Journal of Combinatorial Mathematics and Combinatorial Computing, 117: 177–184
DOI:10.61091/jcmcc117-16
http://www.combinatorialpress.com/jcmcc
Received 28 November 2023, Accepted 2 December 2023, Published 6 December 2023

Article

The proper Ramsey numbers of K_3 against C_3 and C_5

Eric Andrews[1] **and Kyle Walker**[1*]

[1] Department of Mathematics and Statistics University of Alaska, Anchorage Anchorage, AK 99508, USA

* **Correspondence:** kpwalker@alaska.edu

Abstract: For graphs F and H, the proper Ramsey Number $PR(F, H)$ is the smallest integer n so that any $\chi'(H)$-edge-coloring on K_n contains either a monochrome F or a properly colored H. We determine the proper Ramsey number of K_3 against C_3 and C_5.

Keywords: Graph theory, Edge coloring, Ramsey theory, Proper Ramsey number
Mathematics Subject Classification: 05C55.

---

1. Introduction

In this paper, we give values for the proper Ramsey number of K_3 against C_3 and C_5. Let F and H be graphs and $\chi'(H)$ the chromatic index of H. The *proper Ramsey number*, $PR(F, H)$, of F and H is defined to be the smallest positive integer n so that every $\chi'(H)$-edge coloring of K_n contains either a monochrome F or a properly colored H. Necessarily, if $PR(F, H) = n$, then there exists at least one coloring of K_{n-1} which contains neither. Such a coloring is called a *critical coloring* with respect to $PR(F, H)$.

English et al. [1] determined $PR(F, H)$ for a variety of graph pairs in which $\chi'(H) = 2$, including for K_3 against C_4. Extending that work, Olejniczak [2] found the remaining values of $PR(K_3, C_n)$ where n is even. The problem we address here concerns a pair of graphs for which $\chi'(H) = 3$, namely to determine $PR(K_3, C_3)$ and $PR(K_3, C_5)$.

Note that the proper Ramsey number is one of several variations on the ordinary Ramsey number. It is closely related to the *edge-chromatic Ramsey number* introduced by Eroh [3]. The edge-chromatic Ramsey number $CR(F, H)$ is defined to be the the smallest positive integer n so that if the edges of K_n are colored with any number of colors, then the colored graph contains either a monochrome F or a properly colored H. Eroh [3] shows that this number exists only for certain classes of graph pairs. The proper Ramsey number, on the other hand, is a restriction of edge-chromatic Ramsey number, in the sense that we color K_n with a fixed number of colors. Olejniczak [2] shows that the proper Ramsey number exists for all pairs of graphs F and H. Further variations on the Ramsey number are discussed in Chartrand and Zhang [4].

In the following, graphs are assumed to be finite, simple, and undirected. We refer to Chartrand and Zhang [5] for our graph-theoretic notation.

2. Essential Preliminaries

At several points in this paper, we shall need to select a vertex of a graph so that it is incident to the greatest number of edges of the same color. We therefore adopt the following definition.

Definition 1. *Let G be a graph, and let $\{c_i\}$ be the indexed family of colors of edges in G. If v is a vertex of G, then the* color degree of v with respect to the color c_i *is the number of edges incident to v that have the color c_i. We denote the color degree by $d_{c_i}(v)$.*

Note that there must be some i and some x for which $d_{c_i}(x) \geq d_{c_j}(v)$ for all j and for every vertex v in G. That is, there is some vertex incident to the greatest number of like-colored edges. The proof is trivial. Let $S = \{d_{c_i}(v) : v \in V(G)\}$ be the set of all color degrees of all vertices in G. Then S is a set of integers bounded above by $\Delta(G)$. It therefore has a greatest element. We denote this greatest element $\Delta_c(G)$ and call it the *maximal color degree*.

We shall also require the following facts about K_4 and K_5

Lemma 1. *Every 2-edge-coloring of K_4 that does not contain a monochrome K_3 contains a properly colored P_4 beginning with each color.*

Proof. Let G be a 2-edge-colored copy of K_4 containing no monochrome K_3. Consider an arbitrary vertex x of K_4. Without loss of generality, suppose that x is incident to more red edges than blue edges. Then x is incident to either three or two red edges.

Suppose that x has three red edges. Then G must contain either a red or a blue K_3. Let xa, xb, and xc be the three red edges. Then ab and bc must be blue. But then ac produces a monochrome K_3 regardless of whether it is red or blue. So x must not have three red edges.

Suppose that x has two red edges. Let xa and xb be red and xc blue. Then ab must be blue to avoid a red K_3. So (c, x, b, a) is a properly colored P_4 beginning with a blue edge. On the other hand, bc and ac cannot both be blue, since a blue K_3 would result. So one is red. Thus one of (x, a, b, c) or (b, x, c, a) is a properly colored P_4 beginning with a red edge. □

Lemma 2. *Any 2-edge-coloring of K_5 not containing a monochrome K_3 decomposes K_5 into two Hamiltonian cycles of different colors.*

Proof. Let G be a 2-edge-colored copy of K_5. We first show that $d_c(v) = 2$ for every color and vertex v in G. Suppose otherwise. Then $d_{c_i}(v_j) \geq 3$ for some i and j. Without loss of generality, suppose that this vertex is labeled a and the color is red. Then there are at least three vertices, label them b, c, and d, adjacent to a through a red edge. If any of the edges bc, cd, or bd is red, a monochrome K_3 results. So none of them are red. But then they must all be blue, and again a monochrome K_3 results. Thus each monochrome subgraph of G is 2-regular.

Consider then, the red subgraph of G. This subgraph must be C_5, since there is no other 2-regular graph on 5 vertices. The same argument applies to the blue subgraph. Since these subgraphs have disjoint edge sets and identical vertex sets, they decompose G into two Hamiltonian cycles of different colors. □

3. Results

Theorem 1. $PR(K_3, C_3) = 11$.

Proof. We first show that $PR(K_3, C_3) \geq 11$. Consider the following coloring of K_{10}. Consider K_{10} as $K_5 + K_5$. Decompose each K_5 subgraph into two Hamiltonian cycles; in each K_5 subgraph, color one cycle red and the other blue. Color the remaining edges (those between the K_5 sugraphs) green. This coloring contains no monochrome K_3. The red and blue subgraphs are each copies of C_5, which do

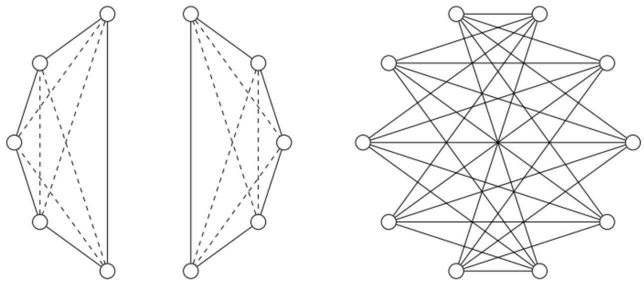

Figure 1. This coloring on K_{10} shows that $PR(K_3, C_3) \geq 11$.

not contain K_3. The green sugraph is $K_{5,5}$, which contains no odd cycles by virtue of being bipartite. Thus, the green subgraph contains no K_3

Furthermore, this coloring contains no proper C_3. Consider any 3-cycle in this graph. Either all of the vertices lie in one K_5 subgraph, or two lie in one K_5 and the third lies in the other. In the first case, all of the edges of the cycle are either red or blue, so it cannot be properly colored. In the other, two adjacent edges are green, so it cannot be properly colored. This coloring is illustrated in Figure 1.

We now show that $PR(K_3, C_3) \leq 11$. The proof is by contradiction. Let G be a 3-edge-colored copy of K_{11} containing no monochrome K_3 and no proper C_3. We show that this assumption leads to a contradiction. Let v be a vertex of maximal color degree, and X_1, X_2, and X_3 the subgraphs of $G - v$ so that each X_i is joined to v entirely edges of a single color, say X_1 by red edges, X_2 by blue, and X_3 by green. Since G contains no monochrome K_3, the edges within X_i must each be of a color different than that of the edges joining X_i to v. Since G contains no proper C_3, the edges between X_i and X_j must each be one of two colors, specifically those of the edges joining X_i and X_j to v. For instance, the edges between X_1 and X_2 must all be either red or blue, since coloring any such edge green yields a proper C_3.

We will suppose that $|X_1| \geq |X_2| \geq |X_3|$, which can always be accomplished by a relabeling of colors. Note that $|X_1| + |X_2| + |X_3| = |G - v| = 10$. It follows that the possible orders of these subgraphs correspond to the three-part integer partitions of 10. We consider these possibilities in order of decreasing $|X_1|$.

Case 1. *Suppose that $|X_1| \geq 6$. Then X_1 contains a 2-edge-colored K_6. It follows that G must contain a monochrome K_3, since $R(K_3, K_3) = 6$. But we assumed otherwise, so this is a contradiction.*

Case 2. *Suppose that $|X_1| = 5$. Then either $|X_2| \geq 3$ or $|X_3| \geq 3$. Without loss of generality, suppose that $|X_2| \geq 3$. Since X_1 is a 2-edge-colored copy of K_5 containing no monochrome K_3, it can only be colored in one way. By Lemma 2, X_1 has one blue Hamiltonian cycle and one green Hamiltonian cycle. Similarly, X_2 must contain a proper P_3, and so it has at least one red edge adjacent to a green edge. From the discussion above, all the edges between X_1 and X_2 are either red or blue.*

Let $V(X_1) = \{a, b, c, d, e\}$ so that (a, b, c, d, e, a) is the blue Hamiltonian cycle and (a, c, e, b, d, a) is the green Hamiltonian cycle, and let p, q, and r belong to $V(X_2)$ so that pq is red and qr green. Consider the set of edges from X_1 to p. If any three of them are blue, then a blue K_3 results, for in any set of three vertices in $V(X_2)$, two of them are adjacent through a blue edge. Thus, at least three of the edges from X_1 to p are red. For the same reason, at least three of the edges from X_1 to q are red. But this means that there exists at least one vertex of X_1 with red edges to both p and q, yielding a red K_3, so this is impossible.

Case 3. *Suppose that $|X_1| = 4$. There are two possibilities.*

Subcase 3.1. *Suppose $|X_2| = |X_3| = 3$. By Lemma 1, X_1 contains both a blue-green-blue and a green-blue-green P_4. We will first examine the subgraph $X_1 + X_2$. Let $V(X_1) = \{a, b, c, d\}$ so that ab and cd*

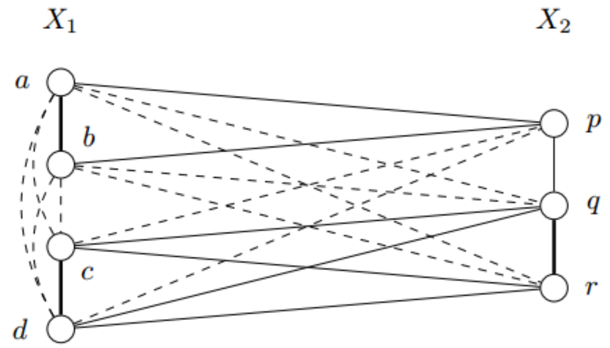

Figure 2. Coloring of $X_1 + X_2$ when $|X_1| = 4$, $|X_2| = |X_3| = 3$ and edge cq is red. Solid edges are red, dashed edges blue, and bold edges green. Note the number of edges of each color in X_1.

are green and bc is blue. Let $V(X_2) = \{p, q, r\}$ so that pq is red and qr is green. Recall that all the edges between X_1 and X_2 are either red or blue. Consider the set of edges $\{bq, cq\}$. At least one of these edges is red, otherwise $G[\{b, c, q\}]$ is a blue K_3. (Recall that $G[X]$ is the subgraph of G induced by the set of vertices X.) Without loss of generality, suppose that cq is red. Then we must have the following:

1. *cr must be red, otherwise (c, q, r, c) is a proper C_3.*
2. *dq must be red, otherwise (d, q, c, d) is a proper C_3.*
3. *dr must be red, otherwise (c, d, r, c) is a proper C_3.*
4. *cp must be blue, otherwise $G[\{c, p, q\}]$ is a red K_3.*
5. *dp must be blue, otherwise (c, d, p, c) is a proper C_3.*
6. *bp must be red, otherwise $G[\{b, c, p\}]$ is a blue K_3.*
7. *ap must be red, otherwise (a, b, p, a) is a proper C_3.*
8. *bq must be blue, otherwise $G[\{b, p, q\}]$ is a a red K_3.*
9. *aq must be blue, otherwise (a, b, q, a) is a proper C_3.*
10. *br must be blue, otherwise (b, q, r, b) is a proper C_3.*
11. *ar must be blue, otherwise (a, b, r, a) is a proper C_3.*

It then follows that the remaining edges of X_1 must all be blue to avoid proper 3-cycles in (a, d, p, a), (a, c, q, a), and (b, d, p, b). The resulting graph is shown in Figure 2. Thus, X_1 has four blue edges and two green edges.

Next consider the subgraph $X_1 + X_3$. Relabel the vertices of X_1 so that $V(X_1) = \{a, b, c, d\}$, ab and cd are blue, and bc is green. (We are guaranteed the existence of such a path by Lemma 1.) Let $V(X_3) = \{x, y, z\}$ so that xy is red and yz is blue. Note that the starting conditions here are isomorphic to those in the case of $X_1 + X_2$ through the permutation of the colors blue and green. It follows that $X_1 + X_3$ must be colored in the same way as $X_1 + X_2$ with the colors blue and green swapped. Hence, X_1 must have four green edges and two blue edges.

But then X_1 must have both four green edges and four blue edges, a contradiction since X_1 has only six edges.

Subcase 3.2. *Suppose that $|X_2| = 4$ and $|X_3| = 2$. We may assume that every vertex has a similar distribution of edges. That is, we may assume that the edges out of each vertex fall into color classes of size 4, 4, and 2, since otherwise we could apply a previous case by making a different choice of v.*

Let $V(X_1) = \{a, b, c, d\}$ so that ab and cd are green and bc blue. Let $V(X_2) = \{p, q, r, s\}$ so that pq and rs are green and qr red. We are guaranteed the existence of these paths by Lemma 1. Lastly, let

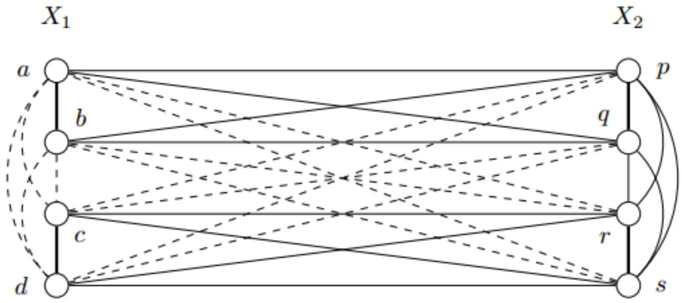

Figure 3. Coloring of $X_1 + X_2$ when $|X_1| = |X_2| = 4$ and edge bq is red. Solid edges are red, dashed edges blue, and bold edges green.

$V(X_3) = \{x, y\}$. *As in the previous subcase, we will begin with one edge and show that the choice of color for that edge fully determines the rest of the coloring.*

Consider the set of edges $\{bq, br, cq, cr\}$. *At least one of these edges must be red, since otherwise a blue* K_3 *results. This scenario is entirely symmetrical with respect to these edges, so we are free to suppose that the edge bq is red. We must then have the following:*

1. *br must be blue, otherwise $G[\{b, q, r\}]$ is a red K_3.*
2. *bp must be red, otherwise (b, p, q, b) is a proper C_3.*
3. *aq must be red, otherwise (a, b, q, a) is a proper C_3.*
4. *ap must be red, otherwise (a, b, p, a) is a proper C_3.*
5. *ar must be blue, otherwise $G[\{a, q, r\}]$ is a red K_3.*
6. *cr must be red, otherwise $G[\{b, c, r\}]$ is a blue K_3.*
7. *cq must be blue, otherwise $G[\{c, q, r\}]$ is a red K_3.*
8. *dr must be red, otherwise (c, d, r, c) is a proper C_3.*
9. *cs must be red, otherwise (c, s, r, c) is a proper C_3.*
10. *ds must be red, otherwise (c, d, s, c) is a proper C_3.*
11. *bs must be blue, otherwise (b, r, s, b) is a proper C_3.*
12. *cp must be blue, otherwise (c, p, q, c) is a proper C_3.*
13. *dp must be blue, otherwise (c, d, p, c) is a proper C_3.*
14. *dq must be blue, otherwise (c, d, q, c) is a proper C_3.*
15. *as must be blue, otherwise (a, b, s, a) is a proper C_3.*

It then follows that the remaining edges within X_1 *are all blue to avoid proper* 3*-cycles in* (a, d, q, a), (a, c, p, a), *and* (b, d, s, b). *Similarly, the remaining edges within* X_2 *are all red to avoid proper* 3*-cycles in* (p, a, s, p), (p, c, r, p), *and* (q, d, s, q). *Thus,* $X_1 + X_2$ *must be colored as shown in Figure 3 in order to avoid both monochrome* K_3 *and proper* C_3.

Since the remaining edges out of X_1 *can only be red or green, each vertex of* X_1 *must have one red edge and one green edge into* X_3, *otherwise the color classes would not have the correct number of edges. Similarly, each vertex of* X_2 *must have one blue edge and one green edge into* X_3. *(Recall, we may assume that every vertex has two edges of one color, and four each of the two remaining colors.) Specifically, exactly one of the edges in* $\{dx, dy\}$ *must be red, and the other green. Similarly, exactly one of the edges in* $\{sx, sy\}$ *must be blue and the other green.*

Finally, consider the edge xy in X_3. *This edge cannot be blue, for then* (d, x, y, d) *would be a proper* C_3. *Neither can it be red, for then* (s, x, y, s) *would be a proper* C_3. *This is a contradiction.*

We have shown that $|X_1|$ cannot take on any value between 4 and 10. But since it must take on one of these values, we have arrived at a contradiction. Therefore $PR(K_3, C_3) \leq 11$.

We conclude that $PR(K_3, C_3) = 11$. □

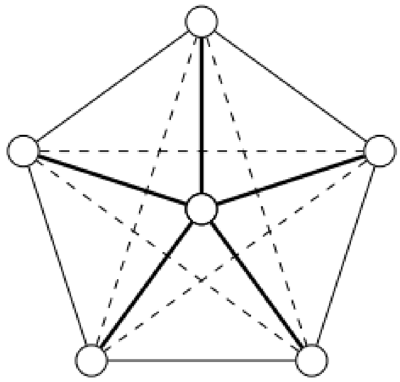

Figure 4. This coloring on K_6 shows that $PR(K_3, C_5) \geq 7$.

We now move on to the result for C_5.

Theorem 2. $PR(K_3, C_5) = 7$.

Proof. We first exhibit a critical coloring on K_6 that contains neither a monochrome K_3 nor a properly colored C_5. To construct this coloring, consider K_6 as $K_5 + K_1$. Decompose the K_5 subgraph into two Hamiltonian cycles, and color one cycle red and the other blue. Color the remaining edges (those between the K_5 and K_1 subgraphs) green. This coloring clearly contains no monochrome K_3: the red and blue subgraphs are each copies of C_5, while the green subgraph is the star $K_{1,5}$. On the other hand, any 5-cycle in the graph either contains the K_1 vertex or it does not. If it does, then it must contain adjacent green edges. If it does not, then it contains only red and blue edges. But C_5 cannot be properly colored with only two colors, so the graph contains no proper C_5. Thus $PR(K_3, C_5) \geq 7$. We next show that $PR(K_3, C_5) \leq 7$. Let G be a 3-edge-colored copy of K_7 containing neither a monochrome K_3 nor a properly colored C_5. We show that this assumption produces a contradiction. Let v be a vertex of G with maximal color degree, and let X_1, X_2, and X_3 be the subgraphs of $G - v$ so that each X_i is joined to v entirely by edges of the same color. We will assume that $|X_1| \geq |X_2| \geq |X_3|$ and that X_1, X_2, and X_3 are joined to v by red, blue, and green edges, respectively. These assumumptions can can always be satisfied by a relabeling of colors. Since $|X_1| + |X_2| + |X_3| = |G - v| = 6$, the possible orders of these subgraphs must correspond to the three-part integer partitions of 6. We consider the possible cases in order of decreasing $|X_1|$.

Case 1. *Suppose that $|X_1| = 6$. Then X_1 contains a 2-edge-colored K_6. It follows that G must contain a monochrome K_3, since $R(K_3, K_3) = 6$. This is a contradiction.*

Case 2. *Suppose that $|X_1| = 5$. Then, without loss of generality, $|X_2| = 1$. By Lemma 2 the subgraph X_1 must be decomposed into Hamiltonian cycles of different colors (blue and green) to avoid a monochrome K_3. Let $V(X_1) = \{a, b, c, d, e\}$ so that the blue cycle is (a, b, c, d, e, a) and the green cycle (a, c, e, b, d, a). Let $V(X_2) = \{p\}$.*

Note that for each vertex in X_1, there is a proper P_4 in $G - p$ beginning at v and ending on that vertex, whose final edge is blue. In alphabetical order of final vertex, (v, c, e, a), (v, d, a, b), (v, a, d, c), (v, c, e, d), and (v, a, d, e) are proper such paths.

Then the edge ap must be blue to avoid a proper C_5; choosing red or green yields a proper C_5 in (v, c, e, a, p, v). But the same consideration applies to every vertex of X_1, so every edge out of X_1 into X_2 must be blue. But this yields a monochrome K_3, such as $G[\{a, b, p\}]$, so this is impossible. This case is illustrated in Figure 5.

Case 3. *Suppose that $|X_1| = 4$. Then there are two possibilities.*

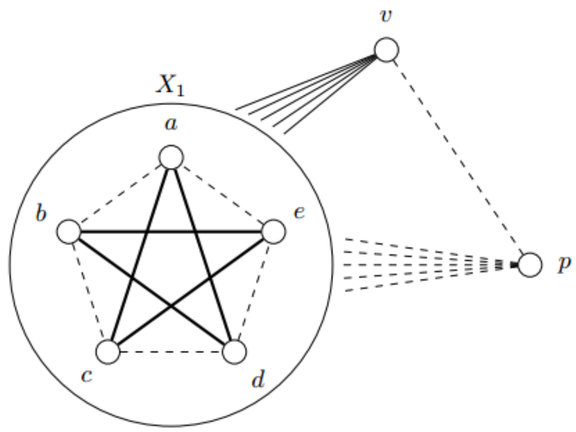

Figure 5. The coloring that results if $|X_1| = 5$ in the proof of Theorem 2. This clearly contains a monchrome K_3, contrary to our assumption.

Subcase 3.1. *Suppose that $|X_2| = 2$. By Lemma 1, X_1 contains a proper P_4 that begins and ends with blue edges. Let $V(X_1) = \{a, b, c, d\}$, so that (a, b, c, d) is the proper P_4, and $V(X_2) = \{p, q\}$. Note that pq is not blue, for then $G[\{p, q, v\}]$ is a blue K_3.*

Consider the edges in $\{ap, aq, dp, dq\}$. If any one of these edges is not blue, then a proper C_5 results. Note that these are symmetrically situated, so we may consider just one. Suppose that dp were not blue. Then (v, b, c, d, p, v) would be a proper C_5. So indeed, each of the four edges indicated must be blue. But then ad must be green to avoid a blue K_3 in $G[\{a, d, q\}]$. But now a proper C_5 cannot be avoided, since (v, a, d, p, q, v) is a proper C_5. So this subcase is impossible.

Subcase 3.2. *Suppose that $|X_2| = |X_3| = 1$. Again, by Lemma 1, X_1 contains a proper P_4 that begins and ends with blue edges. Let $V(X_1) = \{a, b, c, d\}$, so that (a, b, c, d) is the proper P_4, and let $V(X_2) = \{p\}$ and $V(X_3) = \{x\}$. Consider the edges bx and cx. If either of these edges is not green, then a proper C_5 results, namely either of (v, a, b, c, x, v) or (v, d, c, b, x, v). But they also cannot both be green, since then $G[\{b, c, x\}]$ is a green K_3. So this subcase is also impossible.*

Case 4. *Suppose that $|X_1| = 3$. There are two possibilites. Either $|X_2| = 3$ or $|X_2| = 2$ and $|X_3| = 1$.*

Subcase 4.1. *Suppose $|X_2| = 3$. Let $V(X_1) = \{a, b, c\}$ and $V(X_2) = \{p, q, r\}$. Note that X_1 and X_2 are 2-edge-colored copies of K_3 not containing a monochrome K_3, so each contains a proper P_3. We may assume that ab is green and bc blue. Then cr must be blue, otherwise (v, a, b, c, r, v) is a proper C_5. But note that (r, q, p) is a proper P_3 with no blue edges. It follows that (v, c, r, q, p, v) is a proper C_5. So cr cannot be blue. Hence, cr must be both blue and not blue, a contradiction.*

Subcase 4.2. *Suppose that $|X_2| = 2$ and $|X_3| = 1$. Let $V(X_2) = \{p, q\}$ and $V(X_3) = \{x\}$. Note that X_1 must contain a proper P_3 since it does not contain a monochrome K_3. We may assume that ab is blue and bc green. But then,*

1. *cx must be green, otherwise (v, a, b, c, x, v) is a proper C_5.*
2. *bx is red or blue, otherwise $G[\{b, c, x\}]$ would be a green K_3.*
3. *ap must be blue, otherwise (v, c, b, a, p, v) is a proper C_5.*
4. *bp is red or green, otherwise $G[\{a, b, p\}]$ is a blue K_3.*

Now consider ac. This edge cannot be blue, for then (v, a, c, b, x, v) is a proper C_5. But neither can this edge be green, for then (v, c, a, b, p, v) is a proper C_5. Since this edge must be either blue or green, this is a contradiction.

Case 5. *Suppose that* $|X_1| = |X_2| = |X_3| = 2$*. Let* $V(X_1) = \{a, b\}$*,* $V(X_2) = \{p, q\}$*, and* $V(X_3) = \{x, y\}$*. Since we assumed that* v *has maximal color degree, it follows that every vertex of* G *has the same number of edges of each color, namely 2. There are two subcases.*

Subcase 5.1. *Suppose that two of the edges in* $\{ab, pq, xy\}$ *share a color. Without loss of generality, suppose that* ab *and* xy *are both blue. In that case,* ax *must be blue, since otherwise* (v, b, a, x, y, v) *is a proper* C_5*. Since* ay*,* bx*, and* by *are symmetrically situated, they must all be blue as well. But then* $G[\{a, b, x\}]$ *is a blue* K_3*, a contradiction.*

Subcase 5.2. *Suppose that each of* ab*,* pq*, and* xy *are different colors. Without loss of generality, suppose* ab *is blue. Then* pq *must be green and* xy *must be red. Note that each of vertices* a *and* b *must be incident to two green edges, as we established at the beginning of this case. The vertices of* X_2 *and* X_3 *each have one green edge already. Since every vertex must be incident to two green edges, it follows that there is a green edge out of* X_1 *to each vertex in* X_2 *and* X_3*. But this always produces a proper* C_5*. Since this configuration is symmetric, it suffices to check one case. Indeed, if* ax *is green, then* (v, b, a, x, y, v) *is a proper* C_5*. This is a contradiction.*

We have shown that $|X_1|$ cannot take on any of its possible values, and thus arrived at a contradiction. It follows that a 3-edge-colored K_7 must contain either a monochrome K_3 or a properly colored C_5, so $PR(K_3, C_5) \leq 7$. Since we already showed that $PR(K_3, C_5) \geq 7$, we conclude that $PR(K_3, C_5) = 7$. □

4. Concluding Remarks

These two theorems further the determination of the proper Ramsey number of K_3 against C_n. We add the small odd cycles C_3 and C_5 to the even cycles treated by Olejniczak in [2]. There are only five remaining cases: C_7, C_9, C_{11}, C_{13}, and C_{15}. Since the three color Ramsey number $R(K_3, K_3, K_3) = 17$ (see [6]), any 3-edge-colored copy of K_{17} necessarily contains a monochrome K_3, so any 3-edge-colored graph large enough to contain a proper C_{17} necessarily contains a monochrome K_3 already. We may therefore stop at C_{15}. Thus, finding these last five values would complete the determination of $PR(K_3, C_n)$.

5. Conflict of Interest

There was no conflict of interest in this study.

References

1. English, S., Johnston, D., Olejniczak, D. and Zhang, P., 2017. Proper Ramsey numbers of graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing, 101*, pp.281-299.
2. Olejniczak, D., 2019. *Variations in ramsey theory.* Ph.D. Thesis. Western Michigan University, Kalamazoo, MI.
3. Eroh, L.L., 2000. *Rainbow ramsey numbers*. Ph.D. Thesis. Western Michigan University, Kalamazoo, MI
4. Chartrand, G. and Zhang, P., 2021. New directions in Ramsey theory. *Discrete Mathematics Letters, 6*, pp.84-96.
5. Chartrand, G. and Zhang, P., 2012. *A first course in graph theory*. Dover Publications, Garden City, NY
6. Radziszowski, S., 2011. Small ramsey numbers. *The Electronic Journal of Combinatorics, 1000*, pp.DS1-Aug.