On the set edge-reconstruction conjecture

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ABSTRACT. We prove that the set edge-reconstruction conjecture is true for graphs with at most two graphs in the set of edge-deleted subgraphs.

The conjecture that every graph with at least three vertices can be reconstructed from its deck (i.e., multiset) of vertex-deleted subgraphs is due to Kelly and Ulam (see [4] and [7]). There are now several variations on this theme, and the following version should probably be attributed to Harary [2]:

The set edge-reconstruction conjecture. *Every graph with at least four edges can be reconstructed from its set of edge-deleted subgraphs.*

This is equivalent to the conjecture that all line graphs are set (vertex-) reconstructible, see Hemminger [3] and Manvel [6].

What distinguishes set reconstruction from ordinary reconstruction is that you are only given *one* graph from each isomorphism class represented in the collection of edge-deleted (or vertex-deleted) subgraphs. Thus you are given a *set*, not a multi-set, of graphs, and you do not know how many

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times each of the graphs in the set occurs as an edge-deleted (vertex-deleted) subgraph.

The purpose of this note is to show that the set edge-reconstruction conjecture holds for a graph G, if the set contains at most two graphs, i.e., if G has at most two non-isomorphic edge-deleted subgraphs. Using some results by Manvel [6], this comes out easily as a by-product of recent work the authors have done with G. Sabidussi [1].

Theorem 1. *If a graph G with at least four edges has at most two non-isomorphic edge-deleted subgraphs, then G is set edge-reconstructible.*

Proof: Let G be as stated. Manvel [6] observed that graphs with at least two non-trivial components are set edge-reconstructible, so we may assume that G has only one non-trivial component (*trivial* components are isolated vertices). Furthermore, the degree sequence of G can be derived from the set of edge-deleted subgraphs ([6], although the proof there has a flaw; we give a proof repairing the flaw after this proof); so the number of isolated vertices of G can be found, and as many isolated vertices deleted from each graph of the set. This leaves us with the task of reconstructing a connected graph G' whose degree sequence we know. Manvel [5] proved that trees are set edge-reconstructible, so we assume that G' is not a tree.

Suppose first that all edge-deleted subgraphs of G' are isomorphic; then all edges of G' have the same pair of degrees at their end-vertices. Hence G' is either regular, in which case it is clearly set edge-reconstructible, or it is biregular and bipartite with each edge joining, say, a vertex of degree α to a vertex of degree β, where we assume $\alpha < \beta$. In the last case, let e be any edge of G'. Then $G' - e$ has a unique vertex u of degree $\alpha - 1$. Since G' is not a tree, $G' - e$ is connected, and G' is obtained by adding an edge joining u to the unique vertex of degree $\beta - 1$ at odd distance from u (possibly $\beta - 1 = \alpha$). So G' is set edge-reconstructible in this case.

Suppose next that G' has exactly two isomorphism classes of edge-deleted subgraphs, say represented by graphs G_1 and G_2. By Lemmas 1–3 of [1], G' is of one of the following types:

1. G' is regular.

2. G' is biregular with distinct degrees α and β, and letting

$$A = \{v \in V(G') \mid d_{G'}(v) = \alpha\}, \quad B = \{v \in V(G') \mid d_{G'}(v) = \beta\},$$

each edge is either an (A, B)-edge or a (B, B)-edge. Further, there exists an integer δ, $0 < \delta \leq \beta$, so that each vertex of B has either 0 or δ neighbours in A.

Let B_0 be the set of vertices of B with 0 A-neighbours, B_δ the set of B-vertices with δ A-neighbours. If $B_0 \neq \emptyset$ then all (B, B)-edges are (B_0, B_δ)-edges, and so G' is bipartite with bipartition $(A \cup B_0, B_\delta)$.

Note also that if $\delta = \beta$ then $B_0 = \emptyset$ and G' is bipartite with bipartition (A, B).

3. G' has exactly three distinct degrees α, β and γ, and letting

$$A = \{v \in V(G') \mid d_{G'}(v) = \alpha\}, \quad B = \{v \in V(G') \mid d_{G'}(v) = \beta\},$$
$$C = \{v \in V(G') \mid d_{G'}(v) = \gamma\},$$

each edge is either an (A, B)-edge or a (B, C)-edge. Further, there exists an integer δ, $0 < \delta < \beta$, such that each vertex in B has δ neighbours in A and $\beta - \delta$ neighbours in C.

Note that in this case G' is bipartite with bipartition $(A \cup C, B)$.

We now consider these three cases one by one.

1. G' is regular.

Then G' is obviously reconstructible from any one of G_1 and G_2.

2. G' is biregular.

We can tell from a comparison of the degree sequences of G', G_1 and G_2 whether or not G' has (B, B)-edges. If not, that is if $\delta = \beta$, then since at least one of G_1 and G_2 is connected (as G' is not a tree), we may assume that G_1 is, and we can reconstruct G' from G_1 by joining the unique vertex u of minimum degree, say $\alpha - 1$, to the sole vertex of degree $\beta - 1$ at odd distance from u.

So assume that G' has (B, B)-edges. We may assume that G_1 is obtained by deleting an (A, B)-edge, G_2 by deleting a (B, B)-edge. If $\beta - 1 \neq \alpha$ then G' is clearly reconstructible from G_2, so we assume $\alpha = \beta - 1$. If $\alpha = 1$ then G' is a path with 4 edges, so we also assume $\alpha \geq 2$. Now consider $G_1 = G' - e$. One end-vertex of e must be the unique vertex u of degree $\alpha - 1$ in G_1. Let the other be x. G' is not an odd cycle, so it contains an odd cycle if and only if G_1 or G_2 does. So we can tell whether G' is bipartite. If it is, then G_1 is clearly connected (if all (A, B)-edges of G' were bridges, then, as $\alpha > 1$, any end block of G' would be contained in the graph spanned by B, contradicting that all vertices of B_δ would be cut-vertices of G'). Thus, if G' is bipartite we can identify x as the sole vertex of degree α of odd distance from u in G_1, and G' can be reconstructed from G_1 by adding an edge joining u to x.

So we may assume that G' is not bipartite; in particular, $B_0 = \emptyset$. Let F be the subgraph of G_1 induced by the vertices of degree α; all edges of F must be incident with x. If some vertex v has degree at least 2 in F, then $x = v$ and G' is obtained from G_1 by adding an edge joining u to v. Otherwise, F has at most one edge.

If F has exactly one edge, say with end-vertices v_1 and v_2, then $\delta = 2$, and x is the vertex of $\{v_1, v_2\}$ belonging to B; let the other be y. If v_1 and v_2

have exactly the same neighbour set in $G_1 - v_1v_2$, then the graphs obtained by joining u to one of v_1 and v_2 are clearly isomorphic and isomorphic to G'. So assume that v_1 and v_2 have different neighbour sets in $G_1 - v_1v_2$. Then each neighbour of x in $G_1 - v_1v_2$ has exactly three neighbours of degree at most α, and y cannot have this property under the assumption of different neighbour sets, so we can indeed distinguish x, and obtain G' by adding the edge xu to G_1.

Suppose finally that F has no edges. Then $\delta = 1$, x has the property that each of its neighbours in G_1 has exactly two neighbours of degree at most α, and any other vertex of degree α with this property must have the same neighbour set as x; hence x can be identified up to isomorphism, and G' is reconstructible.

3. G' has exactly three distinct degrees.

Let G_1 be obtained from G' by deleting an (A, B)-edge, G_2 by deleting a (B, C)-edge. As G' is not a tree, we may assume that G_1 is connected. Let $G_1 = G' - e$, where e has end-vertices a of degree α in G' and b of degree β in G'.

Case (i). $\alpha - 1 \notin \{\beta, \gamma\}$. Then a is the unique vertex of degree $\alpha - 1$ in G_1, and b is the sole vertex of degree $\beta - 1$ in G_1 at odd distance from a. So G' can be reconstructed from G_1 by joining a and b.

Case (ii). $\alpha - 1 = \beta$. Then $\alpha \geq 3$, and so b can be identified as the sole vertex of degree $\beta - 1$ at even distance from at least two vertices of degree β. And a is the unique vertex of degree β at odd distance from b. Again, G' can be reconstructed by joining these uniquely determined vertices.

Case (iii). $\alpha - 1 = \gamma$. Here b is the sole vertex of degree $\beta - 1$ at even distance from the vertices of degree β, and a has the property that each of its neighbours in G_1 has degree β and has exactly $\beta - \delta + 1$ neighbours of degree γ. If any other vertex in G_1 of degree γ has the same property, then it must have the same neighbour set as a. Therefore the same graph, up to isomorphism, is obtained no matter which vertex with the property is joined to b, and this graph is a reconstruction of G'.

In all cases, we have shown how to reconstruct G', and so G' is set edge-reconstructible. This completes the proof of the theorem.

Manvel's result on the degree sequence

We finally outline a new proof of the result of Manvel that the degree sequence of a graph is set edge-reconstructible. The proof in [6] fails where it claims that exactly one graph in the set of edge-deleted subgraphs of a graph G has maximum degree less than $\Delta(G)$ (the maximum degree of G) if and only if G has exactly two vertices of maximum degree, and these are neighbours; a wheel with at least 5 vertices is a counterexample to this. It contains a similar flaw where it states that if every edge-deleted subgraph of G has either n or $n+1$ vertices of degree $\Delta(G)$, and if no edge of G joins two vertices of degree $\Delta(G)$, then there will be at least two such graphs with n vertices of degree $\Delta(G)$; a graph obtained from $K_{2n+2,n+1}$ by adding $n+1$ independent edges joining vertices of the class with $2n+2$ vertices is a counterexample.

The proof below contains basically no new ideas; we have chosen to refer to Manvel's proof for several details.

Theorem 2. *The degree sequence of a graph with at least 4 edges is set edge-reconstructible.*

Proof: Let G be the graph, and let S be its set of edge-deleted subgraphs. Following [6], we note that we can assume that G is not a star and some isolated vertices, and so we can find $\Delta(G)$ from S; further, we call a graph in S *deficient* if its maximum degree is less than $\Delta(G)$, and we partition the proof into three cases according to whether S contains none, one, or more deficient graphs.

S contains no deficient graphs. Let $n \geq 1$ be the minimum number of vertices of degree $\Delta(G)$ in a graph in S. If some graph in S has $n+2$ vertices of degree $\Delta(G)$, or if all graphs in S have exactly n vertices of degree $\Delta(G)$, we finish as in [6]. So suppose that all graphs in S have n or $n+1$ vertices of degree $\Delta(G)$, with $n+1$ occurring for some graph in S. As in [6], we check from S whether G has an edge joining two vertices of degree $\Delta(G)$, and we finish if this is the case. So assume now that G has no edge joining two vertices of degree $\Delta(G)$. If there are two graphs in S with n vertices of degree $\Delta(G)$ with distinct degree sequences, then writing these in nondecreasing order one above the other, and taking the larger term in each position, and then replacing one $\Delta(G)-1$ by $\Delta(G)$ gives the degree sequence of G. If S does not contain two such graphs (as is the case if S contains only one graph with n vertices of degree $\Delta(G)$), then all vertices adjacent to a vertex of maximum degree in G have the same degree in G, and we can find this degree d as the highest degree of a vertex adjacent to a vertex of degree $\Delta(G)$ in a graph of S with $n+1$ vertices of degree $\Delta(G)$; we then get the degree sequence of G from the degree sequence of a graph in S with n vertices of degree $\Delta(G)$ by replacing one $\Delta(G)-1$ by $\Delta(G)$ and one $d-1$ by d.

S contains exactly one deficient graph G_d. Then G has a unique vertex of maximum degree, or it has exactly two vertices of maximum degree, these being adjacent. If S contains a graph with two vertices of degree $\Delta(G)$, then clearly G has two such vertices, and the degree sequence of G is obtained from that of G_d by replacing two degrees

$\Delta(G) - 1$ by $\Delta(G)$. So now suppose that all graphs in S different from G_d have exactly one vertex of degree $\Delta(G)$. Then, if G has two vertices of degree $\Delta(G)$, each edge must be incident with at least one of them; so any other vertex can have degree at most 2. It follows that if S contains a graph with more than two vertices of degree at least 3, then G has a unique vertex v of maximum degree, and as there is just one deficient graph all neighbours of v must have the same degree. This degree c can be found as the highest degree of a neighbour to v in a non-deficient graph from S. Then the degree sequence of G is obtained from the degree sequence of G_d by replacing one $\Delta(G) - 1$ by $\Delta(G)$ and one $c - 1$ by c.

So we can finally assume that each graph in S has at most two vertices of degree at least 3. If $\Delta(G) \geq 4$, we look at G_d: if it has just one vertex of degree $\Delta(G) - 1$, then G has a unique vertex of maximum degree, and the degree c of its neighbours can be found as above, as can the degree sequence of G. We therefore investigate the case where G_d has exactly two vertices a and b of degree $\Delta(G) - 1$. We claim that we can easily tell from S whether G has one or two vertices of maximum degree, because the following holds:

If both a and b have degree $\Delta(G)$ in G, then each non-deficient graph of S has an edge joining a vertex of degree $\Delta(G)$ to a vertex of degree $\Delta(G) - 1$.
If only one of a and b has degree $\Delta(G)$ in G then no graph of S has such an edge.

For if a and b both have degree $\Delta(G)$ in G, then $G_d = G - ab$, and as each edge is incident with a or b, the first statement above follows. And suppose that, say, a has degree $\Delta(G)$ and b has degree $\Delta - 1$ in G. As S contains only one deficient graph, if a and b are neighbours in some graph of S and therefore in G, all neighbours of a in G would have degree $\Delta(G) - 1 \geq 3$, contradicting that no graph in S has more than two vertices of degree at least 3.

Whichever of the situations above holds, we easily find the degree sequence of G.

So we can now assume that $\Delta(G) \leq 3$. If $\Delta(G) = 3$, then for G to have two vertices of maximum degree precisely 5 edges are required, so if G does not have exactly 5 edges we know that it has a unique vertex of maximum degree and can find the degree sequence as above; if G has 5 edges, the three possibilities for G (not counting variations in the number of isolated vertices) with two vertices of maximum degree can be checked directly and are set edge-reconstructible — in any other case G has a unique vertex of maximum degree, and the degree sequence can be found in the usual way. If $\Delta(G) = 2$, then since G has at least four edges it cannot have two vertices of degree 2 (they would both have degree 2 in some non-deficient graph as well), and so the degree sequence consists of one 2, some 1s, and possibly some 0s.

S **contains more than one deficient graph.** Then G has a unique vertex of maximum degree, and the reconstruction of the degree sequence is done as in [6].

This completes the proof of Theorem 2.

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