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Article

Perfect Matching and Zero-Sum 3-Magic Labeling

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Abstract: A mapping $l : E(G) \to A$, where A is an abelian group written additively, is called an edge labeling of the graph G. For every positive integer $h \geqslant 2$, a graph G is said to be zero-sum h-magic if there is an edge labeling l from $E(G)$ to $\mathbb{Z}_h \backslash \{0\}$ such that $s(v) = \sum_{uv \in E(G)} l(uv) = 0$ for every vertex $v \in V(G)$. In 2014, Saieed Akbari, Farhad Rahmati and Sanaz Zare proved that if r ($r \neq 5$) is odd and G is a 2-edge connected r-regular graph, G admits a zero-sum 3-magic labeling, and they also conjectured that every 5-regular graph admits a zero-sum 3-magic. In this paper, we first prove that every 5-regular graph with every edge contained in a triangle must have a perfect matching, and then we denote the edge set of the perfect maching by EM, and we make a labeling $l : E(EM) \to 2$, and $E(E(G) - EM) \to 1$. Thus we can easily see this labeling is a zero-sum 3-magic, confirming the above conjecture with a moderate condition.

Keywords: 5-regular graph; zero-sum 3-magic; perfect matching; Tutte's Condition
Mathematics Subject Classification: 05C78; 05C70.

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1. Introduction and Basic Definitions

Graphs G considered here are all connected, finite and undirected with multiple-edges without loops. Finding a matching in a bipartite graph can be treated as a network flow. The study of matchings in a bipartite graph can be traced back to the early of the 20th century. In 1935, P. Hall proved a well-known result (**Hall's Theorem**) which provided a necessary and sufficient condition for an X, Y-bipartite graph to have a matching that saturates X [1, 3.1.11 Theorem]. When $|X| = |Y|$, **Hall's Theorem** is the **Marriage Theorem**, proved originally by Frobenius in 1917 [2]. Later in 1947, Tutte [3] investigated the problem concerning the existence of a perfect matching in a general graph and provided a necessary and sufficient condition (**Tutte's Condition**) for such a graph to have a perfect matching [Lemma 1]. If every vertex in a graph has the same degree r then this graph is referred to as a *r-regular* graph. In case of regular graphs, Peterson [4] proved that every 3-regular graph without a bridge (an edge whose deletion disconnects the graph) has a perfect matching. In 1981, Naddef and Pulleyblank considered K-regular graphs with specified edge connectivity and showed some classical theorems and some new results concerning the existence of matchings can be proved by using polyhedral characterization of Edmonds [5]. It is not hard to see that a 5-regular 4-edge connected graph has a perfect matching (see [5] for detail). However very little is known about the existence of a perfect matching in a general 5-regular graph. In this paper we provide a sufficient condition for a 5-regular graph to have a perfect matching and our main result is as follows.

Theorem 1. *Every 5-regular graph in which each edge is contained in a triangle has a perfect matching.*

Our motivation for this work is towards resolving a conjecture of Saieed Akbari, Farhad Rahmati and Sanaz Zare [6]. Before discussing this conjecture we require some more notation. A mapping $l : E(G) \to A$, where A is an abelian group written additively, is called an edge labeling of the graph G. Given an edge labeling l of the graph G, the symbol $s(v)$, which represents the sum of the labels of edges incident with v, is defined to be $s(v) = \sum_{uv \in E(G)} l(uv)$, where $v \in V(G)$. For every positive integer $h \geqslant 2$, a graph G is said to be *zero-sum h-magic* if there is an edge labeling from $E(G)$ into $\mathbb{Z}_h \backslash \{0\}$ such that $s(v) = 0$ for every vertex $v \in V(G)$. The *null* set of a graph G, denoted by $N(G)$, is the set of all natural numbers $h \in \mathbb{N}$ such that G admits a zero-sum h-magic labeling.

Recently, Choi, Georges and Mauro [7] proved that if G is a 3-regular graph, then $N(G)$ is $\mathbb{N} \backslash \{2\}$ or $\mathbb{N} \backslash \{2, 4\}$. Saieed Akbari, Farhad Rahmati and Sanaz Zare [6] extend this result by showing that if G is an r-regular graph, then for even r ($r > 2$), $N(G) = \mathbb{N}$ and for odd r ($r \neq 5$), $\mathbb{N} \backslash \{2, 4\} \subseteq N(G)$. Moreover, they proved that if r ($r \neq 5$) is odd and G is a 2-edge connected r-regular graph, then $N(G) = \mathbb{N} \backslash \{2\}$, implying G admits a zero-sum 3-magic labeling. Thus they proposed the following conjecture.

Conjecture 1. *Every 5-regular graph admits a zero-sum 3-magic labeling.*

As a consequence of our main result, we obtain the following result, partially confirming Conjecture 1.

Corollary 1. *Every 5-regular graph in which each edge is contained in a triangle admits a zero-sum 3-magic labeling.*

By Theorem 1, every 5-regular graph in which each edge is contained in a triangle has a perfect matching. We denote the edge set of the perfect maching by EM, and we make a labeling $l : E(EM) \to 2$, and $E(E(G) - EM) \to 1$. It is easily to see this labeling is a zero-sum 3-magic. Thus every 5-regular graph in which each edge is contained in a triangle admits a zero-sum 3-magic labeling, proving Corollary 1.

We next introduce some basic definitions which will be used in the proof of our main result. If X and Y are disjoint vertex sets of G with $x \in X$ and $y \in Y$, an edge$\{x, y\}$ is usually written as xy (or yx), and such an edge xy is also called an $X - Y$ edge. The set of all $X - Y$ edges in the edge set E is denoted by $E(X, Y)$. We denoted by $G(X, Y)$ the subgraph of G with vertex set $X \bigcup Y$ and the edge set $E(X, Y)$. An *odd* (or even) *component* of a graph is a connected component of odd (or even) number of vertices, and an *odd* (or even) *vertex* is a vertex with odd (or even) degree.

Definition 1. *For a graph G, for any vertex set S of G, and any odd component Q of $G - S$, consider a connected component of $G(S, V(Q))$ which has odd number of edges in $E(S, V(Q))$. The vertices of this component induce a subgraph P in G, and we call it an odd component induced subgraph.*

Definition 2. *Let S be any vertex set and $v \in S$. The vertex v is called removable from S if $q(G-S') = q(G - S) - 1$, where $q(G - S)$ denotes the number of odd components of $G - S$ and $S' = S - v$.*

Definition 3. *Let S be any vertex set, and let $v, w \in S$. The vertex pair (v, w) is called removable from S if $q(G - S') = q(G - S) - 2$, where $S' = S - v - w$.*

Definition 4. *Let S be any vertex set, Q be one of the odd components of $G - S$, we call Q the first type of odd component (with respect to S) if there is an odd component induced subgraph P, such that there exists a removable vertex or a removable vertex pair in $S \cap V(P)$. Otherwise, Q is called the second type of odd component (with respect to S).*

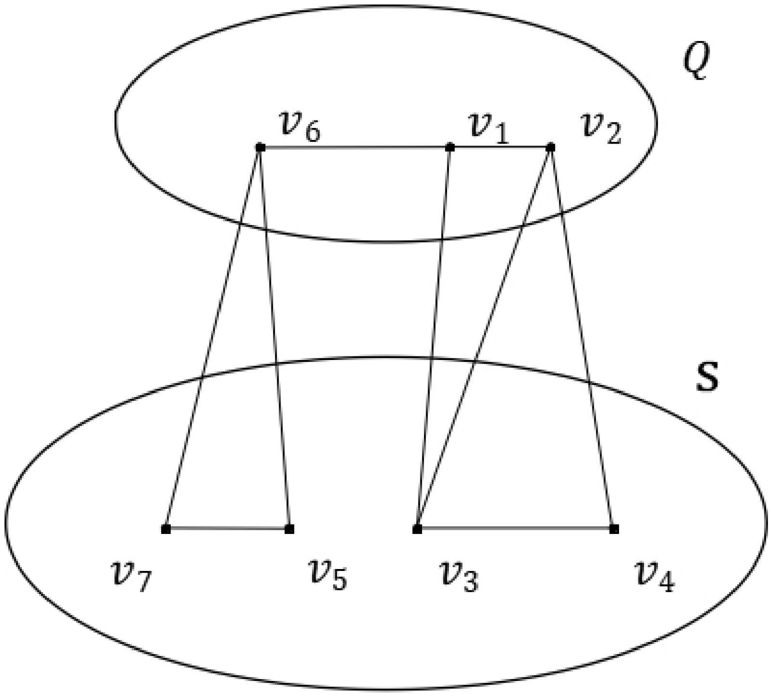

Figure 1. An example of an odd component induced subgraph P whose vertex set is $\{v_1, v_2, v_3, v_4\}$, and the edge set is $\{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4\}$.

Definition 5. *Let S be any vertex set, Q be an odd component of $G - S$ and P be an odd component induced subgraph. A connected subgraph K of P is called a 2-claw substructure, if the following conditions hold:*
1. $V(K) \cap S$ has 3 vertices $\{v, v_1, v_2\}$, and each vertex is connected to exactly 3 odd components of $G - S$.
2. The degree of v in P is 3.
In this case the vertices $\{v, v_1, v_2\}$ are called roots of K.

2. The Proof of Theorem 1

We begin with several important lemmas. The first one is well known as Tutte's Theorem [3] and is essential in the present paper.

Lemma 1. *(Tutte's Theorem) A graph G has a perfect matching if and only if $q(G - S) \leqslant |S|$ for all $S \subseteq V(G)$, where $q(G - S)$ denotes the number of odd components of $G - S$.*

If a graph G has no perfect matching, then there must exist $S \subseteq V(G)$ with $q(G - S) > |S|$. Such a set S is called an *antifactor* set of G.

From now on we always assume that G is a 5-regular graph in which each edge is contained in a triangle.

Lemma 2. *For any vertex set S, every odd component of $G - S$ has at least one odd component induced subgraph.*

Proof. For any vertex set S, and any odd component Q of $G - S$, we assume that $|E(S, V(Q))| = d$, and the number of vertices in Q is p. Since Q is an odd component, we know that p is odd. Let q be the sum of the degrees of each vertex contained in Q. Clearly q is even by the handshaking lemma. Since G is a 5 - regular graph, we have

$$q = 5p - d.$$

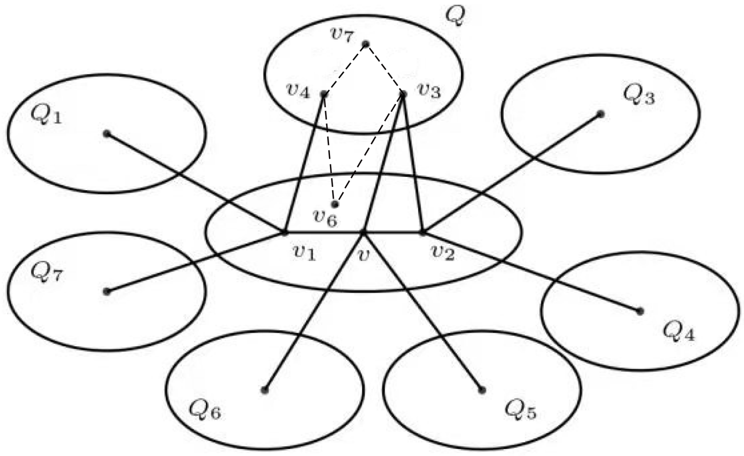

Figure 2. An example for 2-claw substructure whose the vertex set is $\{v_1, v_2, v_3, v_4, v\}$, and the edge set is $\{v_1v, v_1v_4, v_2v, v_2v_3, v_3v\}$.

Thus d is odd as q is even and p is odd. That is, the sum of the edges of all connected portions of $G(S, V(Q))$ is odd, so $G(S, V(Q))$ has at least one connected subgraph having an odd number of edges. The vertices of this connected subgraph induce a subgraph in G which is the desired odd component induced subgraph. □

Lemma 3. *For every vertex set S and every odd component induced subgraph P, there exists at least one vertex $v \in S \cap V(P)$, such that $d_P(v) \geqslant 3$.*

Proof. Assume that for every vertex v in $S \cap V(P)$, we have $d_P(v) \leqslant 2$. Since v has an edge in P and this edge is also in a triangle, the fact that the triangle is a connected graph implys that it is in P, so we must have $d_P(v) \neq 1$, and thus $d_P(v) = 2$ for every vertex v in $S \cap V(P)$. Suppose that there are k vertices in $S \cap V(P)$, and s edges in $E(S) \cap E(P)$. Then $|E(S, V(Q)) \cap E(P)| = 2k - 2s$, which is even, a contradiction. Thus the lemma holds. □

Lemma 4. *If a vertex v in S is connected to d odd components (where $d \leqslant 2$), then v is a removable vertex .*

Proof. If $v \in S$ is connected to only one odd component Q, then we remove v out of S, and denote the new vertex set by $S_1 = S - v$. Since Q is an odd component, after adding a vertex v to Q, it becomes an even component. Note that there is no change in parity for all other odd components. Thus $q(G - S_1) = q(G - S) - 1$, so v is a removable vertex.

If v is connected to 2 odd components Q_i and Q_j, then we also remove v out of S, and denote the new vertex set by $S_1 = S - v$. Since v is connected to Q_j and Q_i, at this time, Q_i, Q_j and v become a new connected component, and the number of its vertices is $|Q_i| + |Q_j| + 1$ which is odd. Note that the number of odd components is reduced by one, that is, $q(G - S_1) = q(G - S) - 1$, so v is a removable vertex. □

We remark that Lemma 4 does not hold when $d \geqslant 3$. However, we may obtain a similar result for a vertex pair when $d = 3$.

Lemma 5. *If a vertex pair (v, w) in S is connected to exactly 3 odd components, then (v, w) is a removable vertex pair.*

Proof. Assume that the vertex pair (v_1, v_2) is connected to 3 odd components Q_i, Q_j and Q_k. We move (v_1, v_2) out of S, and denote the new vertex set by S_1. Clearly $|S_1| = |S| - 2$. As before, there is no change in parity for all odd components except Q_i, Q_j and Q_k. Since the vertex pair (v_1, v_2) is connected to Q_i, Q_j and Q_k, the components Q_i, Q_j and Q_k and the vertex pair (v_1, v_2) become a new connected component, and the number of its vertices $|Q_i|+|Q_j|+|Q_k|+2$ is odd. So the number of odd components is reduced by two, and we have $q(G-S_1) = q(G-S) - 2$. Thus the vertex pair (v_1, v_2) is a removable vertex pair. □

Lemma 6. *Let Q be a second type of odd component in $G-S$ and P_j be the jth odd component induced subgraph of Q. Then for all j, P_j has a 2-claw substructure.*

Proof. According to Lemma 2, Q has an odd component induced subgraph P_j. Let v be a vertex of $V(P_j) \cap S$ such that $d_{P_j}(v)$ is maximum among all vertices in $V(P_j) \cap S$. By Lemma 3, we have

$$3 \leqslant d_{P_j}(v) \leqslant 5.$$

We now divide the rest of the proof into 3 cases.

Case 1. $d_{P_j}(v) = 5$.

In this case, since G is a 5-regular graph, all the edges connected to v are in P_j, so there is only one odd component adjacent to v. By Lemma 4, v is a removable vertex, and thus Q is a first type of odd component, yielding a contradiction.

Case 2. $d_{P_j}(v) = 4$.

In this case, since G is a 5-regular graph, only one of the edges of v is not in P_j, so there are at most 2 odd components adjacent to v. By Lemma 4, v is a removable vertex, and thus Q is a first type of odd component, yielding a contradiction.

Case 3. $d_{P_j}(v) = 3$.

Since $d_{p_j}(v) = 3$ and G is 5-regular, we know that v is connected to at most two other odd components. We may always assume that v is connected to two other odd components Q_s and Q_t. For otherwise, by Lemma 4, v is removable, and thus Q is the first type, yielding a contradiction.

Subcase 3.1. *v has* 0 *edge in* $E(S) \cap E(P_j)$.

In this subcase, as mentioned before, v is connected to two other odd components Q_t and Q_s. We assume that $e_{Q_t}(v)$ is connected to Q_t, where $e_{Q_t}(v)$ denotes the neighbors of v in Q_t, and $e_{Q_s}(v)$ is connected to Q_s. Let us consider $e_{Q_t}(v)$. By the assumption of G, $e_{Q_t}(v)$ must be an edge of a triangle, so there must exist another edge e_t connected to v in this triangle. If $e_t = e_{Q_s}(v)$, then we deduce that Q_t is connected to Q_s, yielding a contradiction, because Q_t and Q_s are different odd components. The same argument shows that $e_t \notin E(P_j)$, so e_t is the 6th edge of v, yielding a contradiction.

Subcase 3.2. *v has* 1 *edge in* $E(S) \cap E(P_j)$.

In this subcase, v has 2 edges in $E(V(Q), S)$. Let $v_1, v_2 \in Q$ be the 2 vertices, which are adjacent to v. And $v_3 \in S$ be the vertex which is adjacent to v. Let $v_4 \in Q_t$ and $v_5 \in Q_s$, which are adjacent to v.

Since the edge $e(v_4, v)$ is in a triangle, we may let v_q be such that vv_4v_q is a triangle. Thus v_q is a neighbor of v, and so $v_q \in \{v_i | i = 1, 2, 3, 4, 5\}$. As proved before $v_q \notin \{v_i | i = 1, 2, 4, 5\}$ (as Q, Q_t, Q_s are different odd components, so any two of Q, Q_t, Q_s are not connected), so $v_q = v_3$. For the same reason, v_3 is connected to v_5 in Q_t. Since $v_3 \in V(P_j)$, there exists a vertex $v_6 \in Q$ adjacent to v_3. If (v, v_3) is connected to 3 different components, by Lemma 5, (v, v_3) is a removable vertex pair, yielding a contradiction.

Otherwise, the last neighbor v_p of v_3 is in another odd component. Since the edge $e(v_3, v_p)$ is in a triangle, v_3 must have another neighbor v_q which is adjacency to v_p. As before $v_q \notin \{v_i | i = p, 4, 5, 6\}$. Thus $v_q = v$, which implies that v has 6 neighbors, yielding a contradiction.

Subcase 3.3. *v has* 2 *edge in* $E(S) \cap E(P_j)$.

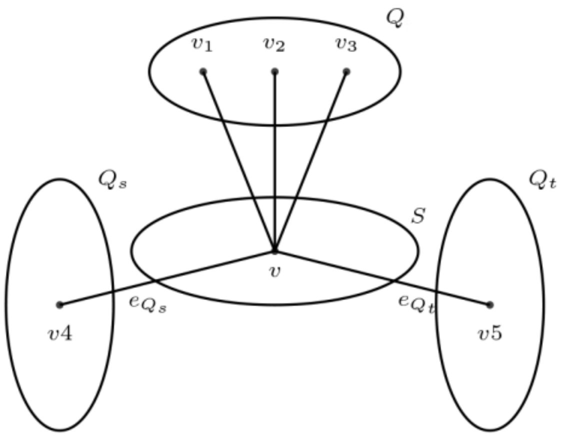

Figure 3. $d_{P_j}(v) = 3$ and v has 0 edge in $E(S) \cap E(P_j)$

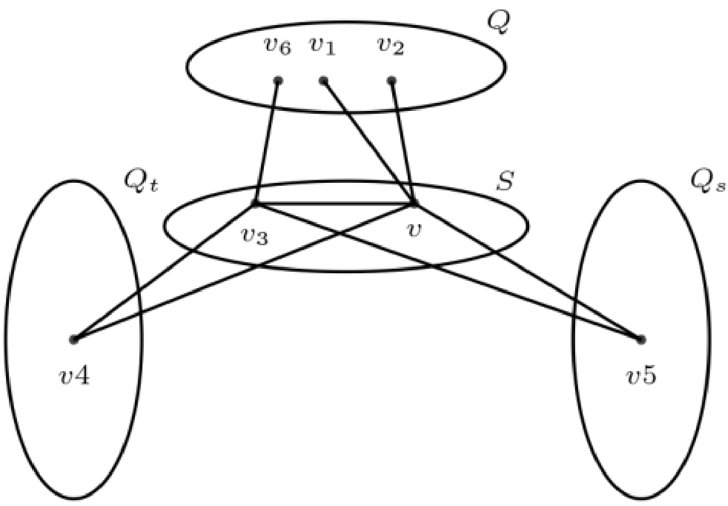

Figure 4. $d_{P_j}(v) = 3$ and v has 1 edge in $E(S) \cap E(P_j)$

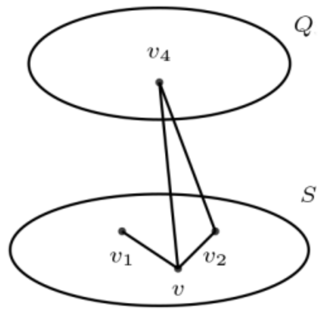

Figure 5. $d_{P_j}(v) = 3$ and v has 2 edge in $E(S) \cap E(P_j)$

In this subcase, we assume that 2 vertices v_1 and v_2 are the neighbors of v and in $S \cap V(P_j)$. If one of the vertex in $\{v, v_1, v_2\}$ is connected to at most 1 other odd component, then according to Lemma 4, then this vertex is removable, so Q is a first type of odd component, yielding a contradiction.

Next we may assume that v is connected to two odd components other than Q. and $v_i \in \{v_1, v_2\}$ is connected to three odd components other than Q. Let e_1, e_2 and e_3 be the edges joining v_i. Without loss of generality, we may assume there is an edge e_j connected to e_1. As we proved before, the third vertex of the triangle containing e_1 must be v. So v has a neighbor in this component outside of P_j. Similar results hold for e_2 and e_3. Thus we have 3 neighbors of v outside of P_j. This together with $d_{p_j}(v) = 3$ shows that v has six neighbors, yielding a contradiction.

In conclusion, we have just shown that every vertex in $\{v, v_1, v_2\}$ is connected to Q and 2 other odd components. Therefore, P_j has a 2-claw substructure.

Subcase 3.4. *v has 3 edges in $E(S) \cap E(P_j)$.*

In this subcase, v has no neighbour in Q, that is $E(v) \cap E(Q, S) = \varnothing$, contradicting the definition of P_j.

□

We are now ready to prove Theorem 1.

Proof of Theorem 1:

Assume that the vertex set S is a minimal counterexample to Tutte's condition, i.e., S is a minimal set such that $|S| < q(G-S)$. Then every odd component of $G-S$ is of the second type. For otherwise, if Q is the first type of odd component of $G-S$, then there exists an odd component induced subgraph P, which has a removable vertex v (or a removable vertex pair (v_1, v_2)). Let S_1 be the new vertex set after removing v (or (v_1, v_2)) from S. Then $S_1 = S - v$ (or $S_1 = S - v_1 - v_2$). So we may replace S by S_1. By the definitions 2 and 3, we have $q(G-S_1) = q(G-S) - 1$ (or $q(G-S_1) = q(G-S) - 2$). That is $q(G - S_1) - |S_1| = q(G - S) - |S|$. So $|S_1| < q(G - S_1)$, yielding a contradiction to the minimality of S.

Suppose that Q is a second type of odd component and P is an odd component induced subgraph. By Lemma 6, P has a 2-claw substructure which has a set of 3 roots. Let S_p be the set of all such roots corresponding to all odd components of $G - S$. Each vertex in S_p has 3 edges connecting itself to the odd components. Let $E(S_p)$ denote the set of all these edges. Then $|E(S_p)| = 3|S_p|$. Since P has a 2-claw substructure, there are at least three edges form Q to S_p. So this gives at least $3q(G-S)$ edges (from all odd components to S_p). Since each of these edges is in $E(S_p)$, we obtain $3q(G-S) \leqslant 3|S_p|$. Since $|S| \geqslant |S_p|$, we have:

$$3|S| \geqslant 3|S_p| \geqslant 3q(G-S).$$

Then, $|S| \geqslant q(G-S)$, yielding a contradiction. Therefore, Tutte's condition holds, and thus by Lemma 1 (Tutte's Theorem), G has a perfect matching.

□

Acknowledgments

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