# A Local Ore-Type Condition for Graphs of Diameter Two to Be Hamiltonian

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ABSTRACT. A graph is said to be in $L_1$ if $\deg u + \deg v \geq |N(u) \cup N(w) \cup N(v)| - 1$ for each induced path $uwv$ of order three. We prove that a 2-connected graph $G$ in $L_1$ of diameter two is hamiltonian, or $K_{d,d+1} \subset G \subset K_d + (d+1)K_1$ for some $d \geq 2$. This theorem generalizes a couple of known sufficient conditions for a graph to be hamiltonian. We also discuss the relation between this theorem and several other degree conditions for hamiltonicity.

We only consider finite simple graphs in this paper. For basic graph-theoretic notation we follow that of [3]. For a graph $G$ we define $\sigma(G)$ to be:

$$\sigma(G) = \begin{cases} \min\{\deg x + \deg y \mid x, y \in V(G), x \neq y, xy \notin E(G)\} & \text{if } G \text{ is not complete} \\ 2|V(G)| - 1 & \text{if } G \text{ is complete.} \end{cases}$$

There are a number of sufficient degree conditions for a graph to be hamiltonian. One of the oldest among them is Ore's theorem.

**Theorem A.** [8]. *A graph of order $p$ with $\sigma(G) \geq p$ is hamiltonian.* □

Later, it has been found that the bound of $\sigma(G)$ in the above theorem can be relaxed by one if we allow a class of the exceptions, which is clearly defined.

**Theorem B.** [1, 6, 7, 9]. *A 2-connected graph $G$ of order $p$ with $\sigma(G) \geq p - 1$ is either hamiltonian, or $K_{d,d+1} \subset G \subset K_d + (d+1)K_1$ for some $d \geq 2$.*

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Note that a graph of order $p$ with $\sigma(G) \geq p-1$ has diameter at most two.

As an extension of Theorem A in a different direction, Asratian and Khachatrian [5] have defined the local Ore-type condition. (In [5] Asratian's last name was transcribed as Hasratian.) For $i \geq 0$ a graph $G$ is in $L_i$ if $\deg u + \deg v \geq |N(u) \cup N(w) \cup N(v)| - i$ for each induced path $uwv$ of order three, where $N(x)$ is the neighborhood of a vertex $x$. They have proved the following generalization of Theorem A.

**Theorem C.** [5]. *A graph in $L_0$ of order at least three is hamiltonian.*

A graph in $L_0$ can have an arbitrarily large diameter. Hence one possible interpretation of Theorem C is that though a graph satisfying the condition of Theorem A has diameter at most two, this fact plays no role in its hamiltonicity.

On the other hand, the situation in Theorem B is somewhat different. As Asratian and Khachatrian have observed (see [2]), all claw-free graphs are in $L_1$. Since not all 2-connected claw-free graphs are hamiltonian, not all 2-connected graph in $L_1$ are hamiltonian. Therefore, a different mechanisim besides local Ore-type condition works under Theorem B.

In this paper we prove the following theorem.

**Theorem 1.** *Let $G$ be a 2-connected graph of diameter two. If $G$ is in $L_1$, then either $G$ is hamiltonian, or $K_{d,d+1} \subset G \subset K_d + (d+1)K_1$ for some $d \geq 2$.*

This theorem seems to give a "circumstantial evidence" that Theorem B is a result of combined mechanisms of a local Ore-type condition and a diameter condition.

Theorem 1 is stronger than Theorem B. Let $H_1 = K_1 + 2K_2$ and let $x$ and $y$ be a pair of nonadjacent vertices of degree two in $H_1$. Let $H_2 \simeq K_m$ $(m \geq 1)$ and let $G$ be a graph obtained from $H_1$ and $H_2$ by joining $\{x, y\}$ and every vertex in $H_2$. Then $G$ is 2-connected graph of diameter two. Moreover, it is in $L_1$ (actually it is claw-free). However, $\sigma(G) = 4 < |V(G)| - 1$.

Since every claw-free graph is in $L_1$, we also have the following theorem by Gould as a corollary.

**Theorem D.** (Gould [4]). *Every 2-connected claw-free graph of diameter two is hamiltonian.*

We introduce some additional notation before we prove Theorem 1. We denote by $\vec{C}$ a cycle $C$ with a given orientation. Let $u$, $v \in V(C)$. By $u \vec{C} v$ we denote the consecutive vertices on $C$ from $u$ to $v$ in the direction specified by $\vec{C}$. The same vertices, in reverse order, are given by $v \overleftarrow{C} u$. We also apply the same notation for a path. We use $u^+$ to denote the successor of $u$ on $\overrightarrow{C}$ and $u^-$ to denote its predecessor. We write $u^{++}$ instead of $(u^+)^+$.

For a subgraph $H$ of $G$ and $x \in V(G)-V(H)$, let $N_H(x) = N(x) \cap V(H)$. Thus $N_G(x) = N(x)$. For $A$, $B \subset V(G)$ with $A \cap B = \emptyset$, we denote by $e_G(A, B)$ the number of the edges in $G$ that join a vertex in $A$ and a vertex in $B$

**Proof of Theorem 1:** First, note the inequality

$$\deg u + \deg v \geq |N(u) \cup N(w) \cup N(v)| - 1$$

is equivalent to

$$|N(u) \cap N(v)| \geq |N(w) - (N(u) \cup N(v))| - 1.$$

Assume $G$ is not hamiltonian and let $C$ be a longest cycle in $G$. Let $H$ be a component of $G - V(C)$. Since $G$ is connected, $H$ has a vertex $z \in V(H)$ with $N_C(z) \neq \emptyset$. Let $N_C(z) = \{x_1, \ldots, x_d\}$. We may assume $x_1, \ldots x_d$ appear in the consecutive order along $\overrightarrow{C}$. Then since $z x_i x_i^+$ is an induced path in $G$, we have

$$|N(z) \cap N(x_i^+)| \geq |N(x_i) - (N(z) \cup N(x_i^+))| - 1. \tag{1}$$

Let $A = \{x_1, \ldots, x_d\}$ and $B = \{x_1^+, \ldots, x_d^+\}$. Since $C$ is a longest cycle, $A \cap B = \emptyset$. We count the number of the edges between $A$ and $B$. Since $C$ is a longest cycle, $N(x_i^+) \cap N(z) \not\subset V(C)$ $(1 \leq i \leq d)$. Therefore, $N(x_i^+) \cap A = N(x_i^+) \cap N(z)$. Again, since $C$ is a longest cycle, $B \cup \{z\}$ is an independent set, and hence

$$N(x_i) \cap B \subset N(x_i) - (N(x_i^+) \cup N(z)) - \{z\}. \tag{2}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{d} |N(x_i^+) \cap N(z)| &= \sum_{i=1}^{d} |N(x_i^+) \cap A| = e_G(A, B) \qquad (3)\\ &= \sum_{i=1}^{d} |N(x_i) \cap B| \leq \sum_{i=1}^{d} (|N(x_i) - (N(x_i^+) \cup N(z))| - 1)\end{aligned}$$

By (1) and (3) the equality holds in (1) and (2). In particular, we have

$$N(x_i) - (N(x_i^+) \cup N(z)) - \{z\} = N(x_i) \cap B \subset B. \tag{4}$$

By the same arguments we also have

$$N(x_i) - (N(x_i^-) \cup N(z)) - \{z\} = N(x_i) \cap \{x_1^-, \ldots, x_d^-\} \subset \{x_1^-, \ldots, x_d^-\}. \tag{5}$$

Now we claim $x_i^+ = x_{i+1}^-$ for each $i$, $1 \le i \le d$. (We consider $x_{d+1} = x_1$.)

Assume the contrary. Then we may assume $x_d^+ \ne x_1^-$. Then $x_1^- \in N(x_1) - N(z) - \{z\}$. If $x_1^- \notin N(x_1^+)$, then by (4) we have $x_1^- = x_d^+$, a contradiction. Thus, we have $x_1^- \in N(x_1^+)$. This implies $x_1^+ \in N(x_1) \cap N(x_1^-)$. On the other hand, $x_2^- \notin N(x_1^-)$ since $C$ is a longest cycle. Therefore, $x_1^+ \ne x_2^-$, and there exists a vertex $v \in x_1^{++} \overrightarrow{C} x_2^-$ with $v \notin N(x_1) \cap N(x_1^-)$ and $x_1^+ \overrightarrow{C} v^- \subset N(x_1) \cap N(x_1^-)$. Since $v \in x_1^{++} \overrightarrow{C} x_2^-$, $v \notin N(z)$. Since $\mathrm{diam}(G) = 2$, we have $N(v) \cap N(z) \ne \emptyset$, say $u \in N(v) \cap N(z)$. (Note that even if $d = 1$ the arguments in this paragraph holds by putting $x_2 = x_1$.)

If $u \notin V(C)$, then $v \overrightarrow{C} x_1^- v^- \overleftarrow{C} x_1 z u v$ is a cycle which is longer than $C$, a contradiction. Therefore, we have $u \in V(C)$. This implies $u = x_i$ for some $i$, $1 \le i \le d$.

Assume $i \ne 1$. Since $v \in N(x_i) - N(z) - \{z\}$ and $v \notin \{x_1^+, \ldots, x_d^+\}$, we have $v \in N(x_i^+)$ by (4). Then $z x_i \overleftarrow{C} v x_i^+ \overrightarrow{C} x_1^- v^- \overleftarrow{C} x_1 z$ is a cycle which is longer than $C$, a contradiction. Thus, we have $i = 1$. Then $v \notin N(x_1^-)$.

Now we have $v \in N(x_1) - (N(x_1^-) \cup N(z)) - \{z\}$, and by (5) this is possible only if $d \ge 2$ and $v = x_2^-$, or $d = 1$ and $v = x_1^-$. If $d \ge 2$, $z x_2 \overrightarrow{C} x_1^- x_1^+ \overrightarrow{C} x_2^- x_1 z$ is a cycle longer than $C$. This is a contradiction. If $d = 1$, consider the component $H$ of $G - V(C)$ that contains $z$. Since $G$ is 2-connected and $C$ is a longest cycle, there exists an edge $ab \in E(G)$ with $a \in V(C) - \{x_1, x_1^+, x_1^-\}$ and $b \in V(H) - \{z\}$. Let $P$ be a path joining $z$ and $b$ in $H$. Since $a \ne x_1$, $a^- \ne x_1^-$ and hence $a^- \in N(x_1)$. Then $z \overrightarrow{P} b a \overrightarrow{C} x_1^- x_1^+ \overrightarrow{C} a^- x_1 z$ is a cycle longer than $C$. This is a contradiction, and the claim follows.

By the above claim, we have $V(C) = A \cup B$. Since $z$ can be any vertex in $G - V(C)$ with $N_C(z) \ne \emptyset$, each component in $G - V(C)$ consists of one vertex. This implies $d \ge 2$. Then for each $y \in V(G) - V(C)$, $N(y) = A$ or $N(y) = B$. However, if $N(y) = B$ for some $y \in V(G) - V(C)$, then $z x_2 x_1^+ y x_2^+ \overrightarrow{C} x_1 z$ is a cycle longer than $C$. This is a contradiction, and hence $N(y) = A$ for each $y \in V(G) - V(C)$. Let $|V(G) - V(C)| = m$. By the assumption $m \ge 1$. Since $x_{i+1} \overrightarrow{C} x_i z x_{i+1}$ is a longest cycle, we have $N(x_i^+) = A$ for each $i$, $1 \le i \le d$. Thus, $B \cup (V(G) - V(C))$ is an independent set and $B \cup (V(G) - V(C)) \subset N(x_i)$ for each $i$, $1 \le i \le d$.

Since $z x_1 x_1^+$ is an induced path in $G$, $\deg z + \deg x_1^+ \ge |N(z) \cup N(x_1^+) \cup N(x_1)| - 1$. However, since $N(z) = N(x_1^+) = A$ and $B \cup (V(G) - V(C)) \subset N(x_1)$, we have $N(z) \cup N(x_1^+) \cup N(x_1) = V(G)$ and

$$2d = \deg z_1 + \deg x_1^+ \ge 2d + m - 1.$$

This implies $m \le 1$ and hence $m = 1$. Therefore, we have $K_{d,d+1} \subset G \subset K_d + (d+1)K_1$. □

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