On The Strong Circuit Uniqueness And Its Application To The Circuit Characterization Of Graphs

E.J. Farrell
The Centre For Graph Polynomials
Department of Mathematics
The University of the West Indies
St.Augustine, Trinidad

J.M. Guo
Department of Applied Mathematics
Tongji University
Shanghai, China

Z.Y. Guo
Department of Mathematics
Huazhong University of Science and Technology
Wuhan, China

ABSTRACT. On the basis of circuit uniqueness, the concept of strong circuit uniqueness is introduced, and some graphs with the property of strong circuit uniqueness are identified. The results are then used to prove successfully the circuit uniqueness of the graphs $K_m \cup K_n$ and $K_{m,n}$. This represents an improvement on the previous papers on the same subject.

1 Basic Definitions

The graphs considered here are finite, undirected, and contain no loops and no multiple edges. Let G be such a graph. We define a *circuit (cycle) with one and two nodes* in G to be a node and an edge respectively. Circuits with more than two nodes are called *proper circuits.* A *circuit cover* of G is a spanning subgraph of G in which all the components are circuits.

Let us associate an indeterminate or weight w_α with each circuit α in G and the monomial $W(S) = \prod w_\alpha$, with each circuit cover S; where the product is taken over all the components in S. Then the *circuit polynomial* of G is

$$C(G;\underline{w}) = \sum W(S),$$

where the summation is taken over all the circuit covers of G, and $\underline{w}$ (called the weight vector) is a vector of the indeterminates w_α.

The circuit polynomial was introduced in Farrell [1]. It has been shown in Farrell [2], that both the characteristic polynomial and the matching polynomial are special cases of the circuit polynomial. In this paper, we assign the weight w_r to each cycle with r nodes. Therefore $\underline{w} = (w_1, w_2, \ldots, w_p)$, where p is the number of nodes in G.

We shall say that $C(G;w)$ *characterizes* graph G if and only if $C(G;w) = C(H;\underline{w})$ implies that $H \cong G$. In this case, we also say that G is *circuit unique.* If $C(G;\underline{w}) = C(H;\underline{w})$, then we say that G and H are *cocircuit.* It has been shown that many of the well known families of graphs are circuit unique. These include *chains* (trees with nodes of valencies 1 and 2), stars, wheels, complete graphs, regular complete bipartite graphs (see [5]), all the basic graphs with cyclomatic number 2 (See [6]), short ladders (the graph formed by joining pairs of corresponding nodes of two equal chains) (see [6], and the unions of chains, cycles, and the union of a circuit unique hamiltonian graph with itself (see [7]), etc. It is interesting to note, from the above list, that the characterizing power of circuit polynomial is quite strong.

In the material which follows, the notations P_m, K_m and Z_m will be used for the chain, complete graph and cycle with m nodes respectively. We will denote the complete m by n bipartite graph by $K_{m,n}$. The notation $G \cup H$ will denote the disjoint union of graphs G and H, and $U^s P_r$, the disjoint union of s copies of P_r. Let H be a subgraph of G; then $G - H$ denotes the graph obtained from G by removing the nodes of H.

In this paper, we extend the list of graphs characterized by their circuit polynomial . We show that the graphs $K_m \cup K_n$ and $K_{m,n}$ are circuit unique. We also introduce the idea of strong circuit uniqueness and identify some families of graphs with this property.

2 Preliminaries

We now give some results which have already been established and which will be useful in the material which follows.

The following Lemmas were established in [2].

Lemma 1 (The Fundamental Edge Theorem). *Let G be a graph and xy an edge in G. Let G' be the graph obtained from G by deleting xy, G'' the graph obtained from G by removing nodes x and y and G^* the graph G*

with the restriction that in every cover, xy must be part of a proper cycle. Then

$$C(G) = C(G') + w_2 C(G'') + C(G^*).$$

Lemma 2 (The Component Theorem). *Let G be a graph consisting of components $G_1, G_2, \ldots, G_k$. Then*

$$C(G) = \prod_{i=1}^{k} C(G_i).$$

The following Lemma can be easily proved.

Lemma 3. *Let $C(G)$ be the circuit polynomial of a graph with p nodes and q edges. Then*

(1) The highest power of w_1 in $C(G)$ is w_1^p and this occurs with coefficient 1.

(2) The coefficient of $w_1^{p-2} w_2$ is q.

(3) The coefficient of w_p is the number of hamiltonian cycles in G.

(4) The coefficient of $w_{r_1}, w_{r_2}, \ldots, w_{r_k}$ is the number of spanning subgraphs of G consisting of the disjoint cycles $Z_{r_1}, Z_{r_2}, \ldots, Z_{r_k}$.

The following lemma is given in Farrell and Guo [6].

Lemma 4. *Let G be a nearly regular graph (a graph in which the valencies of any pair of nodes differ by at most 1) and H a graph such that $C(H) = C(G)$. Then H is also nearly regular and H has the same valency sequence as G.*

The following result was proved in [4].

Lemma 5.

$$\frac{\partial C(G)}{\partial w_r} = \sum C(G - Z_r),$$

where Z_r is a cycle with r nodes, and the summation is taken over all such cycles in G.

The following results can be found in [2].

Lemma 6. *The circuit polynomial of the chain P_p is*

$$C(P_p) = \sum_k \binom{p-k}{k} w_1^{p-2k} w_2^k.$$

Lemma 7. *The circuit polynomial of the circuit Z_p is*

$$C(Z_p) = w_1^p + \sum_{r=1} \left[\frac{p}{r} \binom{p-r-1}{r-1} w_1^{p-2r} w_2^r \right] + w_p (p > 2).$$

Lemma 8.

$$C(K_p) = p! \sum \frac{w_1^{j_1}}{j_1!} \frac{(\frac{1}{2}w_2)^{j_2}}{j_2!} \prod_{i=3}^{p} \left(\frac{1}{2i} w_i \right)^{j_i} \frac{1}{j_i!},$$

where the summation is taken over all sets of positive integers j_i such that $\sum_i i j_i = p$.

Lemma 9. *The circuit polynomial of the complete bipartite graph $K_{m,n}$ satisfies the recurrence relation*

$$\begin{aligned} C(K_{m,n}) &= w_1 C(K_{m-1,n}) + n w_2 C(K_{m-1,n-1}) \\ &+ \frac{1}{2} \sum_{s=2} (m-1)_{s-1} (n)_s w_{2s} C(K_{m-s,n-s}), \end{aligned}$$

and its factorial generating function is

$$C(K_{m,n}; w, u, v) = \exp\{(u+v)w_1 + uvw_2 + 1/2 w(u,v)\},$$

where

$$w(u,v) = \sum_{s=2} w_{2s} \frac{(uv)^s}{s}.$$

3 The Strong Circuit Uniqueness

Definition 1: Let H be a graph with p nodes. Let $G_1, G_2, \ldots, G_n$ be n graphs each containing p nodes. If for any positive integer n, the equation $\sum_{i=1} C(G_i) = nC(H)$ implies that $G_i \cong H$ $(i = 1, 2, \ldots, n)$, then H is said to be *strongly circuit unique.*

Obviously, if H is strongly circuit unique, then H must be circuit unique. We can see this by letting $n = 1$ in the definition. But, the converse is not true. This is demonstrated in the following example.

Let H, G_1, G_2, be the graphs as shown in Figure 1.
It can be easily verified that H is circuit unique, and that

$$\begin{aligned} C(H) &= w_1^4 + 4w_1^2 w_2 + w_2^2 + w_1 w_3, \\ C(G_1) &= w_1^4 + 5w_1^2 w_2 + 2w_2^2 + 2w_1 w_3 \end{aligned}$$

and

$$C(G_2) = w_1^4 + 3w_1^2 w_2.$$

Therefore $C(G_1) + C(G_2) = 2\ C(H)$. However, neither G_1 nor G_2 is isomorphic to H.

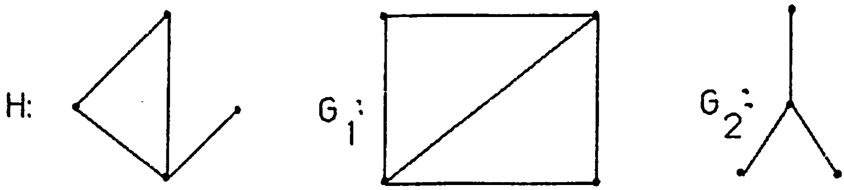

Figure 1

Theorem 1. *$U^p K_1$ is strongly circuit unique.*

Proof: Let $H = U^p K_1$ and $G_1, G_2, \ldots, G_n$ be n graphs, such that $\sum_{i=1} C(G_i) = nC(H) = nw_1^p$. Since each $C(G_i)$ contains the term w_1^p, then clearly $C(G_i) = w_1^p$ $(i = 1, 2, \ldots, n) \Rightarrow G_i$ consists of p isolated nodes. $\Rightarrow G_i \cong H$ $(i = 1, 2, \ldots, n)$. Hence, $U^p K_1$ is strongly circuit unique. □

Theorem 2. *K_p is strongly circuit unique.*

Proof: Let $G_1, G_2, \ldots, G_n$ be n p-node graphs such that

$$\sum_{i=1}^{n} C(G_i) = nC(K_p).$$

Then from the expression of $nC(K_p)$, we know that the graphs $G_1, G_2, \ldots$ and G_n have a total of np nodes and $np(p-1)/2$ edges. We claim that each G_i has $p(p-1)/2$ edges. If not, then $\exists$ a certain G_j $(1 \leq j \leq n)$ such that $|E(G_j)| > \frac{p(p-1)}{2}$ otherwise, the total number of edges of n G_i's cannot reach $np(p-1)/2$. But this is impossible for a simple graph. Therefore for each i, we must have $|E(G_i)| = p(p-1)/2$. Since each G_i has p nodes, we conclude that $G_i \cong K_p$ $(i = 1, 2, \ldots, n)$. □

Theorem 3. *Z_p is strongly circuit unique.*

Proof: This follows immediately from Lemma 8 of [8]. □

The following lemma is immediate from Theorem 1 of [6].

Lemma 10. *Let G be a graph with p nodes, q edges and the valency sequence $\pi(G) = (d_1, d_2, \ldots, d_p)$, where $2q = pd + r$ $(0 \leq r \leq p-1)$, for some positive integer d. Then the sum*

$$\sum_{i=1}^{p} \binom{d_i}{2}$$

attains a minimum value if and only if G is nearly regular and has valency sequence $((d+1)^r, d^r)$.

Theorem 4. *P_p is strongly circuit unique.*

Proof: Let $G_1, G_2, \ldots, G_n$ be n p-node graphs and satisfy the equation

$$\sum_{i=1}^{n} C(G_i) = nC(P_p) = n \sum_{k=1}^{[\frac{p}{2}]} \binom{p-k}{k} w_1^{p-2k} w_2^k.$$

Then from Lemma 3, we know that $G_1, G_2, \ldots, G_n$ must have the following properties:

(1) $\sum_{i=1}^{n} |V(G_i)| = np$, $\sum_{i=1}^{n} |E(G_i)| = n(p-1)$.

(2) None of G_i's contains cycle Z_t, where $t \geq 3$.

(3) The sum of the coefficients of the terms $w_1^{p-4} w_2^2$ in all the $C(G_i)$'s is $n\binom{p-1}{2} - n\sum_{i=1}^{p} \binom{d_i}{2}$, where d_i is the valency of node i in P_p.

From property (2), we know that for each i, $|E(G_i)| \leq p-1$, and so, the equation in property (1), $\sum_{i=1}^{n} |E(G_i)| = n(p-1)$, implies $|E(G_i)| = p-1$ for all i's.

Since each G_i has p nodes, $(p-1)$ edges and cycle free, each G_i must be a tree. Let us consider the valency sequence

$$\pi(G_i) = \left(d_1^{(i)}, d_2^{(i)}, \ldots, d_p^{(i)}\right).$$

of G_i. By property (3), we have

$$\sum_{i=1}^{n} \left[\binom{p-1}{2} - \sum_{j=1}^{p} \binom{d_j^{(i)}}{2} \right] = n\binom{p-1}{2} - n\sum_{t=1}^{p} \binom{d_t}{2}$$

$$\Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{p} \binom{d_j^{(i)}}{2} = n \sum_{t=1}^{p} \binom{d_t}{2} \tag{1}$$

Now, let $H_1 = U^n P_p$, $H_2 = U_{i=1}^n G_i$, $\pi(H_1) = (u_1, u_2, \ldots, u_{np})$ and $\pi(H_2) = (v_1, v_2, \ldots, v_{np})$.

Then it is clear that

$$\sum_{i=1}^{np} \binom{v_i}{2} = \sum_{i=1}^{n} \sum_{j=1}^{p} \binom{d_j^{(i)}}{2}, \quad \sum_{i=1}^{np} \binom{u_i}{2} = n \sum_{t=1}^{p} \binom{d_t}{2}.$$

By relation (1) we have

$$\sum_{i=1}^{np} \binom{v_i}{2} = \sum_{i=1}^{np} \binom{u_i}{2}$$

Since H_1 is nearly regular, it follows from Lemma 10 that H_2 is also nearly regular and $\pi(H_2) = \pi(H_1)$. Hence for all i, $1 \leq i \leq n$, G_i has nodes with valencies 1 and 2 only. Since G_i is a tree then G_i is a chain with p nodes, i.e., $G_i \cong P_p$ for all $1 \leq i \leq n$. Hence the result. □

Theorem 5. *The star $K_{1,k}(k \geq 1)$ is strongly circuit unique.*

Proof: Let $G_1, G_2, \ldots, G_n$ be n $(k+1)$-node graphs, satisfying the equation

$$\sum_{i=1}^{n} C(G_i) = nC(K_{i,k}) = n(w_1^{k+1} + kw_1^{k-1}w_2).$$

Then G_i must have the following properties.

1. $\sum_{i=1}^{n} |V(G_i)| = n(k+1)$ and $\sum_{i=1}^{n} |E(G_i)| = nk$,
2. $\forall\, i$, $1 \leq i \leq n$, G_i is Z_t-free $(t \geq 3)$.
3. $\forall\, i$, $1 \leq i \leq n$, G_i does not contain any 2-matching.

From properties (1) and (2), we know that each G_i is a tree with k edges, by using the similar reasoning as in Theorem 4. From property (3), no G_i has a 2-matching. Therefore, each G_i must be a star $K_{1,k}$, i.e $G_i \cong K_{1,k}$. Hence the result follows. □

Theorem 6. *$K_{m,n}$ and $K_{m,m+1}$ are strongly circuit unique.*

Proof: We first prove the case for $K_{m,m+1}$. Let $G_1, G_2, \ldots, G_n$ be n $(2m+1)$-node graphs satisfying $C(G_i) = nC(K_{m,m+1})$. Then G_i must have the following properties.

1. $\sum_{i=1}^{n} |V(G_i)| = n(2m+1)$ and $\sum_{i=1}^{n} |E(G_i)| = nm(m+1)$ and
2. Each G_i is odd-cycle free.

From (2), we know that each G_i is bipartite with $(2m+1)$ nodes. It is easy to see that $|E(G_i)| \leq m(m+1)$, $1 \leq i \leq n$. Since $\sum_{i=1}^{n} |E(G_i)| = nm(m+1)$, we conclude $|E(G_i)| = m(m+1)$ for all i's.

Since a $(2m+1)$-node bipartite graph with $m(m+1)$ edges must be $K_{m,m+1}$, each G_i must be isomorphic to $K_{m,m+1}$.

The case of $K_{m,m}$ can be easily proved in the same way. □

Theorems 1 to 6 show that many well-known graphs are strongly circuit unique. These results will be used for establishing the circuit uniqueness of some important graphs.

4 The Circuit Characterization of Graphs $K_m \cup K_n$ and $K_{m,n}$

The following lemmas are taken from [7] and [6].

Lemma 12. *$K_n \cup K_n$ is circuit unique.*

Lemma 13. *$K_{n+1} \cup K_n$ is circuit unique.*

We extend these two lemmas to the following theorem.

Theorem 6. *$K_m \cup K_n$ is circuit unique for all $m, n \geq 1$.*

Proof: We may well assume $m \geq n+2$, since the cases of $m = n$ and $m = n+1$ have been settled by Lemmas 12 and 13.

Let $G = K_m \cup K_n$. Suppose that H is a graph such that $C(H) = C(G) = C(K_m)C(K_n)$. Then by Lemmas 3 and 8, H must have the following properties

1. $|V(H)| = m+n, |E(H)| = \frac{m(m-1)}{2} + \frac{n(n-1)}{2}$.
2. H contains $f(K_m)$ m-cycles $Z_m^{(i)}$, $i = 1, 2, \ldots, f(K_m)$, where $f(K_m)$ is the total number of Hamilton cycles in K_m. From Lemma 8 we know that $f(K_m) = \frac{(m-1)}{2}$.
3. H does not contain any cycle Z_t, where $t \geq m+1$.
4. The number of triangles contained in H is $\binom{m}{3} + \binom{n}{3}$.
5. The number of cycle covers of the form $Z_m \cup Z_n$, contained in H, is $f(K_m)f(K_n)$, where if $n = 1, 2$, we define $f(K_n) = 1$.

By applying Lemma 5 to $C(G)$ and $C(H)$, we obtain:

$$\begin{aligned}\frac{\partial C(G)}{\partial w_m} &= \frac{\partial}{\partial w_m}\left[C(K_m)C(K_n)\right] \\ &= \frac{\partial}{\partial w_m}\left[(w_1^m + \ldots f(K_m)w_m)\, C(K_n)\right] = f(K_m)C(K_n)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial C(H)}{\partial w_m} &= \sum_{i=1}^{f(K_m)} C(H - Z_m^{(i)}) \\ &\Rightarrow \sum_{i=1}^{f(K_m)} C(H - Z_m^{(i)}) = f(K_m)C(K_n).\end{aligned}$$

By using the strong circuit uniqueness of K_n, we deduce that for any i, $1 \leq i \leq f(K_m)$, $H - Z_m^{(i)} = K_n$. Fixing a certain i, say $i = t$, we have $H - Z_m^{(t)} \cong K_n = H_2$. We denote by H_1 the induced subgraph of $V(Z_m^{(t)})$

in H and call the edges between H_1 and H_2 link edges. We discuss the following cases:

Case 1. No link edge exist. Then $H = H_1 \cup H_2$. Since H_1 has m nodes and $\frac{m(m-1)}{2}$ edges, $H_1 \cong K_m$.

Case 2. There are link edges, but there is only one node in H_1 incident with the link edges. Suppose that there are i link edges ($i \geq 1$). Then we have

$$|E(H_1)| = \frac{m(m-1)}{2} - 1. \Rightarrow H_1 \not\cong K_m.$$

by Property (2) H contains $f(K_m)$ m-cycles. Since $m \geq n+2$, all these cycles must be contained in H_1. So H_1 has $f(K_m)$ cycles. $\Rightarrow H_1 \cong K_m$. This is a contradiction. Thus Case (2) is impossible.

Case 3. There are link edges, $n \geq 2$, and there are at least 2 nodes in H_1 incident with link edges, but only one node in H_2 incident with link edges. This case is illustrated in Figure 1.1.

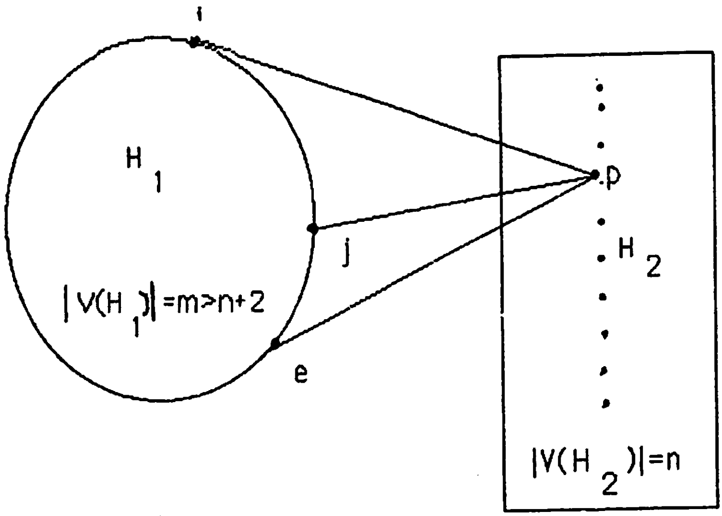

Figure 1.1

By property (5), H contains $f(K_m)f(K_n)$ cycle covers of the form $Z_m \cup Z_n$. It is easy to see that all the nodes of Z_m in this kind of cover must belong to $V(H_1) \cup \{p\}$. Now, if p is a node of such a Z_m, then the remaining n nodes in $(H_1 \cup \{p\} - Z_m) \cup (H_2 - \{p\})$, which is disconnected, cannot contain Z_n. So p cannot be a node of Z_m in the $Z_m \cup Z_n$-type covers. Thus all the Z_m's in $Z_m \cup Z_n$-type covers must be contained in H_1. Hence all the Z_n's in the $Z_m \cup Z_n$-type covers are in H_2. But, $H_2 \cong K_n$. Therefore there are $f(K_n)$ $Z_n s$ in H_2. By property (5), H_1 must contain $f(K_m)$ $Z_m s$. Thus $H_1 \cong K_m$. So link edges do not exist. Hence $H \cong K_m \cup K_n$.

Case 4. There are link edges.

(i) $n = 1$, and there are at least 2 nodes in H_1 incident with link edges; or,

(ii) $n \geq 2$, and there are at least 2 nodes both in H_1 and in H_2 that are incident with link edges.

In this case, there must exist two different nodes, say i and $j \in V(H_1)$, that are adjacent to different nodes, say v_i and $v_j \in V(H_2)$, when $n \geq 2$. (When $n = 1$, $v_i = v_j$, the following proof still applies.) The situation is illustrated in Figure 2 (In the figure, we assume that $i < j$ and the nodes of $Z_m^{(t)}$ are numbered clockwise along the cycle).

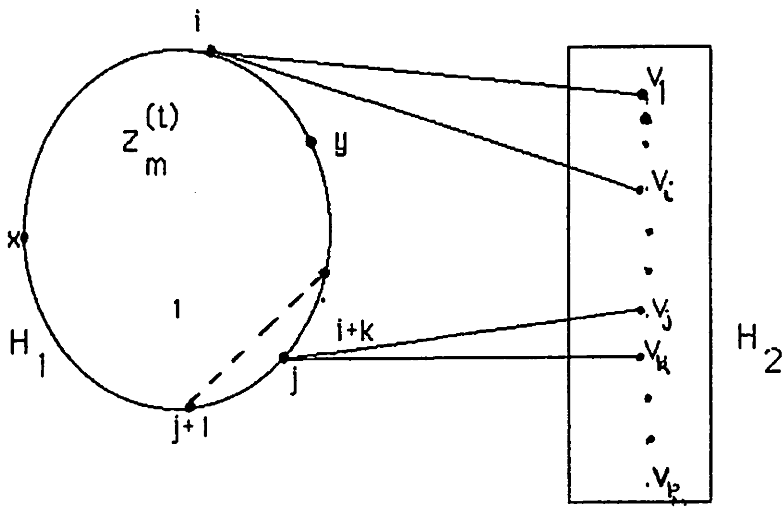

Figure 2

Case 4.1. $n + 1 < m < 2n + 2$.

If $j - 1 < n + 1$, then the length of the path $i \to y \to j$, denoted by $L(i \to y \to j)$, is $< n+1$. $\Rightarrow$ the length of cycle $L(i \to v_i \to v_j \to j \to x \to i) > m$, where the path $v_i \to v_j$ passes through all the n nodes $\in V(H_2)$. This contradicts to Property (3).

If $j - 1 \geq n + 1$, then $(i - j) + m < -(n + 1) + (2n + 2) = n + 1$. Hence the length of the path $L(i \to x \to j) < n + 1$. $\Rightarrow$ the length of the cycle $L(i \to y \to j \to v_j \to v_i \to i)$ is greater than m which again is a contradiction.

Therefore, when $n + 1 < m < 2n + 2$, there cannot be any link edges in H. Thus $H_i \cong K_m \Rightarrow H \cong K_m \cup K_n$.

Case 4.2. $m \geq 2n+2$.

Case 4.2.1. There are only two nodes, say i and $j(j > i)$, in H_i, incident with link edges.

First assume $m \geq 7$. From the discussions in Case 4.1, we know that in order to avoid the appearance of $Z_t(t > m)$ in H, $j-1$ must be $\geq n+1$ and $(i-j)+m$ must also be $\geq n+1$. That is, length of the $L(i \to y \to j)$ and $L(j \to x \to i)$ must both be $\geq n+1$ (See Figure 2).

Now look at the node $j+1$. We claim that the n edges $(j+1)(i+1), (j+1)(i+2), \ldots, (j+1)(i+n) \in (E(H_i)$. If contrary, i.e $\exists\ k$, $1 \leq k \leq n$, such that $(j+1)(i+k) \in E(H_i)$, then $L(j+1 \to i+k \to j \to v_j \to v_i \to i \to x \to j+1) > m$ — a contradiction. Likewise, we can prove for all k, $1 \leq k \leq n$, $(i-1)(j-k) \in E(H_i)$. Hence there are at least $2n$ edges $\in E(H_1)$. (Note: All the additions and subtractions appeared here and later are taken in modulo m.)

Since $m \geq 7$, either $i+2 \neq j-1$ or $j+2 \neq i-1$. Without loss of generality, we assume $j+2 \neq i-1$. Then at least one of the three edges $(j+1)i$, $(j+2)(i+1)$, and $(j+1)(i-1) \in E(H_1)$. Otherwise, the cycle $j+1 \to i \to v_i \to v_j \to j \to y \to i+1 \to j+2 \to x \to i-1 \to j+1$ must have length $> m$; a contradiction. Hence H_1 has at most $\frac{m(m-1)}{2} - (2n+1)$ edges. But the number of link edges ending at i and j cannot be greater than $2n$.

$$\Rightarrow E(H) \leq \frac{m(m-1)}{2} + (2n+1) + 2n + \frac{n(n-1)}{2}$$
$$= \frac{m(m-1)}{2} + \frac{n(n-1)}{2},$$

contradicting property (1).

Next, let $m \leq 6$. Since $m \geq 2n+2$, we need only consider the 4 possibilities, $m = 4$ and $n = 1$; $m = 5$ and $n = 1$; $m = 6$, $n = 1$ and $m = 6$, $n = 7$. In each, there are 2 nodes in H_1 incident with link edges. It is not difficult to show that each case leads to a contradiction.

Case 4.2.2. There are at least 3 nodes in H_1 incident with link edges. The situation is shown in Figure 3.

With the similar arguments as in Case 4.1, we know that $L(a_1 \to x_1 \to a_2) \geq n+1$, $L(a_2 \to x_2 \to a_3) \geq n+1, \ldots, L(a_1 \to x_i \to a_1) \geq n+1$. (Note: It is possible for several different nodes in H_1 to be joined to one node in H_2. If this happens, we need only consider one of these link edges and disregard all the rest. By so doing, the different nodes in $H_1, v_1, a_1, a_2, \ldots, a_i$, may be regarded to be joined to different nodes in H_2. (When $n = 1$, these nodes belong to $V(H_2)$ coincide with each other; and the correctness of the following proof will not be affected.)

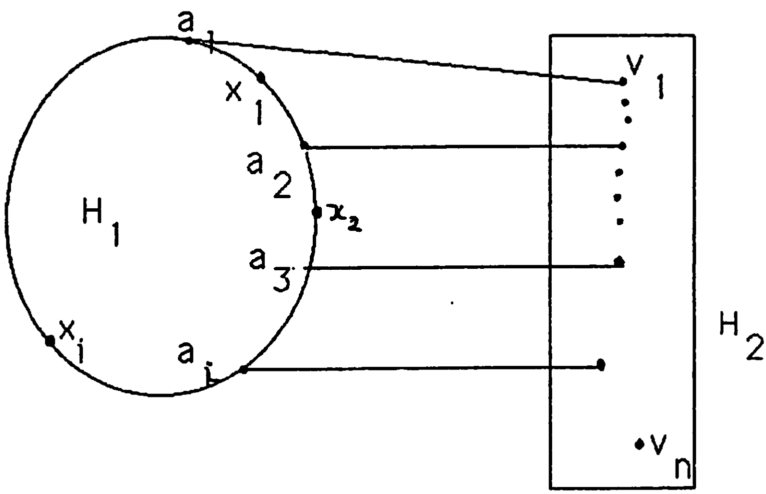

Figure 3

We first consider the nodes a_1 and a_2. In the similar manner as in case 4.1, we may prove that for all k_1, $1 \leq k_1 \leq n$, the edge $(a_1 - 1)(a_2 - k_1) \in E(H_1)$, and $\forall\ k_2$, $1 \leq k_2 \leq n$, the edge $(a_2 + 1)(a_1 + k_2) \in E(H_1)$, and moreover, at least one of the three edges $(a_2 + 2)(a_1 + 1)$, $(a_2 + 1)(a_1 - 1)$ and $(a_2 + 1)a_1 \in E(H_1)$

Suppose that there are exactly $i = 3$ nodes in H_1 with link edges. In this case, m must be $\geq 2i = 6$, because of Property (3).

We next consider nodes a_2 and a_3. If $m \geq 7$, then either $L(a_1 \to x_1 \to a_2)$, or $L(a_2 \to x_2 \to a_3)$, or $L(a_3 \to x_3 \to a_1)$, ≥ 3. Without loss of generality, we assume $L(a_3 \to x_3 \to a_1) \geq 3$. $\Rightarrow a_3 + 1 \neq a_1 - 1$. Then for all k_3, $1 \leq k_3 \leq n$, $(a_3 + 1)(a_2 + k_3) \in E(H_1)$. If $m = 6$, then $n = 1$, and the graph is shown in Figure 4. Since the number of triangles contained in this graph $\leq \binom{m}{3} - (m-2) + 3 = \binom{m}{3} - 1 < \binom{m}{3}$.

It follows that this graph cannot be isomorphic to H. And so, we need not consider it.

Now suppose $i \geq 4$. Then $m \geq 2i \geq 8$, because of Property (3). Similarly, for a_2 and a_3, for all k_3, $1 \leq k_3 \leq n$, $(a_3 + 1)(a_2 + k_3) \in E(H_1)$; for a_3 and a_4, for all k_4, $1 \leq k_4 \leq n$, $(a_4 + 1)(a_3 + k_4) \in E(H_1)$; and go on this way up to nodes a_{i-1}, a_i, for all k_i, $1 \leq k_i \leq n$, $(a_i + 1)(a_i + 1)(a_{i-1} + k_i) \in E(H_1)$. Hence, $|E(H_1)| \leq \frac{m(m-1)}{2} - (ni + 1)$, $i = 3, 4, \ldots$, while the number of link edges cannot exceed ni. $\Rightarrow |E(H_i)| \leq \frac{m(m-1)}{2} + \frac{n(n-1)}{2} - 1$, contradicting property (1).

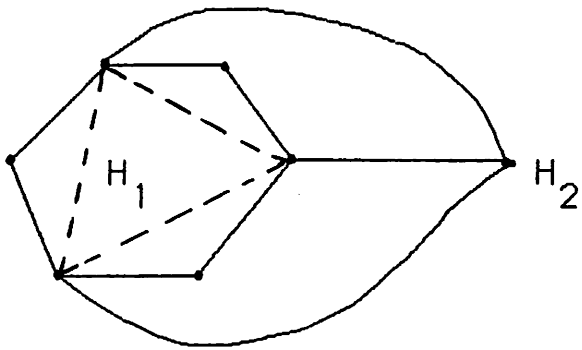

Figure 4

Therefore only case 1 is possible, and that completes our proof. □

The following result is taken from [8].

Lemma 11. *$K_p \cup Z_m$ is circuit unique, where $m \neq p$.*

Farrell and Guo [5] proved the circuit polynomials characterize $K_{m,m}$ and $K_{m,m+1}$ and conjectured that $K_{m,n}$ is circuit unique. We now prove that this conjecture is true.

Theorem 7. *$K_{m,n}$ is circuit unique.*

Proof: The case for $m = n$ is settled in [5]. And so, we assume that $m \neq n$. Without loss of generality, we let $m > n$.

Let G be a graph such that $C(G) = C(K_{m,n})$. By Lemma 3, G has the following properties:

(1) $|V(G)| = m + n$, $|E(G)| = mn$.

(2) G contains $f(K_{m,n})$ cycles of length $2n$, where $f(K_{m,n})$ is the number of cycles of length $2n$ in $K_{m,n}$.

(3) G does not contain any cycle of length $> 2n$.

(4) G does not contain any odd cycles.

By property (4), G is a bipartite graph. By Lemma 5,

$$\frac{\partial C(K_{m,n})}{\partial w_{2n}} = \sum C(K_{m,n} - Z_{2n}) = f(K_{m,n})C(U^{m-n}K_1).$$

On the other hand, $\frac{\partial C(G)}{\partial w_{2n}} = \sum_{j=1}^{f(K_{m,n})} C(G - Z_{2n}^{(j)}.$ $\Rightarrow \sum_{j=1}^{f(K_{m,n})} C(G - Z_{2n}^{(j)}) = f(K_{m,n})C(U^{m-n}K_1)$.

By Theorem 1, we know that for any j, $1 \le j \le f(K_{m,n})$,

$$G - Z_{2n}^{(j)} \cong U^{m-n}K_1.$$

For a fixed j, let the induced subgraph of $V(Z_{2n}^{(j)})$ in G be G_1. Number the $2n$ nodes on $Z_{2n}^{(j)}$ by $0, 1, \ldots 2n-1$, in clockwise order. Let $G_2 = G - Z_{2n}^{(j)} \cong U^{m-n}K_1$. Obviously, G_1 does not contain any odd cycles either. $\Rightarrow G_1$ is bipartite. Let $V_1(G_1)$ and $V_2(G_1)$ be the two parts of the node set $V(G_1)$ in this bipartition. Since G_1 has a Hamilton cycle $Z_{2n}^{(j)}$, it follows that $|V_1(G_1)| = |V_2(G_1)| = n$. $\Rightarrow |(EG_1)| \le n^2$.

Hence, there are at least $mn - n^2 = (m-n)n$ link edges between G_1 and G_2. Since $|V(G_2)| = (m-n)$, it follows that $\exists$ a node $v_1 \in G_2$ such that v_1 is incident with at least n link edges, i.e. $d(v_1) > n \ldots$ (2), (See Figure 5).

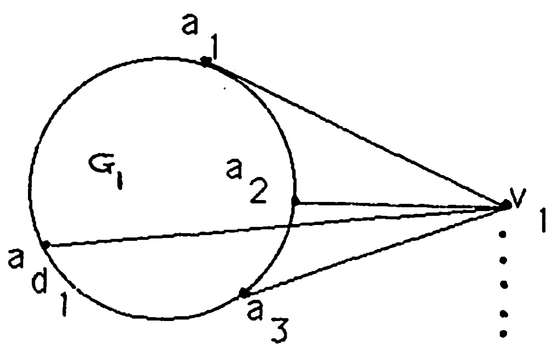

Figure 5

Write $d(v_1) = d_1$. Without loss of generality, we assume the end nodes of the d_1 link-edges incident with V_1 are $a_1, a_2, \ldots, a_{d_1}$ such that $a_1 \le a_1 \cdots \le a_{d_1}$. If two of these nodes are adjacent then Z_n could be extended via v_1 to the cycle Z_t, where $t > 2n$: contradicting Property (3). Therefore no two a_i's are adjacent. Clearly then $|V(G_1)| \ge 2d_1$. $\Rightarrow 2d_1 \le 2n$. Therefore $d_1 \le n$. Combining with Relation (2), it follows that $d_1 = n$, and each pair of n nodes are separate by exactly one node. This means that either v_1 is joined to each node in $V_1(G_1)$ or is joined to each node belonging to $V_2(G_1)$. Let us assume the former.

Now consider the set $A_1 = V(G_2) - \{v_1\}$. $|A_1| = m - n - 1$. There are at least $(m-n)n - n = n(m-n-1)$ edges linking G_1 to A_1. Thus $\exists$ a node $v_2 \in A_1$ such that $d(v_2) \ge n$. Similarily, we can deduce that v_2 is joined to every node $\in V_1(G_1)$, or is joined to every node $\in V_2(G_1)$. (See Figure 6.)

But v_2 cannot be joined to any node in $V_2(G_1)$ since there would be a cycle $a_1 \to v_1 \to a_2 \to a_1 + 1 \to v_2 \to a_2 + 1 \to x \to a_1$ of length $2n+2$, contradicting Property (3). (See Figure 6)

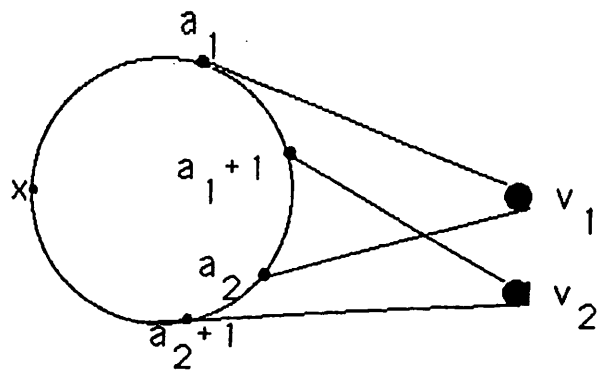

Figure 6

We continue this procedure up to $A_{m-n-1} = V(G_2) - \{v_1, \ldots, v_{m-n-1}\} = \{v_{m-n}\}$, and it follows that $d(v_{m-n})$ is adjacent with every node $\in V_1(G_1)$.

Hence, there are exactly $(m-n)n$ link edges between G_1 and G_2. Thus $|E(G_1)| = mn - (m-n)n = n^2$. Since G_1 is a bipartite graph with $2n$ nodes, it follows $G_1 \cong K_{m,n}$. Because each node in $V(G_2)$ is adjacent with $V_1(G_1)$ and $|E(G_2)| = 0$, it follows that $G \cong K_{m,n}$.

Farrell and Whitehead proved in [3] that if graph G is matching unique, so is $\overline{G}$. We pose the following question:

If graph G is circuit unique, is $\overline{G}$ also circuit unique?

If G and $\overline{G}$ are both circuit unique, then we say G is *two-way circuit unique.* Combining the results in Theorem 6 and Theorem 8, we have the following:

Theorem 8. *$K_{m,n}$ is two-way circuit unique.*

We note that if G is characteristically unique, then G is circuit unique. The converse need not be true. It is known that $K_{m,n}$ is not characteristically unique. We have proved, however that $K_{m,n}$ is circuit unique. Furthermore, we note here that it has been proved (Xu [9]) that $K_{m,n}$ is chromatically unique.

References

[1] E.J. Farrell, On a General Class of Graph Polynomials. *J. Comb. Theory B* **27** (1979), 111–122.

[2] E.J. Farrell, On a Class of Polynomials Obtained from Circuits in a Graph and Its Application to Characteristic Polynomials of Graphs. *Discrete Math.* **25** (1979), 121–133.

[3] E.J. Farrell and E.G. Whitehead Jr. Connections between the Matching and Chromatic Polynomials. *Internat. J. Math. and Math. Sci.* **15** No.4 (1992), 757–766.

[4] E.J. Farrell and J.C. Grell. Some Analytical Properties of the Circuit Polynomial of a Graph. *Caribb. J. Math.* **2(3)**, 69–76.

[5] E.J. Farrell and J.M. Guo. On the Characterizing Properties of the Circuit Polynomial. Submitted.

[6] E.J. Farrell and J.M. Guo. Circuit Characterization of Nearly Regular Graphs. *Match.* **25** (1990), 115–130.

[7] E.J. Farrell and J.M. Guo. Circuit Characterizations of Unions of Graphs. *JCMCC* **2** (1987), 161–177 .

[8] J.M. Guo and Z.Y. Guo, On the Circuit Uniqueness of the Unions of Cycle and Some Other Graphs. *Applied Math.* **3** No.3 (1990), 21–27.

[9] S. Xu. The Chromatic Uniqueness of Complete Bipartite Graphs. *Discrete Math.* **94** (1991),. 153–159.