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Article

Some Excluded Minors for the Spindle Surface

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Abstract: We identified, via a computer search, 143 excluded minors of the spindle surface, the space formed by the identification of two points of the sphere. Per our search, any additional excluded minors must have at least 12 vertices and 28 edges. We also identified 847 topological obstructions for the spindle surface. We conjecture that our lists of excluded minors and topological obstructions are complete.

Keywords: excluded minor, topological obstruction, spindle surface, pseudosurface

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1. Introduction

We say a *surface* is a compact, connected 2-manifold without boundary, and a *pseudosurface* is the result after performing a finite number of point identifications (of finitely many points) of one or more surfaces if the resulting space is also connected. The points that have been identified with other points we call the *pinchpoints* of the pseudosurface. The *spindle surface*, or *pinched sphere*, is the pseudosurface, with one pinchpoint, obtained from identifying two points on a sphere. Every surface is also a pseudosurface (with zero pinchpoints).

We say that a graph G can be *embedded* in a pseudosurface P if G can be drawn in P such that, if we think of G as a 1-complex, no two points in G occupy the same point in P. We say a graph is *pinched-planar* if it can be embedded in the spindle surface.

We assume basic familiarity with graph theory terminology as found in [1]. If a graph H can be obtained from G by deleting edges or vertices, or by suppressing vertices of degree two, we say that H is a *topological minor* of G. We say H is a *minor* of G if H can be obtained from G by deleting edges or vertices, or by contracting edges. Hence, topological minors are also minors.

If G embeds in a pseudosurface P, then so does every topological minor of G. We say a pseudosurface P is *minor-closed* if for every graph G that embeds in P, it follows that every minor of G also embeds in P. It's easy to see that the spindle surface is minor-closed, but not all pseudosurfaces are. In particular the *bananas surface* B_2, the pseudosurface created by identifying two spheres at their respective north and south poles, is not minor-closed [2].

We call a graph G a *topological obstruction* of a pseudosurface P if G does not embed in P but every proper topological minor of G does. We say G is an *excluded minor* of a minor-closed pseudosurface P if G does not embed in P but every proper minor of G does. Note that every excluded minor is also a topological obstruction. In fact, each topological obstruction of a minor-closed pseudosurface can be contracted to an excluded minor.

The collection of excluded minors for any minor-closed pseudosurface must be finite by the Robertson-Seymour Theorem [3]. However, the complete collection of excluded minors is known only for two surfaces, the sphere and the projective plane. The set $\{K_5, K_{3,3}\}$ is the complete collection of both topological obstructions [4] and excluded minors [5] for the sphere. A list of 35 excluded minors and 103 topological obstructions for the projective plane was identified by Glover, Huneke, and Wang [6], and Archdeacon proved their list was complete in [7]. The collection of excluded minors for the torus is not known, but Myrvold and Woodcock have identified 17,535 excluded minors and 250,815 topological obstructions [8]. Mohar and Škoda have investigated the excluded minors of the torus and Klein bottle of low connectivity [9, 10]. Note that the spindle surface can also be obtained by identifying all points on a given meridian of a torus, from which it follows that any graph that can be embedded on the spindle surface can also be embedded on the torus [11]. Similarly, it can be shown that if a graph embeds in the spindle surface, then it can be embedded in the Klein bottle. (While the spindle surface is a pseudosurface, these manners of constructing it are not done with a finite number of point identifications.)

Research on excluded minors and topological obstructions for pseudosurfaces that are not surfaces has also been conducted. Archdeacon and Bonnington in [12] found the complete list of the 21 cubic topological obstructions of the spindle surface. Širáň and Gvozdiak showed that B_2 has infinitely many topological obstructions [13], and with Bodendiek, Gvozdjak and Širáň they identified the 82 which have connectivity at most two [14]. In [15], Boza, Dávila, Fedriani, and Moyano demonstrated an infinite family of pseudosurfaces, each with infinitely many topological obstructions. A graph G is *outer-embeddable* in a pseudosurface P if there is an embedding of G in P with all vertices on the boundary of a single face. Boza, Fedriani, and Núñez in [16] showed that, in general, the problem of a graph's outer-embeddability in a pseudosurface is NP-complete. In [17], they showed that the set of outer-embeddable graphs in B_2 is minor-closed, and they produced a complete list of the 38 minor-minimal graphs that are not outer embeddable in B_2. In [18], the same authors explore a weakened notion of outer-embeddabilty in pseudosurfaces arising from three spheres.

Types of embeddings and embeddability of graphs in pseudosurfaces from algebraic perspectives have also received interest [19–21].

A graph G is *apex* if deleting some vertex makes it planar, or if G is itself planar. If a graph is embedded in the spindle surface, then deleting the vertex (if any) at the pinchpoint gives a plane graph. So pinched-planar graphs are apex. Apex graphs have received considerable attention, for example [22, 23], but their list of excluded minors is still unknown. We find pinched-planar graphs to be an interesting minor-closed subclass of apex graphs that are embeddable in both the torus and the Klein bottle for which the problem of finding the excluded minors appears tractable.

Our contribution is the following: through a computer search, we have identified 143 excluded minors and 847 topological obstructions for the spindle surface. We conjecture that the list of excluded minors is complete.* If correct, our conjecture would answer a question of Archdeacon [24, Problem 6.5]. Our results may be of interest to researchers interested in excluded minors for classes of graphs that are close to being planar [25, 26].

2. Additional Background

From now on, we will work exclusively with the following reformulation of embeddability in the spindle surface.

*Bodendiek and Wagner thought that the number of topological obstructions (which they called $<_1$-minimal graphs) for the spindle surface was “about 100” in [11].

Proposition 1. *Any planar graph is pinched-planar. A non-planar graph is pinched-planar if and only if it can be obtained by identifying two vertices of a planar graph.*

Sketch of Proof: The vertex created by the identification of the two selected vertices of a graph embedded in the sphere is embedded at the pinchpoint of the spindle surface. □

We make essential use of [11, Theorem 2], which we rephrase for our purposes.

Theorem 1. [11, Theorem 2] *A graph is a topological obstruction of the spindle surface if and only if the following three conditions hold:*

1. *G is not pinched-planar,*
2. *the minimum degree of G is at least three, and*
3. *the graph $G - e$ is pinched-planar for each edge e of G.*

We call a graph *Kuratowski* if it is a subdivision of K_5 or $K_{3,3}$. We denote the disjoint union of graphs G and H by $G \dot\cup H$. Identifying a vertex of G with that of H gives a 1-*sum* of G and H. If G and H are vertex-transitive, they have a unique 1-sum, up to isomorphism, which we denote by $G \bigoplus_1 H$.

It is easy to give a complete description of topological obstructions and excluded minors of connectivity less than two. In fact, these are the same as for the torus [8, Figure 7].

Theorem 2. *There are three disconnected excluded minors for the spindle surface: $K_5 \dot\cup K_5$, $K_5 \dot\cup K_{3,3}$, and $K_{3,3} \dot\cup K_{3,3}$.*

Proof. That these three graphs are excluded minors is easily checked. Let G be a disconnected excluded minor for the spindle surface. If some component of G were planar, then deleting it would give a pinched-planar graph with an embedding in the spindle surface containing a face in which the planar component could itself be embedded. So every component of G is nonplanar and hence has either a K_5- or $K_{3,3}$-minor. By minimality in the minor order, G must have exactly two components and must be either $K_5 \dot\cup K_5$, $K_5 \dot\cup K_{3,3}$, or $K_{3,3} \dot\cup K_{3,3}$. □

By considering blocks and vertex identifications instead of components and disjoint unions, one can show the following result.

Theorem 3. *There are three excluded minors of connectivity one for the spindle surface: $K_5 \bigoplus_1 K_5$, $K_5 \bigoplus_1 K_{3,3}$, and $K_{3,3} \bigoplus_1 K_{3,3}$.*

There are no more disconnected topological obstructions, but seven additional topological obstructions are obtained by performing 1-sum operations on Kuratowski graphs.

3. Computer Search

We write $V(G)$ for the vertex-set of a graph G and $E(G)$ for the edge-set. For $v \in V(G)$, we write $d(v)$ for the degree of v, and $N(v)$ for the neighborhood of v. To facilitate our use of Proposition 1, we introduce a definition.

Definition 1. *Given a simple graph G with vertex v and a subset S of $N(v)$, we define the* split of G on v by S, *denoted $G_{v|S}$, as the graph that results from the following steps:*

1. *add a new vertex, say w, to G, and*
2. *for each $x \in S$, delete edge vx but add edge wx.*

So a graph G is pinched-planar if and only if some split $G_{v|S}$ is planar. A naive algorithm for testing embeddability for the spindle surface is to test every split $G_{v|S}$ for planarity.

We optimized this naive algorithm somewhat although ours still has exponential running time. We give pseudocode for our algorithm in Algorithm 1. First note that if $|S| = 0$, then $G_{v|S}$ is obtained from G by adding an isolated vertex. So in this case, G is planar if and only if $G_{v|S}$ is. Next, notice that if $|S| = 1$, then $G_{v|S}$ is obtained from G by deleting an edge e incident to v, then adding a vertex of degree one adjacent to the other vertex incident to e. In this case, $G_{v|S}$ is planar if and only if $G - e$ is planar. Finally, notice that, by symmetry, a split $G_{v|S}$ is isomorphic to $G_{v|N(v)-S}$, so we need only test the splits corresponding to half the subsets S of $N(v)$ for planarity.

To test if a graph G is pinched-planar we first test G for planarity using the Boyer-Myrvold test [27], as implemented in `nauty` version 2.7r1 [28]. If G is planar, it is also pinched-planar. If G is non-planar, then we request a Kuratowski subgraph, say K, of G. We need only test splits on vertices in $V(K)$ for planarity since K is a subgraph of any split $G_{x|S}$, where $x \notin V(K)$. Likewise, we need only test deletions $G - e$, where $e \in E(K)$, for planarity.

Algorithm 1 Testing a graph G for embeddability in the spindle surface

if G is planar **then**
  return True
else
  $K \leftarrow$ a Kuratowski subgraph of G
  for each $e \in E(K)$ **do**
    if $G - e$ is planar **then**
      return True
    end if
  end for
  for each $v \in V(K)$ **do**
    for each $S \subseteq N(v)$ with $2 \leq |S| \leq d(v) - 2$, where $G_{v|N(v)-S}$ has not been tested **do**
      if $G_{v|S}$ is planar **then**
        return True
      end if
    end for
  end for
  return False
end if

The `planarg` program from `nauty`[†] either indicates that a graph is planar or, if not, produces a Kuratowsi subgraph. Using `planarg` as a starting point, we implemented our algorithm in the C programming language. Our source code is available at [29].

Testing whether a graph is a topological obstruction for the spindle surface is straightforward, and we optimized slightly by suppressing vertices of degree two after deleting an edge. Testing whether a graph is an excluded minor is also straightforward, but we did optimize slightly by deleting any multiple edges that arose from contracting an edge.

Given our observations above, to search exhaustively for topological obstructions on n vertices and m edges, we need search only the 2-connected graphs with minimum degree at least three.

To search exhaustively for excluded minors on n vertices and m edges, we need only search the topological obstructions we previously found. Moreover, we can narrow the search space by considering that pinched-planar graphs are sparse in that the number of edges is linear in terms of the number of vertices. More specifically, since a simple planar graph on $n \geq 3$ vertices has at most $3n - 6$ edges, a simple pinched-planar graph on $n \geq 2$ vertices has at most $3n - 3$ edges. So any topological obstruction for the spindle surface on n vertices has at most $3n - 2$ edges. Since the minimum degree

[†]The `planarg` program was written by Brendan McKay and Paulette Lieby, and is itself based on code from the Magma Computational Algebra System [30].

$n\backslash m$	15	16	17	18	19	20	21	22	23	24	25	26	27	total
6	1													1
7	2													2
8	3				1		2	1						7
9	1		2	5	4	7	4	1						24
10	1		4	7	18	21	4	2		1				58
11				5	17	5	8	1	1					37
12				3	2	2	4	1	1					≥ 13
13							1							≥ 1
total	8	0	6	20	42	35	23	6	2	1	0	0	0	≥ 143

Table 1. The number of excluded minors for the spindle surface with N vertices and M edges. A blank entry in the table should be interpreted as a 0. There are no excluded minors with fewer than 15 edges

$n\backslash m$	15	16	17	18	19	20	21	22	23	24	25	26	27	total
6	1													1
7	2													2
8	3	2			1		2	1						9
9	1	4	3	5	4	12	4	1						34
10	1	1	13	16	35	29	30	3		1				129
11			1	28	52	81	38	18	1		1			220
12				4	50	84	58	21	10	2		1		≥ 230
13						27	60	21	13	1	2		1	≥ 125
14							10	46	3	8		1		≥ 68
15									19	1	3			≥ 23
16										6				≥ 6
total	8	7	17	53	142	233	202	111	46	19	6	2	1	≥ 847

Table 2. The number of topological obstructions for the spindle surface with N vertices and M edges. A blank entry in the table should be interpreted as a 0. There are no topological obstructions with fewer than 15 edges

of a topological obstruction must be at least three, a topological obstruction on n vertices must have between $\lceil 3n/2 \rceil$ and $3n-2$ edges. For example, any topological obstruction for the spindle surface on 11 vertices must have between 17 and 31 edges.

The `geng` program of `nauty` can be used to generate all non-isomorphic graphs on a small number of vertices and edges, perhaps subject to additional constraints such as connectivity or minimum degree. We used `geng` to generate all 2-connected graphs with minimum degree at least three on $n \leq 11$ vertices or with $m \leq 27$ edges. We also generated all such graphs with 12 vertices and 28 edges. (These totaled roughly 50 billion graphs.) We then used our algorithm to check whether each generated graph was a topological obstruction, and, finally, we checked which topological obstructions were excluded minors. We used GNU Parallel [31] to parallelize the search. Our computations took roughly two months using four multi-core computers.

We found 143 excluded minors and 847 topological obstructions. See Tables 1 and 2 and Appendices 1 and 2. We summarize our results in the following theorems.

Theorem 4. *The 143 graphs in Appendix 1 are excluded minors for the spindle surface. No excluded minor has fewer than 15 edges. Any additional excluded minors must have at least 12 vertices and at* *least 28 edges.*

Theorem 5. *The 847 graphs in Appendix 2 are topological obstructions for the spindle surface. No topological obstruction has fewer than 15 edges. Any additional topological obstructions must have at least 12 vertices and 28 edges.*

Recall that there are three disconnected excluded minors and three of connectivity one. Our computer search found six excluded minors of connectivity two, 117 of connectivity three, 12 of connectivity four, and two of connectivity five. That we found no excluded minors of higher connectivity is perhaps not surprising since Lipton et al. showed that an excluded minor for the class of apex graphs has connectivity at most five [32].

As a sanity check, we also searched all cubic graphs on $n \leq 24$ vertices and found precisely the topological obstructions in [12]. We point out that 125 of the 143 excluded minors and 701 of the 847 topological obstructions we found are apex graphs. We note that K_6, the Petersen graph, and the five other members of the Petersen family are excluded minors for the spindle surface.

Based on the fact that there are no excluded minors with 25, 26, or 27 edges, we conjecture that our list of excluded minors is complete.

Finally, we performed an additional computation that shows that if our list of excluded minors is complete, then our list of topological obstructions must also be complete. Given an excluded minor G of the spindle surface, we describe how to find the set of topological obstructions that contract to G. Note that if H is a topological obstruction with a set of edges, say E, that contracts to G, then contracting any subset of E from H also yields a topological obstruction. Let us call a graph G' an *inverse-contraction* of G if contracting some single edge of G' gives G. Note that, using the terminology of our Definition 1, any inverse-contraction of G may be obtained by adding edge vw to some split $G_{v|S}$. (Contracting vw gives G.) The set of all topological obstructions that contract to G can be found by finding all inverse-contractions of G, discarding any graphs which are not topological obstructions, then finding all inverse-contractions of the remaining graphs, again discarding any graphs which are not themselves topological obstructions, and repeating this process. Since the minimum degree of a topological obstruction is at least three, this procedure must eventually terminate. Performing this procedure on each of our 143 excluded minors resulted precisely in our set of 847 topological obstructions.

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4. Appendix 1

Here is a list of the 143 known excluded minors for the spindle surface organized by connectivity. Each graph is given as a `graph6` string[‡], a comma, and an indication of whether the graph is apex. This list may be downloaded from [29].

Disconnected

```
I~{?GKF@w, non-apex
Js\o?CB?wF_, non-apex
Ks?GOOF@r_M?, non-apex
```

Connectivity One

```
H~?GW^~, apex
IFw?GKFxw, apex
J??FF?[FFw?, apex
```

Connectivity Two

```
GwC^~{, apex
Hr?G[|n, apex
Hs?GZ|}, apex
IIMCC@NLo, apex
Io?WrAF]G, apex
IS`AA?~No, apex
```

[‡]The `graph6` format translates the adjacency matrix of a graph into a compact description in the ASCII character set. See [33] for more.

Connectivity Three

```
Fs\zw, non-apex
G?B~vo, apex
GqOxs{, non-apex
GS`zro, non-apex
H@Bmtpx, apex
H?B\rrw, apex
HCCR^Zk, apex
H_hZtg~, apex
Hj?L[x~, apex
H@KemZ{, apex
H`K}UNr, apex
HodXrL~, apex
HoSsaKj, non-apex
H@ou^jy, apex
H?{pnNU, apex
HqSp[|}, apex
H@r@xzr, apex
Hs`Hb|~, apex
Hs[JIkv, apex
H?zTb|}, apex
IAHbCyYf_, apex
IaOpSKxf_, apex
I@BLQvoNw, apex
I?BvSpXMW, apex
IC[V?LdbW, apex
I`?CzYSYG, apex
I??CzZw|?, apex
IE?HXjgro, apex
I?{EMGvLo, non-apex
IFaAPKmRO, apex
IF??X^kro, apex
IGBTSo|Uo, apex
IgCXQMpn_, apex
IGFcqWr}?, apex
I`GRQimfO, apex
I?Gu]iw]G, apex
I?hicdxl_, apex
I?hlbaTUg, apex
IIISPlM}?, apex
I`K|ATEcW, apex
I@?Kyzgx?, apex
IMK?G]fr_, apex
I`?N~bLNW, apex
IoCGZhqbw, apex
IoDb?wZf_, apex
IoDPhPHfw, apex
IoDPPXRf_, apex
I?oHhjB|?, apex
I]o?HKVBw, apex
Io[?Ikubw, apex
IoP@pi[Fo, apex
IOP_sxqfg, apex
IOWQkpdfg, apex
IQGPGvLr_, apex
IQGP_^Lr_, apex
IQ?gprFpo, apex
IqHPSpVJo, apex
IQMR?MJRW, apex
IQ?pOvKro, apex
Iq??Xy]Zg, apex
I?rE@wyL_, non-apex
IRQM@cNNW, apex
IsGZAcN^G, apex
IsP@OkWHG, non-apex
IW?{OtZtO, apex
I??wVFQ}?, apex
IX?G}_Nv?, apex
I??xuNgu?, non-apex
I]??XyMRg, apex
J?Ab?wYuFB?, apex
J@?BCZ_Fmw?, apex
JB?G}Qcw?^_, apex
JB?K]QcCzW_, apex
J?BLQow@~__, apex
JB??yIhbVK?, apex
J?C\RJ_C}w?, apex
J@Dc?SMsVB?, apex
JDZE?KaCWR_, apex
J??EdPKLFw?, apex
JE?GV?XhjL?, apex
JEr@@?J@p`_, apex
J@?e?[[uFB?, apex
JFaAPScPG^_, apex
JFO@OWFxCN?, apex
J]?GGRBI_i_, apex
J?GU_X`bNa?, apex
J?`H]ao[?^_, apex
JkGPcPDAgM_, apex
J_?@|`L[CW_, non-apex
Jl`@OgH@gF_, apex
JoC?Jpef_~?, apex
J?ox_E`S]M?, apex
Jo??ygl]CN_, apex
JpG?IhIFcN_, apex
J?_PIRocrW_, apex
J_?@|pSRD@_, apex
JQAKQXIL`b?, apex
JqC_iIIGWU_, apex
JQ?EXW[WKa_, apex
JQ`@GodEmE?, apex
JqQ@`_MBOF_, apex
JR?GSddba[_, apex
JRQKACg@Wd_, apex
JsO_oghP_F_, apex
K``@?aEQWkDH, apex
K``@?aEQWsCh, apex
KB@HSb?g?M@F, apex
K??CrJSjBo?^, apex
K?df_?@B[bKL, apex
K??FMo{o@GbB, non-apex
K_hSb?OG?R_u, apex
KoCO?\IHb_ON, apex
K??@pATRUI^?, apex
KPHAC?TAo{WF, apex
Ks?APKSEAGdC, apex
Ks?GOoUP@CkG, apex
LF`@@@?_?X`Y@w, apex
```

Connectivity Four

```
Fs\vw, non-apex
GEl~~w, apex
GJz\~{, apex
H~AIX[^, apex
H`iZQn~, apex
HoSsZf{, apex
HqAztXZ, apex
H{_yqgj, non-apex
IrQH_YrRo, apex
IscQXXb\?, non-apex
ISPDtlkVG, apex
IukAHLLL_, apex
```

Connectivity Five

```
E~~w, non-apex
G{S~~w, non-apex
```

5. Appendix 2

Here is a list of the 847 known topological obstructions for the spindle surface organized by connectivity. Each graph is given as a `graph6` string, a comma, and an indication of whether the graph is apex. This list may be downloaded from [29].

Disconnected

```
I~{?GKF@w, non-apex
Js\o?CB?wF_, non-apex
Ks?GOOF@r_M?, non-apex
```

Connectivity One

```
H~?GW^~, apex
IFw?GKFxw, apex
I~??W[N~?, apex
J??FF?[FFw?, apex
JFw??KF@~K?, apex
JS\o?CB?}F_, apex
Jv{??KF@zK?, apex
K?ACALEM@o^?, apex
Krd_WCA?WB_N, apex
L?aAA?bGowB_{?, apex
```

Connectivity Two

```
GwC^~{, apex
Hr?G[|n, apex
Hs?GZ|}, apex
HwC^?~|, apex
Hw?W~r}, apex
IF?GW^`z_, apex
IFw?GKfpw, apex
IFW?GMNxo, apex
I~?G]?nFo, apex
I~?GW]pRg, apex
I~?GWXrbo, apex
I]?GX_Nro, apex
IIMCC@NLo, apex
IJa?W^o^o, apex
IJ?G^aM~_, apex
I}k?GLNLo, apex
Io?WrAF]G, apex
IS`AA?~No, apex
I~w?GKVBw, apex
I]??W[{ro, apex
J?Bb?oW_~o?, apex
J?BMP_oA^__, apex
J?B_ooBwNo?, apex
JC^_?CBC~F?, apex
J?C^FA[Wrw?, apex
JC\o?CB_~F?, apex
JF?GW[MwV@?, apex
JFw??KFXcN?, apex
J]?GO[Mp`b?, apex
J]?GW[Kohb?, apex
J]?GW[MoP`_, apex
JIk?GKFkMM?, apex
J`KDsHDPZw?, apex
J@MDqHDP^o?, apex
J]oo?CBAwN_, apex
JoXk?cRCzw?, apex
Jr`G?CBHwv?, apex
JrW?GKW@{N?, apex
JrY??CBDw^?, apex
J@vP@?B?}F_, apex
KBX_sA?_?J_], apex
K?C\D@bWswJ_, apex
K??CJC[FF?^?, apex
KC_ZB@__?JbQ, apex
KDYIg?@?XBwM, apex
K@?G[\MkB_}?, apex
K??GUGqKeG^?, apex
KI_xo?@?[BwM, apex
K`K?GMw]AMOw, apex
K_KsAE?OG[eK, apex
KoDjO_O?WBoN, apex
Ko?Wr@__?q_u, apex
KPTSW?@?XBwM, apex
K?r@`b?K?P_x, apex
KRr?x?@?WB_V, apex
KrY?oGC?wF?N, apex
KrY?oKC?gB_N, apex
KrY?wG@?WB_V, apex
K]??WZ?K?F@b, apex
L??CATIb@gE_{?, apex
L??CCXKR@cEO{?, apex
L??GOkUaB?{?{?, apex
L??H?cdDeOX?{?, apex
L]ooOGB?OC_F?N, apex
L]ooOG@?OD_U?N, apex
L]ooOK@?_@_F?N, apex
Lr`HGo@?_@_F?N, apex
LrY?gWA?O@_F?N, apex
Ms???[_CHCaSR?R??, apex
Ms???KEA_RH_KC[??, apex
Ms???@KR@WAWCoGo?, apex
```

Connectivity Three

```
Fs\zw, non-apex
G?B~vo, apex
GqOxs{, non-apex
GRr@x{, non-apex
GS`zro, non-apex
H?aFbx{, apex
HBGc}ZK, non-apex
H`BHvr}, apex
H@Bmtpx, apex
H?B\rrw, apex
HCCR^Zk, apex
HCS~F?^, non-apex
HeGn?~|, apex
HFGcY^K, non-apex
H_hZtg~, apex
Hj?L[x~, apex
H@KemZ{, apex
H`K}UNr, apex
HodXrL~, apex
HoSsaKj, non-apex
H@ou^jy, apex
H?{pnNU, apex
HQoHhjF, non-apex
HqS`K|~, apex
HqSp[|}, apex
H@r@xzr, apex
Hs`Hb|~, apex
Hs[JIkv, apex
H?zTb|}, apex
IAHbCyYf_, apex
IAKGeMex_, apex
IAM?gZbwo, apex
IaOpSKxf_, apex
I@aqQVo^o, apex
IB_GRMUx_, apex
I@BLQvoNw, apex
I?BvSpXMW, apex
I@C^F?NvG, apex
I]C?G[mro, apex
ICOf?w[wW, non-apex
IC[V?LdbW, apex
I@?cyzKy?, apex
I`?CzYSYG, apex
I??CzZw|?, apex
I?DcvHy~?, apex
IE?HXjgro, apex
I?{EMGvLo, non-apex
IFaAPKmRO, apex
IFG@G^Kro, apex
IF?GXNHz_, apex
IFo_GKjpw, apex
IfrH@CfEw, apex
IF??X^kro, apex
IGBTSo|Uo, apex
IGCxEfIf_, apex
IgCXQMpn_, apex
IgDbKyYMW, apex
IGFcqWr}?, apex
IGFcrGZ}G, apex
I@GGuNSx_, apex
I`GRQimfO, apex
I`GR[yiTW, apex
I?Gu]iw]G, apex
I??gvFS}?, apex
I?GWuNox_, apex
I?@HeUs}?, apex
I?hicdxl_, apex
I?hicfoNw, apex
I?hlbaTUg, apex
I_hSb?~Nw, apex
I?`HW~o{?, non-apex
IIISPlM}?, apex
I`@ItakNw, apex
I@J]E?zMo, apex
I`K|ATEcW, apex
Ik__ghJIo, non-apex
I@KpUfKp_, apex
IkQ?Hs]Jg, apex
I@KxEfIp_, apex
I@?Kyzgx?, apex
IL?OXVKro, apex
IMK?G]fr_, apex
I`?N~bLNW, apex
IoCGZhqbw, apex
IoCWbTef_, apex
IOCZRIRv?, apex
IoDb?wZf_, apex
IoDcotd^G, apex
IoDPhPHfw, apex
IoDPPXRf_, apex
IoGictV^?, apex
I@OGtNSx_, apex
I?oHhjB|?, apex
I]o?HKVBw, apex
Io[?Ikubw, apex
IoP@pi[Fo, apex
IOP_sxqfg, apex
IOWQkpdfg, apex
I_PHduu^?, apex
IpNE?pfFo, apex
I?qapjo^o, apex
IqGBGw[_w, non-apex
IqGh[pTSw, apex
IQGPGvLr_, apex
IQGP_^Lr_, apex
IQ?gprFpo, apex
IqHPSpVJo, apex
Iq?kqcl^G, apex
IQMR?MJRW, apex
IQ?pOvKro, apex
IqQ@`_NBo, non-apex
I`@qSUwNw, apex
Iq??Xy]Zg, apex
IRaAQGbFG, non-apex
I?rE@wyL_, non-apex
IR?M@[]rW, apex
IRQM@cNNW, apex
IsCRRGNBw, apex
IsGZAcN^G, apex
IsP@OkWHG, non-apex
ItPH_Xrbo, apex
I@Uee_mJW, apex
I?U\F@qNw, apex
IWC_WZRso, apex
IWFE?{]MW, apex
I@w?GnUxo, apex
IW?mow\sW, apex
IW?{OtZtO, apex
I??wVFQ}?, apex
I??WvJa}?, apex
I]`?WWbCw, non-apex
IXaAA?^Fo, apex
IX?G}_Nv?, apex
I??xuNgu?, non-apex
I]??XyMRg, apex
J?Ab?wYuFB?, apex
JAc_GMqYUM?, apex
JAf`P?P?}F_, apex
JAI@iX_c[T?, apex
JAJ_?cJefF?, apex
JAKaKqEW[{?, apex
JAk?HDFdVK?, apex
JAw?_MsHuM_, apex
J?AZS`_A^__, apex
J@AZS``aiY_, apex
JBc_GN_A}M_, apex
J@?BCZ_Fmw?, apex
JBf@GCHG]F_, apex
JB__G\gcmM?, apex
JB?GkZaJUS_, apex
```

Connectivity Three, continued

```
JB?G}Qcw?^_, apex
JB?G[]QXNO?, apex
JBg?WL`dMM?, apex
JB?K^B_BW{_, apex
JB?K]QcCzW_, apex
J?BLQow@~__, apex
JBOk?SEc^B?, apex
JBo?pGFhMM?, apex
JB?[QTIhTH?, apex
JBWK?KEc^B?, apex
JB??yIhbVK?, apex
Jc_`A@wDo]?, apex
J_CbLLWFLo_, apex
J??CjQK[Nw?, apex
JCOpOKXwMM?, apex
JcO_yIGKXF?, apex
J?C\RJ_C}w?, apex
JCW?WloxCN?, apex
J?d\B?RwG}?, apex
J@Dc?SMsVB?, apex
Jdh???VBrE?, apex
J_?DjqKR@`_, non-apex
J@@DOW[sVB?, apex
JDS_GL`amM?, apex
JdWcA?F@op_, non-apex
JdWW?Cb`yN?, apex
JDZE?KaCWR_, apex
J?EdAqX\b[?, apex
J??EdPKLFw?, apex
JE?GV?XhjL?, apex
J@?eOW[sVB?, apex
J??EPgK{Nw?, apex
JEr@@?J@p`_, apex
J@?e?[[uFB?, apex
JFaAPScPG^_, apex
JFAKR@_Bw^?, apex
JF?GW]EW^O?, apex
JF?GX[aqMH?, apex
JF?GXSeqMI?, apex
JF?GYSehUP?, apex
JF?HKSdRMS?, apex
JFO@OWFxCN?, apex
J??^F?{u?}?, apex
JF??XKmrEQ?, apex
JgCWTCeS[{?, apex
J]?GGRBI_i_, apex
J@G?g^WxEM?, apex
JG?hubCE[{?, apex
J]?GOBHHom?, apex
J]`GOGBGwf_, apex
```

```
J`GpSpFPco_, non-apex
J@?GuNaVDW_, apex
J?GU_X`bNa?, apex
J]?GW[PHmP?, apex
J?`H]ao[?^_, apex
J@HCOK[sVB?, apex
J@HCWoSo^B?, apex
J?HRTVOFLg_, apex
Jh_?wWcS[F?, apex
J??HWzo{E]?, apex
JiGO?SFelM?, apex
JIgW?cbpcN?, apex
JIg@_WTpcN?, apex
JI?HaYLkd{?, apex
JI_H_gJhMM?, apex
J@IIOgQo^B?, apex
J?Ij[`PE^__, apex
JIk_GCBd]F?, apex
JIk?GKp`mM?, apex
JIk?H_F`mM?, apex
JIk??Ku`uM?, apex
J@IQOoEo^B?, apex
J@IQOWQo^B?, apex
Jj?DcWLBjw?, apex
Jj?DkOLBZw?, apex
J@JGOcQo^B?, apex
JjKCKOF@zw?, apex
JkCaC?NBPc_, apex
JK_@GhJNEE_, apex
JkGPcPDAgM_, apex
J?K@gRDpVa?, apex
J`KhchIOsH_, apex
JKog?cFKmM?, apex
Jk__x?BPWr_, apex
Jk??xXKKKE_, non-apex
J@?LbdKbNw?, apex
J_?@|`L[CW_, non-apex
JL?GW{aqMH?, apex
JL?GXSUpUP?, apex
JL?GY[SolP?, apex
JLog?CFC}M?, apex
Jl`@OgH@gF_, apex
JLO?WKXhMM?, apex
JLQ@W_DA]F_, apex
JLS?GKJhMM?, apex
J?N_?cJsfF?, apex
JoC?Jpef_~?, apex
J@OcOK[sVB?, apex
Jo@cOsZ\Bw?, apex
JoCQ@ARRP[?, apex
```

```
JoC@WXoTSN_, apex
J]o???fDo]?, apex
J?oGGJokre?, apex
JoG?WlW\CN_, apex
JoG?Z_[TcN_, apex
J?oH?kU{FB?, apex
Jo@HWpPK[{?, apex
J?oi``o`sN?, apex
J]?@OjKBo]_, apex
Jo[?_KTXcN?, apex
J?oPGSppNa?, apex
J@OWpRDotK?, apex
J?ox_E`S]M?, apex
Jo??ygl]CN_, apex
JpC?G\SYcN_, apex
JpG?IhIFcN_, apex
Jp?_GtWRcN_, apex
J?_PIRocrW_, apex
JpPC??lEom?, apex
JpS?OKTXcN?, apex
J_?@|pSRD@_, apex
JQAKQXIL`b?, apex
JqC_iIIGWU_, apex
Jq?CYWKKXd?, non-apex
JQ?EXW[WKa_, apex
JQ`@GodEmE?, apex
JqGOO?rRSM?, apex
JqGPGoBP[T_, apex
JqGWGOBW{f_, apex
JQ?HW{oqMH?, apex
JQo@GgJLMM?, apex
JQo_`_M@}M?, apex
Jq_P??xBqM?, apex
JqQ?@KeEwv?, non-apex
JqQ@`_MBOF_, apex
JQ_@WhHLME_, apex
Jq?@wWKcZB?, apex
JQ?@WzCMME_, apex
J?_QXYOWN_?, non-apex
J@r_?CREvF?, apex
Jr_GOdc@wN_, apex
JR?GSddba[_, apex
JR?GUN?Jg|?, apex
Jr`???NHou?, apex
J@rO?cbIeF_, apex
J?roOF@Kxf?, non-apex
JRQG?CfEuF_, apex
JRQKACg@Wd_, apex
JrWOGCBB[F_, apex
JrY?GGBCwV_, apex
```

Connectivity Three, continued

```
Js?BRG[F?F_, non-apex
J@s?GKfsVK?, apex
JsH@goKOWF_, apex
JsO_goKOxF?, apex
JsO@gWKKYF?, non-apex
JsO_oghP_F_, apex
JsO__@pBo]?, non-apex
JSPB_WKK[F?, apex
Js??wxaWOX_, non-apex
Js?@Wx`S_X_, apex
JSXP_OD?}F_, apex
J???thkpfo?, apex
JtPG?C~kq]_, apex
J???vG{pfo?, apex
JWCcIMWU?~_, apex
J@?WfRaba[_, apex
J``@WgWW[F?, apex
J??wPbBovo?, apex
J??wU?Rwfo?, apex
J?@XOaBwVo?, apex
J?xo?CRcvF?, apex
J??^@?XpVo?, apex
JXqA?_F@op_, non-apex
J??XU?RxFo?, apex
J??Y`QBxFo?, apex
JYQ?wG`E[F_, apex
J_?ZAyaMlw?, apex
J@zP?_B?}F_, apex
J?zP`?P?}F_, apex
J??Z?qBxFo?, apex
K?AAHGYEV?^?, apex
K`aAQGbGoxBa, non-apex
KAc`aI?AWL[B, apex
K``AC?hEPKas, non-apex
K``AC?kJ?fAY, apex
K``@?aEQWkDH, apex
K``@?aEQWsCh, apex
K?A?ghIIV?^?, apex
KAH@ICceKqO{, apex
K?[AIIokCM?}, apex
KaMC@?FB_akD, non-apex
K?AQhmGQHP~?, apex
KBAKS`_W?Q_t, apex
KB@HSb?g?M@F, apex
K]???BKIom@w, apex
K`CaC?Ngaa@b, apex
Kc_`A@w@oMBB, apex
K?CcJLeUCov?, apex
KcD`ACg_?b`q, apex
K_CdAIGS?[eL, apex
K??cGdHFF?^?, apex
KcHAC?tAo[EB, apex
K??CJG[Ef?^?, apex
KcL_ACa_?b`q, apex
KCLJ_?@@[B{K, apex
KCOb_ODFCB{K, apex
K?CPPJ??}W]A, apex
K??CrJSjBo?^, apex
KcS`B?E_?b`q, apex
KCS`CDCCW[[B, apex
KCSq@E?@WT[B, apex
KCTPW?@G\BWM, apex
KCUJG?@GXbWM, apex
KC`Xo?@OXB{K, apex
K`d_??BBsfK[, apex
K?df_?@B[bKL, apex
K?df_?H@[akL, non-apex
KDhQP?@?WLwN, apex
K?dPf?G@GD{F, apex
KdY?wG@?XBgV, apex
KDZ???Z@pEWB, non-apex
KEh@K@@GOTae, apex
K?ERV?CAG`{F, apex
K]?EX?O?WZ`m, non-apex
KFGe?C@oGZ`m, non-apex
KF?HHR?o?T?j, apex
KF?KQSj`AGcb, apex
K??FMo{o@GbB, non-apex
K?GaCNOF?s[B, apex
K?GacV?DGw[B, apex
K?GcIJOE_q[B, apex
K]?G?DKKs]@}, apex
K@G]G?PA^AWM, apex
K?GOZAO?}W]A, apex
K?@g_QBcRC~?, apex
KGQgo?BC\aWM, apex
K`G?sGdPYsRo, apex
K_GSKHIQP`eW, apex
K]?GS?w@oMGF, apex
K_GTAHGc?[eL, apex
K??GTHQcbG~?, apex
K??GUJAK`g^?, apex
K??hEbCE_s[B, apex
Khe?hSCOGB_V, apex
K?@H_?FqTa^?, apex
K?HO?]QgaP}G, apex
KhOWsA?G?J_], apex
K`hP?d?_?R_u, apex
K_hSb?OG?R_u, apex
K?hSb@?_?rcu, apex
Kh_XQ?O_?F_m, apex
K??_iagAuW]A, apex
KIa@yW_CK@_N, apex
KiGSC?J@oocd, apex
KIKqCE?_?J`U, apex
KI?|_OG?[BwM, apex
K_iPa?H@QDeE, apex
KKa@A?JCqhDa, apex
KKaA@?RCqWdc, non-apex
KkC`AOE_?b`q, apex
KkCcXd?@GP_N, apex
KkC_?@FIoeP`, apex
KkC_GPK_p__\, apex
KkD@?Kg_?b`q, apex
Kk__?@FEowCX, non-apex
KKhO??rAoMWB, non-apex
KK`@I_g_?Uae, apex
Kk__??NEqIEP, apex
KKo@`HC_?w_], apex
KKo@HHO_?w_], apex
KKoo`D?_?J`U, apex
KkQ@??\AoMEB, apex
K??KQSjtDgM_, apex
KKSs@D?_?R_u, apex
Kk__yoCGGB_V, apex
K?`LA`Og?kdK, apex
K@?LbdoRCWpc, apex
K??]L@_EOp[B, apex
KL_gq@?_?F_m, apex
KL?HOj?o?K`L, apex
Kl?IC?FE_IdD, apex
Kl?KA?iDOJ@R, apex
KLQ?@?FAo{WF, apex
KlQ???Z@oMDB, apex
KLr@G_I?WB_Z, apex
KLr?XCO?gB_N, apex
KMi@GsCOGB_V, apex
KMi?oGC?YFc], apex
K??MQgsiF_?^, apex
KM_XQA?G?F_m, apex
K]`???NAoMEB, non-apex
KoCO?\IHb_ON, apex
KoDrO_G?WBoN, apex
Ko??GkM]BoO{, apex
K??@OhceEE^_, apex
KoHJ_gOAGBoN, apex
KoMA?_bR?Ead, apex
K??}OO`C^_WM, apex
Ko@PPPO_?XaY, apex
Ko@rOoOAGBoN, apex
```

Connectivity Three, continued

```
KoSsbCG@GB_V, apex
KoTO_CBIPHO^, apex
K??OUIaJAc^?, apex
K??OUJAJ@c^?, apex
Ko@@?wYBb_ON, apex
Ko@@?wYJAcON, apex
Ko_Xa?H?qHeE, apex
KoXPc?HCOH_V, apex
Ko?YoG`OXoON, apex
K??@pATRUI^?, apex
KPDCA?RBP[WF, apex
KpG??LWBqkO{, apex
KpGWq@?_?J_], apex
KPHAC?TAo{WF, apex
KpO??AUIo]Ig, apex
K?@PO?FqTa^?, apex
K?pp_?BA\aWM, apex
K??]P?PWKI~?, apex
K`_PQIOW?Q_t, apex
KpSAhWC_GD_N, apex
KPTC??RBPMWF, apex
KqC_GOT_r?`T, apex
K??{QCo?]O}A, apex
KqCPQG__?L`M, apex
K?qc``_S?P_x, apex
KqGO?DEa`IbK, apex
K]Q@gWG?gB_N, apex
KqGWs@?G?J_], apex
KQKsC@?O?V_}, apex
KqOc??\GoMBB, non-apex
K??qOcKkMQ[K, apex
KQPCC?kK_Ya[, apex
KQPCC?qIOZAq, apex
KQPCC?wM?N?y, non-apex
KQQ???fErIRO, apex
KqQH?c_C?L`M, apex
KQQ???jDrIRO, apex
KqQ@Og_C?L`M, apex
KQr@?_B?p``e, non-apex
KqrH@COCHf`}, apex
K]Q?X_H@GE_V, apex
Kr???AMFQUCw, apex
KR`CA?XH_Q_t, apex
KRr@?_G@?F_], non-apex
KRr?X?O?wF?N, apex
KrY?GGQAOF_^, apex
KS`A@?{@oUIB, apex
Ks?APKSEAGdC, apex
K?S_Cd_@k[]A, apex
KsCQPG_O?L`M, apex
K]`@_SEAGI_N, apex
Ks?G?@hX_uAw, non-apex
Ks?GOoUP@CkG, apex
Ks?HA?TEaTCi, apex
KsH???rBowGX, non-apex
KsH???ZHowGX, non-apex
KsL?XcCOGB_V, apex
Ks@@?O[E`Ig[, apex
KsO_??ZDpQGp, apex
KsO_z_G@GB_V, apex
KSTcA?HGOLaU, apex
KsWR_WC?gB_N, apex
Ks???wYP`KL?, non-apex
KSXO??rAoMWB, non-apex
K?@@tBGF?i[B, apex
K[TCG?@CwZCN, apex
K`UC@?FH_QiD, non-apex
KUHAC?XH_Q_t, non-apex
KUIAA?[@oUCF, non-apex
KUoa??F@oqCF, non-apex
K??WAObkeK^?, apex
KWC?GoFNeqWs, apex
KW?cIKWeIpBw, apex
KWECA?VIO[IB, apex
KW?KYWocI`bw, apex
K]`?WOCCXF?], apex
K]`?XcG@GB_V, apex
K??x?o@omD^?, apex
KXo?Wg@?ghwF, apex
K`XP?e?_?R_u, apex
K???X[]rF_]?, apex
K`XSC?B?pHbE, apex
K?XSW?`C\aWM, apex
K?XT_?H@\aWM, apex
K??@YaoBUW]A, apex
KYc?GGJGogwD, apex
KYCPIQ?_?J`U, apex
K?YX?_B?~_W], apex
L?aAA?bCpWB_{?, apex
L?_aOiOWF_@I@h, apex
LAr@@?O_?F_]N?, apex
LB?SIKfaCOHBIE, apex
LbY@C?C??F_yEK, non-apex
L??CATEb@oE_{?, apex
LChb?a?O?Rao?\, non-apex
L?@CHOYCPGx?{?, apex
L?CidB?o?H`WKB, non-apex
L??cIS_EGpW_{?, apex
L]CkA?F?_A`B?N, apex
L[CqAA?A?F_eAp, non-apex
L??CQG[FBCSG{?, apex
LC`QU?_G?L_MM@, apex
L??CSG[FBCKG{?, apex
L??C?STXF?Bo}?, apex
LCTdE????bdE@k, non-apex
LC?ZBAGo?H`WKB, apex
L??CZb?I?g?F~?, apex
LDjAA?Z@@?_R?V, apex
LDWcC@?O?F`wDK, non-apex
LDxRC?B?OG_F?N, apex
L@EcU@?O?d`gGL, non-apex
L_eRB?RA?GcB?N, apex
LFw?G?@P`BO{@}, apex
LF`@@@?_?X`Y@w, apex
L??g_aIQPEW_{?, apex
L??G`CiDUGX?{?, apex
LGed?`?O?F`IDH, apex
LGfE?__G?R_wCL, non-apex
L??G_geIeAX?{?, apex
L??goaGOXEW_{?, apex
L??go_gO[EW_{?, apex
L??go_hOcAw_{?, apex
L??goq?OXDW_{?, apex
L??GP?TCv?Wo}?, apex
LGQQdA_A?K_MEB, apex
L??GT@OcbG?{}?, apex
L]?GWC@?XBber_, apex
L@GYtB??GA_boF, apex
L???hGiFEAX?{?, apex
L@HIeA?_?h`WGL, non-apex
L??H?kUIE?x?{?, apex
L??hOgQqE??L{@, apex
LHoPAA?_?]AM@k, non-apex
LHoPAA?_?]?mCk, non-apex
L@HQUA?_?LbGGL, non-apex
L@HQUA?_?p_wGL, non-apex
L@hUE????FbEHK, non-apex
L?iaqg??H@aEoN, apex
LiC`Oi?_?D_MBB, non-apex
Li?G`ECa?T?iKB, apex
LiGP_Y?_?B_UBB, non-apex
L??IKSgDA`SA{?, apex
Li?pOq?_?D_MBB, non-apex
L??ISMGP@`EA{?, apex
L@j?c@?O?]AMGk, apex
L@`Kd@?O?h_MI`, apex
L???kGiFBAX?{?, apex
LK_hc@?O?b`E@p, non-apex
L?KHeFCG?__boF, apex
LKHM?___?J_YEH, apex
```

Connectivity Three, continued

```
L???KH`S`SR_}?, apex
L]KIK?B?OG_F?N, apex
L????kM[DOWo}?, apex
LK`M?__G?R_wCL, non-apex
L_KtA`?_?D_MBB, non-apex
L?Ku??@AWN]?o[, apex
L?Ku??H@OF}?o[, non-apex
L?LACAoHOe@ooF, apex
L?LCLHQA?_cBoF, apex
LL_gq@?_?B_M@b, apex
L?lu????WfCMo[, apex
L`?MECoBHd@i~?, apex
LMG[C@?G?F_eAp, non-apex
LoDa`a?A?D_MBB, apex
L@O\DB?O?H_YEB, non-apex
LoDcaOOG?B_UBB, apex
L???OkUbBO[?{?, apex
L]o__OE@OC_L?N, apex
L]o__OF@?C_J?N, apex
L]o__O???L_m?{, non-apex
Lo?qW?@CWRN?_], apex
LOsoGCA?Z_GM_], apex
L???oyEHAO{?{?, apex
LpG[A@???F_mEc, apex
LpGYAA???MaUA[, apex
L@PTDB?A?H_YEB, non-apex
LQKsAE?O?B_UBB, non-apex
LqOpOo@?G@_r_N, apex
LqQ???fEpGB@?r, non-apex
LqQ@Go??gBcdEL, non-apex
LqQ@Go??GFceEL, apex
LqYP?_???F_]BK, apex
LRCkC@?O?F_eAp, non-apex
LR`KAC??@P_mCk, non-apex
LrWO[?@?OH_U?N, apex
LrWOS?C?_B_M@b, non-apex
LS`aaOw@?C_J?N, apex
LsCRB?A?O@`fBM, non-apex
LSXO?C@?`d@ioK, apex
L_T_`A?_?FdSDS, apex
L[TCH?@?O@aVBM, apex
L@UeE?FC@?`B?N, apex
L??w@Ca_yFZ?u?, apex
LWCkc@?O?F`EDP, apex
LWIK_`?O?J`WCL, apex
L???[WqSF?@a{@, apex
L??@Wx_cEO?e{@, apex
L???wy?QHH[?{?, apex
L???WZ?gaaEc}?, apex
```

```
L???_xIHcW[?{?, apex
LXQM?_F?O_`B?N, apex
L?YUE?_C?FcaEH, apex
L?YW??BGwfWW{?, apex
M?aAQHOg@_E??{BB?, non-apex
M]_AGWKG?_?B?X?e_, non-apex
M]_AHGSA?O?B?X?e_, apex
M??C?gSAoTK_w?}??, apex
M?`CIQOW@_E??{BB?, non-apex
M`dcb?K?OC?F?M?F_, apex
MDg?gWcoA?G??N?r?, apex
Mdg?WgCGI?G??V?j?, non-apex
MdWcA?Z@?G?J?e?F_, apex
MDWEGG[?OA?BGFoF?, apex
M_Ea@OgS?cO?@F?x?, non-apex
ME?hON@???`BKEoF?, apex
M_E?j@OI@OO?@F?x?, non-apex
M?FDA_gIA_O??r@L?, non-apex
MF?KC@AOPGD?Aq@L?, apex
MgA?i_WI@OO?@F?x?, apex
M???GGRAuGM?i?}??, apex
M`GQSGWHA?O??r?N?, apex
M???GSIgIaQ_X?}??, apex
M??guB?WC_E?AF@w?, non-apex
MhIAG?WCOQO?@F?x?, apex
MhS_WGO?[?O??f?Z?, non-apex
M?iaa_No@?A@?F?F_, apex
MiC_X_CAK?O??V?j?, non-apex
MiGOpGG@K?O??V?j?, non-apex
MiKoOGA?[?O??f?Z?, non-apex
MK?CQICQ@_E?@dAY?, apex
MkhP_OC?W?_P?J?R_, apex
MM_e?WKC@??B?X?e_, apex
M?ND?O`SA_O??N@p?, apex
M]o?gG_A?G_K?b?[_, non-apex
M@@?oggoE?S??xBE?, apex
M[O?iGga?GA@?p?M_, apex
Mo@Q`Wi???aBAE_F_, apex
M`P@OgSIC?O??f?Z?, apex
MpTPS?@?OG?O?N?N?, apex
MQCkb?CAK?G??f?Z?, apex
M`Q?gggW?OO@?L?q_, non-apex
M`Q?gh_a@?@@?p?M_, non-apex
MqG?WWSKA?O??N?r?, non-apex
MqG?WWSWC?A??N?r?, non-apex
MQKsC@?O?F_W?LB@_, non-apex
M`Q@OgSIA?O@?d?Y_, non-apex
Mq_?WWSK@?G@?L?q_, non-apex
Mq_?WWSW@?A@?L?q_, non-apex
```

```
M?r@`boA?C_I@B?F_, non-apex
Mr_?gWKW??_P?R?J_, apex
MrW?G?@@_BaqAY`w?, apex
MrWOOGB_?G?O?N?V?, non-apex
MrY?GGOA?B?Q?X?e_, apex
Ms?B?oWP@G@_?F?w_, non-apex
Ms???\?GGccSF?R??, apex
Ms?GGGBEA@CBCoY??, apex
Ms?GOCEE?aCDPOWC?, apex
MsG?WWSKA?G??N?r?, non-apex
MsG?XGSEA?G??N?r?, non-apex
Ms???K_CgdHGR?J??, apex
Ms???@KQ_eBGHOH_?, apex
Ms???@KR@SCgCoCo?, apex
Ms??OKEB?_aKY?WA?, apex
MSP@OggDC?G??f?Z?, apex
MSP@OgSIC?G??f?Z?, apex
MsP@_W_AGGA@?X?e_, apex
MWCU_WK?O@?BGFoF?, non-apex
MWFE?gIA?_C??V?N?, non-apex
N???`EC`APHAI_DO]??, apex
N??GGOGCXDE_oOo_]??, non-apex
N??G?GXGogCPq?o_]??, non-apex
N??GOCDHOdX?S?g?}??, apex
N??GOCECHAw_oOM?]??, non-apex
N??GOGSI?ccKo_oO]??, non-apex
N??GOKCAMCWCKCKO]??, non-apex
N??GOKEAC_kAcGWO]??, apex
N??GOOSDAECWoCgG]??, non-apex
N??GOPE`_aCGgAS@]??, non-apex
N??GQA?HOdCcCoq?]??, apex
N???GQBOogD@p?M?]??, apex
N???GSGChBW_L?q?]??, non-apex
N????[OFADCSb?w?]??, non-apex
N???OOBDHEW_q?L?]??, non-apex
N@QEE?gD?gD??o?H_BW, non-apex
NqQ@GocE?E?A?H?B_@w, apex
NqQ@HGWE?C?G?J?F?@w, non-apex
N????SECWwP_W_w?]??, apex
N[TCGcKG??_A?J?R?@w, apex
N???WWOC`@gKb?w?]??, apex
N???WXBCCGCBM?oC]??, non-apex
N???WXBC?gCBi?oC]??, non-apex
Os?G??I@OI@OCHCEKGBC?, apex
Os?G??I@OI@OCHGEIGBC?, non-apex
Os?GOGA@?CAICIGQGEBO?, non-apex
Os?GOKG?AKAIGKCCCCA?B, non-apex
Os?GO?@?OIADOcGcAcBO?, apex
Os?G????wEBCHADGGWBO?, non-apex
```

Connectivity Four

```
Fs\vw, non-apex
GEl~~w, apex
GJz\~{, apex
G}_xq[, non-apex
H~AIX[^, apex
H`iZQn~, apex
HoSsZf{, apex
HqAztXZ, apex
H{_yqgj, non-apex
IrQH_YrRo, apex
IscQXXb\?, non-apex
Is_JJd[N?, apex
ISPDtlkVG, apex
IukAHLLL_, apex
```

Connectivity Five

```
E~~w, non-apex
G{S~~w, non-apex
```