered set (poset) \((L; \leq)\) consists of a nonempty set \(L\) and a binary relation \(\leq\) on \(L\), which is reflexive, antisymmetric and transitive. If \(a\) covers \(b\), then we use notation \(b \prec a\). A poset \((L; \leq)\) is a lattice if \(\sup\{a, b\}\) and \(\inf\{a, b\}\) exist for all \(a, b \in L\). Also, if \((L; \leq)\) is a lattice, so is its dual \((L; \geq)\). For a lattice \((L; \leq)\) and \(M \subseteq L\), \((M; \leq)\) is a sublattice of \((L; \leq)\) if \(a, b \in M\) implies that \(\sup\{a, b\}\) and \(\inf\{a, b\}\) lies in \(M\). A poset \((L; \leq)\) is a join-semilattice if \(\sup\{a, b\}\) exists for all \(a, b \in L\). If \((L; \leq)\) is a poset and \(M \subseteq L\) such that \(a, b \in M\) implies that \(\sup\{a, b\} \in M\), then \(M\) is called a join-subsemilattice. In the similar manner, we define meet-semilattice and meet-subsemilattice. Two lattices \((L_1; \leq_1)\) and \((L_2; \leq_2)\) are said to be isomorphic if there exists a bijection \(\phi : L_1 \to L_2\) such that \(a \leq_1 b\) in \(L_1\) implies that \(\phi(a) \leq_2 \phi(b)\) in \(L_2\). A lattice that is isomorphic to its dual is called a self dual lattice. For \(a, b \in L\), we use notations \(a \lor b = \sup\{a, b\}\) and \(a \land b = \inf\{a, b\}\) and we call \(\lor\), the join and \(\land\), the meet. In a bounded lattice \((L; \leq)\), \(a\) is a complement of \(b\) if and only if \(a \land b = 0\) and \(a \lor b = 1\).

If the vertices of \(Q_n\) are denoted as \(0, 1, \cdots , 2^n - 1\) in the usual way, that is, two vertices are adjacent if and only if their corresponding numbers in the binary representation differ in exactly one digit, then \(\{0, 1, \cdots , k - 1\}\) is a good set of \(k\) vertices, denoted by \(G_k\). For every vertex \(i\), the corresponding binary \(n\)-tuple \((i_{n-1}, i_{n-2}, \ldots , i_1, i_0)\) is such that \(i = \sum_{j=0}^{n-1} i_j 2^j\), where \(0 \leq i \leq 2^n - 1\).

\(Q_n\) can be viewed as a set of all binary \(n\)-tuples equipped with three operations \(\text{meet} (\land)\), \(\text{join} (\lor)\) and \(\text{complement} (\lor^c)\) as follows.

For two vertices \(a = (a_{n-1}, a_{n-2}, \cdots , a_0)\), \(b = (b_{n-1}, b_{n-2}, \cdots , b_0)\) of \(Q_n\) we define \(a \land b = (a_{n-1} \land b_{n-1}, a_{n-2} \land b_{n-2}, \cdots , a_0 \land b_0)\), \(a \lor b = (a_{n-1} \lor b_{n-1}, a_{n-2} \lor b_{n-2}, \cdots , a_0 \lor b_0)\), and \(a^c = (a^c_{n-1}, a^c_{n-2}, \cdots , a^c_0)\), where \(0^c = 1\), \(1^c = 0\) and meet-table, join-table are described as follows:

\[
\begin{array}{c|c|c}
\land & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\quad
\begin{array}{c|c|c}
\lor & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

For an element \(a = (a_{n-1}, a_{n-2}, \cdots , a_0)\) in \(Q_n\), we define support of \(a\) as \(\text{supp}(a) = \{j : a_j = 1\}\). Then, \(a = \sum_{j \in \text{supp}(a)} 2^j\). So for \(a, b \in V(Q_n)\),

(i) \(a \leq b\) if and only if \(\text{supp}(a) \subseteq \text{supp}(b)\).

(ii) Similarly, \(a \nless b\) if and only if \(\text{supp}(a) \cap \text{supp}(b) = \emptyset\).

(iii) If \(\text{supp}(a) \cap \text{supp}(b) \neq \emptyset\), \(a \land b = \sum_{j \in \text{supp}(a) \cap \text{supp}(b)} 2^j\).

(iv) \(a \lor b = \sum_{j \in \text{supp}(a) \cup \text{supp}(b)} 2^j\) for \(a, b \in V(Q_n)\).

For \(0 < m \leq 2^n\), \(n \geq 0\), we denote by \(S_m\) the subposet with \(m\) elements \(\{0, 1, \cdots , m - 1\}\) in \(Q_n\) and \(S_m^c = \{0^c, 1^c, \cdots , (m - 1)^c\}\), the set of complements in \(Q_n\) of the elements in \(S_m\). It can be easily observed that \(S_m^c\) is a subposet of \(Q_n\).

In the following figure, we have drawn \(S_m\) in \(Q_3\).
3. Properties of the poset $S_m$, $m \leq 2^n$

Now, we give some useful lemmas.

**Lemma 1.** The underlying graph of $S_m$ is isomorphic to a good set $G_m$ in $Q_n$.

*Proof.* Let $H$ be the underlying graph of an upper good lattice $S_m$. Let $a \in H$. Define a map $\phi : H \to G_m$ which maps $a \in H$ to $\phi(a) = a$. Clearly, $\phi$ is a bijection. It is easy to observe that $a$ is adjacent to $b$ in $H$ if and only if $\phi(a)$ is adjacent to $\phi(b)$ in $G_m$. This completes the proof. $\square$

**Lemma 2.** All lower bounds of every element of $S_m$ in $Q_n$ exist in $S_m$.

*Proof.* If $m = 2^t$ for $1 \leq t \leq n$, then $S_m = Q_t$ which is a sublattice of $Q_n$. Therefore, the result is true when $m$ is a power of 2. So, suppose that $m$ is not a power of 2. Note that $S_m$ is a set of $m$ consecutive nonnegative integers. So, for any $a \in S_m$, all previous nonnegative integers $0, 1, \ldots, a - 1$ lie in $S_m$. In other words, for any $a \in S_m$, all elements $b$ with $supp(b) \subseteq supp(a)$ lie in $S_m$. This proves the result. $\square$

In the similar manner, the following lemma can be proved.

**Lemma 3.** All upper bounds of every element of $S_m^c$ in $Q_n$ exist in $S_m^c$.

**Lemma 4.** $S_m$ is a cover preserving meet subsemilattice of $Q_n$.

*Proof.* Let $a, b \in S_m$. Then, $a \wedge b \leq a$ and $a \wedge b \leq b$. By using Lemma 2, all lower bounds of $a$ and $b$ in $Q_n$ exist in $S_m$ also. So, $a \wedge b \in S_m$. Therefore, $S_m$ is a meet subsemilattice. Suppose $b < a$ in $S_m$. Assume that there exists $d \in Q_n$ such that $b < d < a$. Then, $d \in S_m$, by Lemma 2. It implies that $b \neq a$ in $S_m$, a contradiction. Therefore, $S_m$ is a meet subsemilattice of $Q_n$. $\square$

**Corollary 1.** $S_m^c$ is a cover preserving join subsemilattice of $Q_n$.

*Proof.* Since $Q_n$ is a bounded distributive lattice, an element in $Q_n$ can have only one complement. Let $a, b \in S_m^c$. Then, $a^c, b^c$ lie in $S_m$. By using Lemma 4, $a^c \wedge b^c$ lies in $S_m$. But, $a^c \wedge b^c = (a \vee b)^c$ by using De Morgan’s identities. Therefore, $a \vee b$ lies in $S_m^c$. Since $a^c \wedge b^c$ in $S_m$ is same as $a^c \wedge b^c$ in $Q_n$, $a \vee b$ in $S_m^c$ is same as $a \vee b$ in $Q_n$. Therefore, $S_m^c$ is join subsemilattice of $Q_n$. $S_m^c$ is cover preserving by the similar arguments the proof of the above lemma. $\square
Here, we give a necessary and sufficient condition for $S_m$ to be a lattice.

**Lemma 5.** $S_m$ is a lattice if and only if $m = 2^l$ for some $0 \leq t \leq n$.

**Proof.** Let $m = 2^l$ for some $0 \leq t \leq n$. Since, $S_m$ is cover preserving and $S_m = Q_l$, it is a lattice. Conversely, assume that $m$ is not a power of 2, that is, $m = 2^l + l$ for some $0 \leq t \leq n$ and $0 < l < 2^l$. Consider the binary representation of $l - 1$, say $(l_{n-1}, l_{n-2}, \ldots, l_0)$. Since, $l - 1 < 2^l - 1$, $l_i = 0$ for all $i \leq n - 1$ and there is at least one $0 \leq j \leq t - 1$ such that $l_j = 0$. Let the binary representation of $m - 1$ be $(m_{n-1}, m_{n-2}, \ldots, m_0)$. Since $m - 1 = 2^l + (l - 1)$, $m_t = 1$ and $m_i = l_i$ for all $i \neq t$, $0 \leq i \leq n - 1$. Let $p$ be an element with the binary representation $(p_{n-1}, p_{n-2}, \ldots, p_0)$ such that $p_i = l_i$ for all $i \neq j$, $0 \leq i \leq n - 1$ and $p_j = 1$. Since, $p < 2^l$, $p$ lies in $S_m$. Clearly, $p = 2^l + (l - 1)$. Now, $(m - 1) \lor p = (m_{n-1} \lor p_{n-1}, m_{n-2} \lor p_{n-2}, \ldots, m_0 \lor p_0)$, where $m_i \lor p_i = 0$ for all $t + 1 \leq i \leq n - 1$, $m_t \lor p_t = 1$, $m_i \lor p_j = 1$ and $m_i \lor p_i = l_i$ for all $i \neq j$, $0 \leq i \leq t - 1$. Thus, $(m - 1) \lor p = 2^l + 2^l + (l - 1) > m - 1$. It implies that for $m - 1, p \in S_m$, $(m - 1) \lor p$ does not lie in $S_m$. Thus, if $m$ is not a power of 2, then $S_m$ is not a lattice. \hfill \Box

### 4. An upper good lattice and its properties

Now, we define an upper good lattice in $Q_n$.

**Definition 1.** The set $U_k$ with $k$ elements is an upper good lattice if (i) $k = 1$ or (ii) Suppose that $2^{n-1} < k \leq 2^n$, $n > 0$. Then, consider the hypercube $Q_n$ as union of two copies of $n - 1$ dimensional cubes $Q_{n-1}^{0}, Q_{n-1}^{1}$, such that $V(Q_{n-1}^{0}) = \{0\} \oplus V(Q_{n-1})$ and $V(Q_{n-1}^{1}) = \{1\} \oplus V(Q_{n-1})$. Then, $U_k$ is a subposet of $Q_n$ consisting of elements in $Q_{n-1}^{0} \cup S_{k-2^l-1}^{c}$.

**Observation 1.** For $2^{n-1} < k \leq 2^n$, $n > 0$, $U_k = U_{k-1} \cup (k - 2^{n-1} - 1)^c$. Then, the binary representation of $(k - 2^{n-1} - 1)^c$ has the form $(1, d_{n-2}, \ldots, d_0)$. It implies that $(k - 2^{n-1} - 1)^c$ covers exactly one element $(0, d_{n-2}, \ldots, d_0)$ of $Q_{n-1}^{0}$. Moreover, at least one element of $U_{k-1}$ which is not in $Q_{n-1}^{0}$ covers $(k - 2^{n-1} - 1)^c$.

Dually, a lower good lattice $L_k$ can be defined.

In the following figure, we have drawn upper good lattices in $Q_3$. Corresponding binary representations of the elements are given in the round brackets.

<table>
<thead>
<tr>
<th>$U_1$</th>
<th>$U_2 = Q_1$</th>
<th>$U_3$</th>
<th>$U_4 = Q_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^c = (1, 1)$ in $Q_1$</td>
<td>$0^c = (1, 1)$ in $Q_2$</td>
<td>$0^c = (1, 1)$ in $Q_2$</td>
<td>$1^c = (1, 0)$ in $Q_2$</td>
</tr>
<tr>
<td>$0 = (0, 0)$</td>
<td>$0 = (0, 0)$</td>
<td>$0 = (0, 0)$</td>
<td>$0 = (0, 0)$</td>
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<tr>
<td>$U_5$</td>
<td>$U_6$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0^c = (1, 1, 1)$ in $Q_3$</td>
<td>$0^c = (1, 1, 1)$ in $Q_3$</td>
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</tr>
<tr>
<td>$3 = (0, 1, 1)$</td>
<td>$3 = (0, 1, 1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1 = (0, 0, 1)$</td>
<td>$1 = (0, 0, 1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$2 = (0, 1, 0)$</td>
<td>$2 = (0, 2, 0)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$0 = (0, 0, 0)$</td>
<td>$0 = (0, 0, 0)$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Proof. Let \( U_k \) be an upper good lattice on \( k \) elements, where \( 2^{n-1} < k \leq 2^n \), for some \( n > 0 \).

**Proposition 1.** The underlying graph of an upper good lattice \( U_k \) is isomorphic to a good set \( G_k \).

Proof. Let \( H \) be the underlying graph of an upper good lattice \( U_k \). Let \( a \in H \) with \( a = (a_{n-1}, a_{n-2}, \ldots, a_0) \). Define a map \( \phi : H \to G_k \) which maps \( a \in H \) to \( \phi(a) = (a_{n-1}, a_{n-2}, \ldots, a_0) \). Clearly, \( \phi \) is a bijection. It is easy to observe that \( a \) is adjacent to \( b \) in \( H \) if and only if \( \phi(a) \) is adjacent to \( \phi(b) \) in \( G_k \). This completes the proof.

**Lemma 6.** An upper good lattice \( U_k \) is cover preserving.

Proof. Let \( a, b \in U_k \) with the binary representations \((a_n, a_{n-1}, \ldots, a_0), (b_n, b_{n-1}, \ldots, b_0)\), respectively. If both \( a \) and \( b \) lie in either \( Q_{n-1}^0 \) or \( S_{k-2n+1}^c \), then \( a \) is covered by \( b \) in \( U_k \) if and only if \( a \) is covered by \( b \) in \( Q_n \). Since \( a \) is covered by \( b \) in \( Q_n \), \( a_i = b_i \) for \( 0 \leq i \leq n - 1 \). Thus, \( a \) is covered by \( b \) in \( U_k \). Conversely, suppose that \( a \) is covered by \( b \) in \( U_k \) and not in \( Q_n \) means there exists \( d \in Q_n - U_k \) such that \( a < d < b \). Now, consider \( b' \) whose binary representation is \((0, b_{n-1}, \ldots, b_0)\). Then, \( a_n = 0 \) and \( b_n = 1 \). Suppose that \( a \) is covered by \( b \) in \( Q_n \). Since \( a \) is covered by \( b \) in \( Q_n \), \( a_i = b_i \) for \( 0 \leq i \leq n - 1 \). Thus, \( a \) is covered by \( b \) in \( U_k \).

**Lemma 7.** An upper good lattice \( U_k \) is a join subsemilattice of \( Q_n \).

Proof. Let \( a, b, c \in U_k \) such that \( a \leq c \) and \( b \leq c \). To prove that \( a \lor b \leq c \).

Case 1: If \( a, b \in S_{k-2n+1}^c \), then by using Lemma 1, \( a \lor b \leq c \).

Case 2: If \( a, b \in Q_{n-1}^0 \), then \( a \lor b \leq c \).

Case 3: Let \( a \in Q_{n-1}^0 \) and \( b \in S_{k-2n-1}^c \). By using Lemma 2, \( a \lor b \leq c \). This completes the proof.

**Proposition 2.** An upper good lattice \( U_k \) is a lattice.

Proof. By using Lemma 7, \( U_k \) is a join semilattice. Moreover, the zero element lies in \( U_k \). Thus, it is a lattice.

**Corollary 2.** An upper good lattice \( U_k \) is an upper semimodular lattice.

Proof. Since \( Q_n \) is modular, it satisfies the Upper and the Lower Covering Conditions. By using Lemma 6 and Lemma 7, \( U_k \) is a cover preserving join subsemilattice of \( Q_n \). Therefore, \( U_k \) satisfies the Upper Covering Condition. This proves the result.

We know that \( U_{2^t} = Q_t \) for \( t < n \) is a sublattice of \( Q_n \). Now, we give values of \( k \) other than power two, for which \( U_k \) is a sublattice of \( Q_n \).

**Proposition 3.** Let \( 2^{n-1} < k < 2^n \), for some \( n > 0 \). An upper good lattice \( U_k \) is a sublattice of \( Q_n \) if and only if either \( k \) is a power of \( 2 \) or \( k = 2^{n-1} + 2^m \) for some \( 0 \leq m \leq n - 1 \).
Proof. If \( k \) is a power of 2, it can be proved easily proved that \( U_k \) is sublattice. Suppose that \( k = 2^{n-1} + 2^m \) for some \( 0 \leq m < n - 1 \). In this case, \( S_{k-2^{n-1}} \) is \( \mathcal{Q}_m \). By using Lemma 7, \( U_k \) is a join subsemilattice of \( Q_n \). It is enough to prove that \( U_k \) is a meet subsemilattice of \( Q_n \). Let \( a, b \in U_k \). If both \( a \) and \( b \) lie in either \( Q_{m-1}^0 \) or \( Q_{m}^0 \), then \( a \land b \) exists in \( U_k \) which is same as \( a \land b \in Q_n \). Now, suppose that \( a \in Q_{m-1}^0 \) and \( b \in Q_{m}^0 \) with their binary representations \((0, a_{n-1}, \ldots, a_0), (1, b_{n-1}, \ldots, b_0)\), respectively. Let \( b' \in Q_{m-1}^0 \) be the corresponding element such that \( b' < b \). Then, the binary representation of \( b' \) is \((0, b_{n-1}, \cdots, b_0)\). Since \( a \) and \( b' \) lie in \( Q_{m-1}^0 \) and \( b' \) exists in \( Q_{m-1}^0 \) with the binary representation \((0, a_{n-1} \land b_{n-1}, \ldots, a_0 \land b_0)\). Then, the binary representation of \( a \land b \) is \((0 \land 1, a_{n-1} \land b_{n-1}, \ldots, a_0 \land b_0)\) which is an element of \( Q_{m-1}^0 \). Therefore, \( a \land b \) exists in \( U_k \) and is same as \( a \land b \in Q_n \). Conversely, assume that \( k \neq 2^{n-1} + 2^m \) for some \( 0 \leq m < n - 1 \), that is, \( k = 2^{m-1} + l \) for some \( l \neq 2^m \) for \( 0 \leq m \leq n - 1 \). It means that \( k - 2^{n-1} \) is not a power of 2. By using Lemma 5, \( S_{k-2^{n-1}} \) is not a sublattice of \( Q_n \). We can find two elements in \( S_{k-2^{n-1}} \), say \( p, q \), such that \( p \lor q \) does not exist in \( S_{k-2^{n-1}} \). Since \( p, q \in S_{k-2^{n-1}} \), \( p' \lor q' \) lie in \( S_{k-2^{n-1}} \subseteq \mathcal{U}_k \). But, \( p' \land q' = (p \lor q)^c \). Since \( p \lor q \) does not exist in \( S_{k-2^{n-1}} \), \( p' \lor q' \) does not exist in \( S_{k-2^{n-1}} \). Thus, \( U_k \) is not a sublattice of \( Q_n \) if \( k \neq 2^{n-1} + 2^m \) for some \( 0 \leq m < n - 1 \).

**Corollary 3.** Let \( 2^{n-1} < k \leq 2^n \), for some \( n > 0 \). An upper good lattice \( U_k \) is distributive if and only if either \( k \) is a power of 2 or \( k = 2^{n-1} + 2^m \) for some \( 0 \leq m \leq n - 1 \).

**Proof.** Note that \( Q_n \) is distributive and a sublattice of a distributive lattice is distributive. By using Proposition 3, if \( k \) is a power of 2 or \( k = 2^{n-1} + 2^m \) for some \( 0 \leq m \leq n - 1 \), then \( U_k \) is a sublattice of \( Q_n \) and hence, distributive.

Conversely, assume that \( U_k \) is distributive. So, it is modular. Since \( U_k \) is a finite lattice, it is modular if and only if it satisfies the upper covering condition (UCC) and the lower covering condition (LCC). By using Corollary 2, \( U_k \) satisfies UCC. By using Proposition 3, \( U_k \) satisfies both UCC and LCC if and only if \( k = 2^{n-1} + 2^m \) for some \( 0 \leq m \leq n - 1 \). This completes the proof.

By the similar arguments, one can easily prove the following corollary.

**Corollary 4.** An upper good lattice \( U_k \) is distributive if and only if it is modular.

We know that \( Q_n \) is self dual for all values of \( n \). Now, we find \( k \) different from power of 2, for which \( U_k \) is self dual.

**Proposition 4.** Let \( 2^{n-1} < k < 2^n \), for some \( n > 1 \). An upper good lattice \( U_k \) is a selfdual lattice if and only if either \( k \) is a power of 2 or \( k = 2^{n-1} + 2^m \).

**Proof.** If \( k \) is a power of 2, clearly \( U_k \) is self-dual. Let \( k = 2^{n-1} + 2^m \) for \( n \geq 2 \). In this case, \( S_{k-2^{n-1}} = S_{2^{n-1}} = Q_n \). Let \( a \in U_k \) with \( a = (a_{n-1}, a_{n-2}, \ldots, a_0) \). Define a map \( \phi : (U_k; \leq \rightarrow (U_k; \leq \) which maps \( a \in S_{2^{n-1}} \) to \( \phi(a) = (a'_{n-1}, a'_{n-2}, \cdots, a'_0) \) and \( a \in Q_{n-1}^0 - S_{2^{n-2}} \) to \( \phi(a) = (a_{n-1}, a_{n-2}, a_{n-3}, \cdots, a'_0) \). Clearly, \( \phi \) is a bijection. Now, we prove that \( \phi \) is a homomorphism. Let \( a = (a_{n-1}, a_{n-2}, \cdots, a_0), b = (b_{n-1}, b_{n-2}, \cdots, b_0) \in U_k \). Clearly, \( a_i \leq b_i \) if and only if \( a'_i \geq b'_i \) for all \( 0 \leq i \leq n - 1 \).

**Case 1:** \( a_i \leq b_i \) if and only if \( a'_i \geq b'_i \) for all \( 0 \leq i \leq n - 3, a \leq b \) if and only if \( \phi(a) \geq \phi(b) \).

**Case 2:** \( a \leq b \) if and only if \( a'_i \geq b'_i \).

**Case 3:** \( a \leq b \) if and only if \( a'_i \geq b'_i \) and \( b \leq a \).

**Case 4:** \( a \leq b \) if and only if \( a'_i \geq b'_i \) and \( a \leq b \).
which implies that \( a_{n-1} = 0 \) and \( a_{n-2} = 1 \). In the similar manner, \( b \in S_{2^n-2}^c \) implies that \( b_{n-1} = 1 \) and \( b_{n-2} = 1 \). Thus, \( \phi(a) = (0, 1, a_{n-3}', \ldots, a_0') \), \( \phi(b) = (0, 0, b_{n-3}', \ldots, b_0') \) and \( a_i' \geq b_i' \) for all \( 0 \leq i \leq n - 3 \). Therefore, \( \phi(a) \geq \phi(b) \).

This proves that if \( k = 2^{n-1} + 2^{n-2} \) for \( n \geq 2 \), then \( U_k \) is a self-dual lattice. Conversely, assume that \( U_k \) is a self-dual lattice. By Proposition 7, \( U_k \) is a join subsemilattice. Since \( U_k \) is self-dual, it is meet subsemilattice also. So, \( U_k \) is a sublattice. By Proposition 3, \( k = 2^{n-1} + 2^m \) for some \( 0 \leq m \leq n - 1 \). Note that there are \( n - 1 \) atoms in \( U_k \). In \( S_{2^n} = Q_m \), there are \( m \) atoms and the complement in \( Q_n \) of an atom in \( S_{2^n} \) is the dual atom in \( U_k \). So, the dual atoms in \( U_k \) are \( 2^{n-1} - 1 = (0, 1, 1, \ldots, 1) \) and complements of \( m \) atoms in \( S_{2^m} \). Thus, there are \( m + 1 \) dual atoms in \( U_k \) if \( k = 2^{n-1} + 2^m \) for some \( 0 \leq m \leq n - 1 \). Since \( U_k \) is self dual, the number of atoms in \( U_k \) is same as the number of dual atoms in it. Therefore, \( n - 1 = m + 1 \) which implies that \( m = n - 2 \). This completes the proof.

\[ \square \]

5. Concluding remarks

In this paper, we studied lattice theoretic properties of upper good lattices. These properties follow dually for lower good lattices. We strongly feel that the upper and lower good lattices are the only lattices whose cover graphs are good sets. However, the following problem is open.

**Problem 1.** How many lattices are there whose cover graphs are good sets?

References


