Construction of Orthogonal Group Divisible Designs

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ABSTRACT. In this paper we introduce some concepts relating to idempotent ordered orthogonal quasigroup (IOOQ), ordered orthogonal Steiner triple systems (ordered OSTS) and ordered orthogonal group divisible designs (ordered OGDD), and use them to obtain some construction methods for OGDD.

1 Introduction

A *Steiner triple system* of order n, (or STS(n)), can be defined to be an $(n,3)$-PBD. The necessary and sufficient condition for the existence of an STS(n) is that $n \equiv 1$ or $3 \pmod 6$. Two STS(n) on the same point set, say $(X, \mathbf{A})$ and $(X, \mathbf{B})$, are said to be *orthogonal* provided the following properties are satisfied:

a) $\mathbf{A} \cap \mathbf{B} = \phi$

b) if $\{u, v, w\}$ and $\{x, y, w\} \in \mathbf{A}$ and $\{u, v, s\}$ and $\{x, y, t\} \in \mathbf{B}$, then $s \neq t$.

Orthogonal STS(n) will be denoted by OSTS(n).

Let $(X, \mathbf{G}, \mathbf{A})$ and $(X, \mathbf{G}, \mathbf{B})$ be two 3-GDDs having the same groups. We say that they are *orthogonal* if the following properties are satisfied:

a') if $\{u, v, s\} \in \mathbf{A}$ and $\{u, v, t\} \in \mathbf{B}$, then s and t belong to different groups.

b') if $\{u, v, w\}$ and $\{x, y, w\} \in \mathbf{A}$, and $\{u, v, s\}$ and $\{x, y, t\} \in \mathbf{B}$, then $s \neq t$.

We shall use the abbreviation OGDD to denote orthogonal 3-GDDs. It is easy to see that OSTS(n) are equivalent to OGDD of type 1^n, since condition a') implies that $\mathbf{A} \cap \mathbf{B} = \phi$.

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A *quasigroup* of order v is a pair $(Q, \oplus)$, where Q is a set of cardinality v, and $\oplus\colon Q \times Q \rightarrow Q$ is a binary operation such that $q \oplus r = q \oplus s$ if and only if $r = s$, and $r \oplus q = s \oplus q$ if and only if $r = s$. The quasigroup $(Q, \oplus)$ is said to be *idempotent* if $q \oplus q = q$ for all $q \in Q$. Two quasigroups of order v, $(Q, \oplus)$ and $(Q, \otimes)$, are said to be *orthogonal* if, for every ordered pair $(s, t) \in Q \times Q$, there is a unique ordered pair (q, r) such that $q \oplus r = s$ and $q \otimes r = t$.

The problem of existence of orthogonal Steiner triple systems bas been given quite a lot of attention by various authors since it was posed by O'Shaughnessy in 1968. The problem was completely settled by the joint work of C.J. Colbourn, P.B. Gibbons, R. Mathon, R.C. Mullin and A. Rosa (see [6]). That is, if $v \equiv 1, 3 \pmod 6$, $v \geq 7$ and $v \neq 9$ a pair of orthogonal Stenier triple systems of order v exist.

In this paper, we introduce some concepts relating to idempotent ordered orthogonal quasigroups (IOOQ), ordered orthogonal Steiner triple systems (ordered OSTS(and ordered orthogonal group divisible designs (ordered OGDD), and use them to obtain some construction methods for OGDD. To some extent, the techniques improve the construction methods in [5].

2 Idempotent ordered orthogonal quasigroups

Let $(Q, \oplus)$ be any quasigroup. We define on the set Q, six binary operations, $\oplus_{123}$, $\oplus_{231}$, $\oplus_{312}$, $\oplus_{132}$, $\oplus_{321}$ and $\oplus_{213}$, as follows: $q \oplus r = s$ if and only if

$$q \oplus_{123} r = s, \quad q \oplus_{132} s = r, \quad r \oplus_{213} q = s,$$
$$r \oplus_{231} s = q, \quad s \oplus_{312} q = r, \quad s \oplus_{321} r = q.$$

Two idempotent quasigroups of order v $(Q, \oplus)$ and $(Q, \otimes)$ are defined to be *ordered orthogonal quasigroups* (briefly IOOQ(v)) if $\oplus_{ijk}$ is orthogonal to $\otimes_{jki}$ and $\otimes_{kij}$ for $(ijk) = (123), (231), (312)$. If each operation of $(Q, \oplus)$ is orthogonal to that of $(Q, \otimes)$, we say they are conjugate orthogonal (briefly ICOQ(v)) (see [5]).

Theorem 2.1. *There exists IOOQ(v) for* $v = 4, 5, 8$.

Proof: For each case of $v = 4, 5, 8$, we let $Q = GF(v)$ and θ be a primitive root of $GF(v)$ and define $(Q, \oplus)$ and $(Q, \otimes)$. It is easy to see that $(Q, \oplus)$ and $(Q, \otimes)$ are IOOQ(v).

For $v = 4$, define $q \oplus r = q + (q + r)\theta$ and $q \otimes r = r + (q + r)\theta$ for $q, r \in GF(4)$ where $\theta^2 = \theta + 1$. For $v = 5$, define $q \oplus r = 2r - q$ and $q \otimes r = 2r - q$ for $q, r \in GF(5)$. For $v = 8$, define $q \oplus r = q + (q + r)\theta$ and $q \otimes r = r + (q + r)\theta$ for $q, r \in GF(8)$ where $\theta^3 = \theta + 1$.

Let

$$\text{ICOQ} = \{v\colon \text{ there exist ICOQ(v)}\}$$
$$\text{IOOQ} = \{v\colon \text{ there exist IOOQ(v)}\}$$

Theorem 2.2. *If v is a prime power, $v \neq 2, 3, 4, 5$ or 8, then there exist ICOQ(v) (see [5]).*

It ie easy to see that ICOQ $\subset$ IOOQ and it is not difficult to see that the set IOOQ is PBD-closed. This proof is similar to that of ICOQ (see [5]). From Theorem 2.1 and Theorem 2.2, we have

Theorem 2.3. *There exist IOOQ(v) for $v \in B(P_4)$ where $P_4 = \{v\colon v \geq 4,$ v is a prime power$\}$.*

3 Ordered orthogonal Steiner triple systems

Let $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ be two collections of ordered subsets of X with size 3, $\mathbf{A} = \{\{a,b,c\}\colon (a,b,c) \in \tilde{\mathbf{A}}\}$ and $\mathbf{B} = \{\{a,b,c\}\colon (a,b,c) \in \tilde{\mathbf{B}}\}$. For $\tilde{A} = (a,b,c)$ we say that the three ordered pairs of points $(a,b), (b,c), (c,a)$ are in $\tilde{A}$ and the seat of (a,b) is α, the seat of (b,c) is β and the seat of (c,a) is γ.

We say $(X, \tilde{\mathbf{A}})$ and $(X, \tilde{\mathbf{B}})$ are ordered-OSTS(v) provided that $(X, \mathbf{A})$ and $(X, \mathbf{B})$ are OSTS(v) and that the following property is satisfied:

c) If (a,b) is in a block of $\tilde{\mathbf{A}}$ then (a,b) is also in a block of $\tilde{\mathbf{B}}$ and two seats of (a,b) are different.

Similarly, we can define *ordered-OGDD.*

In this section we will show the OSTS(v) which was established bẏ Mullin and Nemeth [2] can be arranged into an ordered-OSTS(v) For $v \in P_{1,6}$ where $P_{1,6} = \{v\colon v$ is a prime power and $v \equiv 1 \pmod 6\}$.

Lemma 3.1. *If $6k+1$ is a prime power then $1 + \theta^{2k} + \theta^{4k} = 0$ where θ is a primitive root.*

Proof: Since $1 = \theta^{6k} = (1 + \theta^{2k} + \theta^{4k})(1 - \theta^{2k}) = 0$. Let $v = 6k+1$ be a prime power and $X = GF(v)$. Let $x = \theta^{2k}$, $A(i) = \{0, 1, 2 + x^i\}$, $B(i) = \{0, x^i, 1 + x^i\}$ where $i = 1$ or 2. $\mathbf{A}_j(i) = \{\theta^j A(i) + g\colon g \in X\}$ and $\mathbf{B}_j(i) = \{\theta^j B(i) + g\colon g \in X\}$ where $0 \leq j \leq k-1$.

From Lemma 3.1 we have

Lemma 3.2. *$(X, \mathbf{A}(i))$ and $(X, \mathbf{B}(i))$ are OSTS(v) for $i = 1, 2$ where $\mathbf{A}(i) = \cup_{0 \leq j \leq k-1} \mathbf{A}_j(i)$ and $\mathbf{B}(i) = \cup_{o \leq j \leq k-1} \mathbf{B}_j(i)$.*

Now, we want to arrange $\mathbf{A}(i)$ into $\tilde{\mathbf{A}}(i)$ and $\mathbf{B}(i)$ into $\tilde{\mathbf{B}}(i)$ such that $\tilde{\mathbf{A}}(i)$ and $\tilde{\mathbf{B}}(i)$ satisfy the condition c). It is not difficult to see that if we can arrange $\mathbf{A}_j(i)$ and $\mathbf{B}_j(i)$ with $j = 0$ then we also can arrange $\mathbf{A}(i)$ and $\mathbf{B}(i)$. Let

$$\begin{aligned}\tilde{\mathbf{A}}_0(i) &= \{(0,1,2+x^i)+g\colon g \in G_1\}\\ &\cup \{(1,1+x^i,0)+g\colon g \in G_2\}\\ &\cup \{(1+x^i,0,1)+g\colon g \in G_3\}\\ \tilde{\mathbf{B}}_0(i) &= \{(x^i,1+x^i,0)+g\colon g \in G_1'\}\\ &\cup \{(1+x^i,0,x^i)+g\colon g \in G_2'\}\\ &\cup \{(0,x^i,1+x^i)+g\colon g \in G_3'\}.\end{aligned}$$

where $\{G_1, G_2, G_3\}$ and $\{G_1', G_2', G_3'\}$ are two partitions of X.

It is easy to see that

Lemma 3.3. *If $(G_i'+x^i)\cap G_i = \phi$, $G_{i+1}'\cap(G_i+1) = \phi$ and $G_i'\cap G_{i+1} = \phi$ for $i = 1,2,3$ and $i+1$ is taken modulo 3, then $\tilde{\mathbf{A}}_0(i)$ and $\tilde{\mathbf{B}}_0(i)$ satisfy the condition c).*

Theorem 3.4. *There exist ordered-OSTS(v) for $v = 6k+1$ is prime power. Further, there exist ordered OSTS(v) for $v \in B(P_{1,6})$.*

Proof: For $p = 6k+1$ is a prime, if $\theta^{2k} \neq 3s$, $1 \leq 3s \leq p-1$. Let

$$\begin{aligned}G_1 = G_1' &= \{0,3,6,\ldots,p-1\}\\ G_2 = G_2' &= \{2,5,8,\ldots,p-2\}\\ G_3 = G_3' &= \{1,4,7,\ldots,p-3\}\end{aligned}$$

It is easily checked that if $\theta^{2ki} = 2+3s$ $(1 \leq 2+3s \leq p-2)$ then $\{G_1, G_2, G_3\}$ and $\{G_1', G_2', G_3'\}$ satisfy the condition of Lemma 3.3. Since $1+\theta^{2k}+\theta^{4k} = 0$ and $\theta^{2k} \neq 3s$, $1 \leq 3s \leq p-1$ we have $\theta^{2k} = 2+3s$ or $\theta^{4k} = 2+3s$ where $1 \leq 2+3s \leq p-2$. If $\theta^{2k} = 3s$, $1 \leq 3s \leq p-1$, let

$$\begin{aligned}G_1 &= \{0,3,6,\ldots,p-4\}\\ G_2 &= \{2,5,8,\ldots,p-2\}\\ G_3 &= \{1,4,7,\ldots,p-3,p-1\}\\ G_1' &= \{1,4,\ldots,p-3-x,p+2-x,p+2-x+3,\ldots,p-1\}\\ G_2' &= \{0,3,6,\ldots,p-1-x,p+1-x,p+1-x+3,\ldots,p-2\}\\ G_3' &= \{2,5,\ldots,p-2-x,p-x,p-x+3,\ldots,p-3\}\end{aligned}$$

where $x = \theta^{2k}$.

It is easily checked that $\{G_1, G_2, G_3\}$ and $\{G_1', G_2', G_3'\}$ satisfy the condition of Lemma 3.3 with $i = 1$.

For $p^2 = 6k+1$ is a prime power where p is a prime and $p \equiv 5 \pmod 6$. Let $\theta^{2ki} = a\theta + b$, where $a, b \in GF(p)$ since $a \neq 0$ and $1+\theta^{2k}+\theta^{4k} = 0$

without loss of generality we can suppose $a = 3s$ or $a = 3s + 1$ and $a \neq 1$. Let

$$H_1 = \{0, 3, 6, \ldots, p-2\}$$
$$H_2 = \{2, 5, 8, \ldots, p-3\}$$
$$H_3 = \{1, 4, 7, \ldots, p-1\}$$

If $a = 3s$ $3 \leq 3s \leq p-2$. Let $H'_1 = H_3$, $H'_2 = H_1$, $H'_3 = H_2$. If $a = 1 + 3s$ $4 \leq 1 + 3s \leq p-1$. Let

$$H'_1 = \{1, 4, 7, \ldots, p-3-a, p+2-a, p+5-a, \ldots, p-2\}$$
$$H'_2 = \{0, 3, 6, \ldots, p-1-a, p+1-a, p+4-a, \ldots, p-3\}$$
$$H'_3 = \{2, 5, 8, \ldots, p-2-a, p-a, p+3-a, \ldots, p-1\}$$

Let

$$G_i = \{c_i\theta + d_i : c_i \in H_i, d_i \in GF(p)\}$$
$$G'_i = \{c_i\theta + d_i : c_i \in H'_i, d_i \in GF(p)\}$$

for $1 \leq i \leq 3$. Since $\{H_1, H_2, H_3\}$ and $\{H'_1, H'_2, H'_3\}$ satisfy $(H'_i + a) \cap H_i = \phi$, $H'_{i+1} \cap (H_i + 1) = \phi$ and $H'_i \cap H_{i+1} = \phi$ then it is easy to see that $\{G_1, G_2, G_3\}$ and $\{G'_1, G'_2, G'_3\}$ satisfy the condition of Lemma 3.3. Hence we have that if $6k + 1 = p$, a prime of $6k + 1 = p^2$ where $p \equiv 5 \pmod 6$, a prime, then there exist ordered-OSTS$(6k+1)$.

Since the set OSTS $= \{n\colon$ there exists OSTS(n)$\}$ is PBD-closed (see [5]), it is easy to see that the set of ordered-OSTS is also PBD-closed. Therefore we have completed the proof.

4 Construction methods for OGDD

Some constructions for OGDD have been given in [5]. In this section, we give some improvements. Our main tools are ordered-OSTS, ordered-OGDD and IOOQ.

Theorem 4.1. *Suppose there exist ordered-OGDD of type T, and suppose there exist IOOQ(m). Then there exist ordered-OGDD of type $mT = \{mt\colon t \in T\}$.*

Proof: Suppose $(Q, \oplus)$ and $(Q, \otimes)$ are IOOQ(m) and that $(X, \mathbf{G}, \tilde{\mathbf{A}}_1)$ and $(X, \mathbf{G}, \tilde{\mathbf{A}}_2)$ are ordered-OGDD of type T. We will construct ordered-OGDD on point set $X \times Q$ having groups $\mathbf{H} = \{G \times Q\colon G \in \mathbf{G}\}$.

For every block $\tilde{A} = (x, y, z) \in \tilde{\mathbf{A}}_1$, construct the m^2 blocks $\tilde{\mathbf{B}}_1(\tilde{A}) = \{((x, a), (y, b), (z, a \oplus b))\colon a, b \in Q\}$. Define $\tilde{\mathbf{B}}_1 = \cup_{\tilde{A} \in \tilde{\mathbf{A}}_1} \tilde{\mathbf{B}}_1(\tilde{A})$.

For every block $\tilde{A} = (x, y, z) \in \tilde{\mathbf{A}}_2$, construct the m^2 blocks $\tilde{\mathbf{B}}_2(\tilde{A}) = \{((x,a),(y,b),(z,a \otimes b)) \colon a, b \in Q\}$. Define $\tilde{\mathbf{B}}_2 = \cup_{\tilde{A} \in \tilde{\mathbf{A}}_2} \tilde{\mathbf{B}}_2(\tilde{A})$. We will show that $(X, \mathbf{H}, \tilde{\mathbf{B}}_1)$ and $(X, \mathbf{H}, \tilde{\mathbf{B}}_2)$ are ordered-OGDD.

First, since $\tilde{\mathbf{A}}_1$ and $\tilde{\mathbf{A}}_2$ satisfy the condition c) and $(Q, \oplus)$ and $(Q, \otimes)$ are quasigroups, then it is clear that $\tilde{\mathbf{B}}_1$ and $\tilde{\mathbf{B}}_2$ satisfy the condition c). Next, since $\mathbf{A}_1$ and $\mathbf{A}_2$ satisfy the condition a'), then it is easy to see that $\mathbf{B}_1$ and $\mathbf{B}_2$ satisfy the condition a'). Finally we prove that $\mathbf{B}_1$ and $\mathbf{B}_2$ satisfy the condition b). Suppose that $\{(u,a),(v,b),(w,c)\}$ and $\{(x,d),(y,e),(w,c)\} \in \mathbf{B}_1$ are distinct blocks and that $\{(u,a),(v,b),(t,f)\}$ and $\{(x,d),(y,e),(t,f)\} \in \mathbf{B}_2$ are distinct blocks. Then $\{u,v,w\}$ and $\{x,y,w\} \in \mathbf{A}_1$ and $\{u,v,t\}$ and $\{x,y,t\} \in \mathbf{A}_2$. Since $\mathbf{A}_1$ and $\mathbf{A}_2$ satisfy the condition b), then we have $\{u,v\} = \{x,y\}$ and $w \neq t$. Without loss of generality, suppose $u = x$, $v = y$ and $(u,v,w) \in \tilde{\mathbf{A}}_1$, $(t,u,v) \in \tilde{\mathbf{A}}_2$. Hence $((u,a),(v,b),(w,c))$ and $((u,d),(v,e),(w,c)) \in \tilde{\mathbf{B}}_1$ and $((t,f),(u,a),(v,b))$ and $((t,f),(u,d),(v,e)) \in \tilde{\mathbf{B}}_2$. That is, $c = a \oplus_{123} b$, $f = a \otimes_{231} b$, $c = d \oplus_{123} e$ and $f = d \otimes_{231} e$. Since $\oplus_{123}$ os orthogonal to $\otimes_{231}$ then $(a,b) = (d,e)$. But then the blocks $\{(u,a),(v,b),(w,c)\}$ and $\{(x,d),(y,e),(w,c)\}$ are identical, a contradiction. Therefore $\mathbf{B}_1$ and $\mathbf{B}_2$ satisfy the condition b). We have completed the proof.

Corollary 4.2. *Suppose there exist ordered-OSTS(u) and IOOQ(v). Then there exist ordered-OGDD of type v^u.*

Proof: Ordered-OSTS(u) are equivalent to ordered-OGDD of type 1^u. Apply Theorem 4.1.

Corollary 4.3. *There exist ordered-OGDD of type v^u for $u \in B(P_{1.6})$ and $v \in B(P_4)$.*

If the condition c) in the definition of ordered-OSTS or ordered-OGDD is replaced by

c') If (a,b) is in a block of $\tilde{\mathbf{A}}$ then (a,b) is also in a block of $\tilde{\mathbf{B}}$.

Then $(X, \tilde{\mathbf{A}})$ and $(X, \tilde{\mathbf{B}})$ are called cyclic-OSTS(v). Similarly we can define cyclic-OGDD.

For two idempotent quasigroups of order v, $(Q, \oplus)$ and $(Q, \otimes)$, if $\oplus_{ijk}$ is orthogonal to $\otimes_{ijk}$, $\otimes_{jki}$ and $\otimes_{kij}$ for $(ijk) = (123), (231), (312)$, we say they are half conjugate orthogonal (briefly half-ICOQ(v)).

From the proof of Theorem 2.1, we have there exist half-ICOQ(v) for $v = 4$ and 8. Hence we have

Theorem 4.4. *There exist half-ICOQ(v) for $v \in B(P_4 \setminus \{5\})$.*

Theorem 4.5. *Suppose there exist cyclic-OGDD of type T and suppose there exist half-ICOQ(m). Then there exist cyclic-OGDD of type $mT = \{mt \colon t \in T\}$.*

Proof: The proof is similar to that of Theorem 4.1.

Remark: It is easy to see that we can arrange an OGDD into a cyclic-OGDD more easily than into an ordered-OGDD. Can we arrange any OGDD into a cyclic-OGDD? There is an example from [6] which can not be arranged into a cyclic-OGDD. That is, on $Z_q \times \{0,1\}$ form two GDD by developing the following started blocks modulo $(9,-)$:

GDD1			GDD2		
00	20	81	01	21	80
00	40	71	01	41	70
00	10	51	01	11	50
00	11	21	01	10	20
01	21	51	00	20	50
00	30	60	01	31	61

Each of the first five starter blocks develops into nine blocks while the sixth generates only three distinct blocks. It is easy to see that these two GDD form OGDD. However, if we let (00 20 81) $\in \tilde{\mathbf{A}}$, then (00 20 50), (20 81 41) $\in \tilde{\mathbf{B}}$, then (20 50 80) $\in \tilde{\mathbf{A}}$, then (50 80 30) $\in \tilde{\mathbf{B}}$, then (20 31 41), (80 30 61) $\in \tilde{\mathbf{A}}$, then (80 31 41), (80 70 61) $\in \tilde{\mathbf{B}}$, then (70 80 31) $\in \tilde{\mathbf{A}}$. That is, (70 80) is in a block of $\tilde{\mathbf{A}}$ but (70 80) is not in a block of $\tilde{\mathbf{B}}$.

5 Further results

Lemma 5.1. *Suppose there is a $(u, K, 1)$-PBD. If there exist ordered-OGDD of tyoe v^k for any $k \in K$. Then there exist ordered-OGDD of tyoe v^u.*

Proof: Let $\mathbf{A}$ be a collection of blocks of a $(u, K, 1)$-PBD on Z_u. For any $A \in \mathbf{A}$ let $\mathbf{B}_1(A)$ and $\mathbf{B}_2(A)$ be two collections of blocks of ordered-OGDD of type v^k, where $|A| = k \in K$, on $A \times Z_v$. It is easy to see $\mathbf{B}_1$ and $\mathbf{B}_2$ are two collections of blocks of ordered-OGDD of type v^u on $Z_u \times Z_v$, where $\mathbf{B}_1 = \cup_{A \in \mathbf{A}} \mathbf{B}_1(A)$ and $\mathbf{B}_2 = \cup_{A \in \mathbf{A}} \mathbf{B}_2(A)$.

Theorem 5.2. *There exist ordered-OGDD of type 2^u for $u \in B(P_{1,6})$.*

Proof: Let $u = 6k+1$ be a prime power and θ be a primitive root of $GF(u)$. On $Z_u \times \{a, b\}$, define a GDD of type 2^u by taking $\{i \times \{a, b\} : i \in Z_u\}$ to form the u groups, and develop the ordered starter blocks:

$$
\begin{array}{lll}
((0,b), & (-\theta^{j-\ell},a), & (\theta^{j-\ell},a)) \\
((\theta^{2k+j-\ell},a), & (0,b), & (-\theta^{2k+j-\ell},a)) \\
((-\theta^{4k+j-\ell},a), & (\theta^{4k+j-\ell},a), & (0,b)) \\
((0,b), & (\theta^{j},b), & (\theta^{j}+\theta^{2k+j},b))
\end{array}
$$

where $j = 0, 1, \ldots, k-1$ and $\theta^\ell = 2$ modulo $(u, -)$ to form the triples. Form a second GDD by developing

$$\begin{array}{lll}
((0,a), & (-\theta^{j-\ell},b), & (\theta^{j-\ell},b)) \\
((\theta^{2k+j-\ell},b), & (0,a), & (-\theta^{2k+j-\ell},b)) \\
((-\theta^{4k+j-\ell},b), & (\theta^{4k+j-\ell},b), & (0,a)) \\
((0,a), & (\theta^j,a), & (\theta^j+\theta^{2k+j},a))
\end{array}$$

where $j = 0, 1, \ldots, k-1$ and $\theta^\ell = 2$.

Since $\theta^{4k} \neq 1$ and $1 + \theta^{2k} + \theta^{4k} = 0$ then $\theta^{2k} \neq -2$, that is, $\theta^{2k-\ell} + 1 \neq 0$, then $\{\pm(\theta^{2k-\ell}+1)\theta^j, \pm(\theta^{2k-\ell}+1)\theta^{2k+j}, \pm(\theta^{2k-\ell}+1)\theta^{4k+j} : j = 0, 1, 2, \ldots, k-1\} = GF(u) \setminus \{0\}$. Therefore it is easliy verified that the two GDD are ordered-OGDD. From Lemma 5.1 we have completed the proof.

From Theorem 4.1 and Corollary 4.3 we have

Corollary 5.3. *There exist ordered-OGDD of type* v^u *and* $(2v)^u)$ *for* $u \in B(P_{1,6})$ *and* $v \in B(P_4)$.

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