On Perfect Dominating Sets in Hypercubes and Their Complements *

Joaquim Borges
Department d'Informàtica
Universitat Autònoma de Barcelona
08193-Bellaterra (Spain)

Italo J. Dejter
Department of Mathematics and Computer Sciences
University of Puerto Rico
Rio Piedras
Puerto Rico 00931

ABSTRACT. The complements of the perfect dominating sets of the n-cube, for $n \leq 8$, are characterized as well as some outstanding vertex-spanning edge-partitions of them involving the Fano plane, as a contribution to the study of distance-preserving regular subgraphs of hypercubes.

1 Introduction

1.1 Hypercubes

Given a positive integer n, the n-cube Q_n is the graph whose set of vertices is the set of n-tuples over $\mathbb{Z}_2 = \{0, 1\}$ such that any two vertices which as n-tuples differ in exactly one coordinate determine an edge of Q_n. Those edges of Q_n whose endvertices differ in the i-th coordinate, where $1 \leq i \leq n$, constitute a 1-factor of Q_n, referred to as the i-th parallel 1-factor of Q_n. The n-cubes defined above are called *hypercubes*. They constitute a family of graphs that, because of their symmetry properties, became of most importance in modern computer architecture science (see [4]), even though they were conceived initially as the natural setting of error-correcting codes. The study of distance preserving regular subgraphs of hypercubes was also

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considered in [4]. We attempt to contribute to the previous work by considering the complements of perfect dominating set of hypercubes, that we pass to define.

1.2 Perfect Dominating Sets

Given a graph G and a subset S of vertices of G, we say that S is a *perfect dominating set* (PDS) of G if every vertex of G not in S is neighbor to exactly one vertex of S.

Examples. The set of vertices of G itself is a PDS by definition, but we are interested only in proper PDS's.

Each parallel 1-factor in Q_n separates Q_n into two subcubes which are $(n-1)$-cubes, for $n \geq 1$. The set of vertices of each one of these subcubes is a proper PDS of Q_n, (see Lemma 3.1).

Perfect codes provide a large class of PDS's. There are two classes of these codes: linear and nonlinear. Some nonlinear codes are defined in the literature, see for example [3].

A *perfect 1-error correcting code* P in Q_n is a set of isolated vertices of Q_n such that:

1. the minimum distance in P is 3 and
2. the distance $d(v, P)$ from any vertex v not in P to P is equal to 1.

In this case, it is necessary and sufficient for the existence of such a code that n be of the form $2^r - 1$, where r is a nonnegative integer. Such a code is said to be *linear* if it is a linear subspace of Q_n looked upon as the vector space of dimension n over the Galois field of two elements.

Clearly, a perfect 1-error correcting code is a PDS. □

2 Known Examples of Nonisolated PDS's

2.1 Weichsel's PDS's, Results and Conjecture

Weichsel [4] considered PDS's in hypercubes as a way of generating distance-preserving regular subgraphs. The motivation for this starts with the NP-completeness of whether a graph can be embedded as a subgraph of the n-cube. Naturally enough, interest has centered into the study of particular classes of subgraphs of hypercubes, like the induced regular subgraphs that we treat in this paper. (Notice that every distance-preserving subgraph of a simple graph G is a vertex-induced subgraph of G, but that the converse of this assertion is not true).

We are interested in PDS's because their removal results in a subgraph in which each vertex has degree one less than in the original graph. In fact, we have the following lemma, ([4]). If G is a graph, then let V(S) stand for the set of vertices of a subgraph S of G.

Lemma 2.1 *If G is a k-regular graph, then a PDS of G can be described as follows: If D is a $(k-1)$-regular induced subgraph of G, then $V(G)-V(D)$ is a PDS of G.*

We will identify a PDS S in Q_n with the induced subgraph $Q_n[S]$. Thus, this induced subgraph will be denoted simply by S. Thus, S could stand either for a PDS or for its induced subgraph.

Examples. A way to construct new PDS's out of old ones, was provided by Weichsel as follows: Let S stand for a PDS in Q_n. A PDS in the $(n+r)$-cube, where r is also a positive integer, is produced by assigning to each vertex $s \in S$ expressed by an n-tuple $(s_1, \ldots, s_n)$, all the vertices of Q_{n+r} whose first n coordinates are exactly $s_1, \ldots, s_n$, precisely in this order. The totality of all the assigned vertices in the $(n+r)$-cube, for all the vertices of S in the n-cube, constitutes a PDS in Q_{n+r} whose components are r-subcubes that use the last r parallel 1-factors of Q_{n+r}, namely the parallel $(n+1)$-th 1-factor, ..., and the parallel $(n+r)$-th 1-factor. If $r = 1$, this PDS generating technique will be referred as *doubling*, and in general, as *multiple doubling*. The technique and perfect 1-error correcting codes yield the family of nonisolated examples of PDS's that Weichsel conceived firstly. Based on these examples, Weichsel proved the following, [4]. □

Theorem 2.1 *All components of a PDS in the n-cube are subcubes of dimensions $\leq n$.*

Weichsel also formulated the next conjecture, known as the Uniformity Conjecture.

Conjecture 2.1 *All the components of a PDS in a hypercube should have the same dimension.*

To support this conjecture, Weichsel proved the following, [4].

Theorem 2.2 *Given a component r-subcube of a PDS of Q_n, it holds that $n - r$ is congruent either to 1 or 3 modulo 6.*

One of us, (Borges) found some additional conditions to narrow the possibility of PDS's not satisfying the Uniformity Conjecture and in particular showed that this conjecture holds for the n-cube, for $1 \leq n \leq 9$. Thus, for $n \leq 9$, all PDS's in Q_n have their components being subcubes of the same dimension.

In Section 3, we characterize PDS's in n-cubes, for $n \leq 8$, up to PDS's in Q_8 formed by a maximal number of components. In particular, we know of three different ways in which this maximal number of components is given: One arising from Weichsel's construction and those contained in the following two subsections.

2.2 Component Parallelism and the Felzenbaum PDS

Felzenbaum showed that, in the cases for which the Uniformity Conjecture is satisfied, the subcubes which are components of a PDS, all of the same dimension, are not necessarily parallel to each other. His example in support of this can be given as in [2] or as follows.

Example. The 8-cube Q_8 may be viewed as a product $Q_4 \times Q_4$ of a "giant" 4-cube G and a "tiny" 4-cube T, where each vertex of G may be looked upon as "containing inside" a copy of T. We may represent the first four coordinates of a vertex of Q_8 as the coordinates of G and the last four coordinates as the coordinates of T. To obtain the Felzenbaum PDS, we consider in G its eight vertices of even weight. These can be partitioned into four pairs of opposite vertices. Say that these pairs are denoted V_i, for $i = 1, 2, 3, 4$. On the other hand, it is easy to determine a 1-factor of T constituted by four pairs of edges, each formed by two opposite edges. Let these pairs be denoted by E_i, for $i = 1, 2, 3, 4$. We may reproduce the Felzenbaum PDS by selecting the endvertices of the edges in E_i from the copy of T associated with each vertex of the pair V_i for $i = 1, 2, 3, 4$. □

2.3 Projectivity Properties on Nonisolated PDS's

Observe that the Felzenbaum PDS in Q_8 induces a subgraph S of Q_8 whose components are 1-cubes (that is edges with their endvertices) along only four of the eight coordinate directions of the 8-cube. Also, there is a graph covering map from Q_8 onto Q_4 (namely onto T) that maps the Felzenbaum PDS onto a 1-factor of Q_4, namely the 1-factor cited in the Example above, that has two opposite edges in each coordinate direction of Q_4. (This type of properties are also shared by a generalization of the Felzenbaum PDS that takes place in Q_{16}, see [2]).

On the other hand, each one of Weichsel's nonisolated PDS's S (given above) on the n-cube Q_n admits a graph homomorphism from Q_n onto a smaller dimensional cube Q_{n-r} under which the components of S collapse onto the components of a PDS S' of Q_{n-r}. In particular, the components of S' may be the vertices of a perfect 1-error correcting code in Q_{n-r}, in which case $n - r$ must be one less than a power of 2.

We will show, in the following subsection, that the pair of projectivity properties mentioned in the last two paragraphs, satisfied respectively by the Felzenbaum PDS and those obtained by Weichsel's construction are not always the case for PDS's in hypercubes, by means of an example due to Dejter and Weichsel, [2].

2.4 The Dejter-Weichsel PDS

It is convenient for us now to introduce the *Fano plane*, constituted by the points $0,1,2,3,4,5,6$ and by the lines (where customary parentheses and commas to indicate vectors are deleted) $124, 235, 346, 450, 561, 602, 013$. We see that these defining lists are cyclic modulo 7 and that the correspondence Φ that associates to each Fano point its corresponding Fano line, in the order of the above enumerations, has some nice duality properties. In fact, from

$$\Phi(0) = 124, \quad \Phi(1) = 235, \quad \dots, \quad \Phi(6) = 013$$

we get the complementary point-line correspondence

$$\Psi(0) = 653, \quad \Psi(1) = 064, \quad \dots, \quad \Psi(6) = 542$$

and that $\Phi(-i)$ is composed by the negatives of the coordinates of $\Psi(i)$, and viceversa. All of this may be used to show that the following edges in Q_8 and those obtained from them by (all-coordinate) complementation have their endvertices forming a PDS as claimed at the end of the last subsection, having as components 1-cubes along the eight coordinate directions of Q_8. The expression of these edges is given by means of Rado's notation for elements of the 8-cube looked upon as a Boolean lattice:

$$(\emptyset, 7), (124, 1240), (235, 2351), (346, 3462),$$

$$(450, 4503), (561, 5614), (602, 6025), (013, 0136).$$

3 PDS's in Q_n, for $n \leq 8$

3.1 Some Enumerations of PDS's

Lemma 3.1 *For every $n \geq 1$, Q_n has $2n$ PDS's each with only one component isomorphic to Q_{n-1}.*

Proof: Clearly, Q_n has $2n$ $(n-1)$-subcubes. Let R be any one of these subcubes. Without loss of generality, let each vertex in R have the first coordinate equal to zero. Then any vertex $x \notin R$ has a 1 in the first coordinate position and it is at distance one from exactly one vertex in R. Hence R is a PDS with exactly one component. □

Lemma 3.2 *For every $n \geq 3$, Q_n has $4\binom{n}{3}$ PDS's, each one with two parallel, opposite components isomorphic to Q_{n-3}.*

Proof: We can choose 3 coordinate positions in $\binom{n}{3}$ different ways, and we can choose 8 different binary 3-vectors for these coordinate positions. Each choice defines an $(n-3)$-subcube. Let R be one of these subcubes. Let T be the $(n-3)$-subcube that is parallel and opposite to R, i.e. T fixes the same 3 coordinate positions but with different values at the 3 places. Clearly, any vertex not in R and not in T is at distance one from a vertex either in R or in T. Thus, R and T form a PDS in Q_n. As we have seen before, Q_n has $8\binom{n}{3}$ $(n-3)$-subcubes, each pair of opposite subcubes being a PDS. Hence, there are $4\binom{n}{3}$ such PDS's. □

3.2 Characterization of PDS's

In this subsection we will characterize all PDS's in n-cubes, when $n \leq 8$.

Lemma 3.3 *If S is a PDS in Q_n and S has a component T that is isomorphic to Q_{n-3}, then S is formed exactly by two parallel components, T and R. Moreover, R is isomorphic to Q_{n-3}.*

Proof: Let S be a PDS in Q_n with a component that is isomorphic to Q_{n-3}. We may assume without loss of generality that T fixes the first 3 coordinate positions equal to 0. Let R be the parallel and opposite $(n-3)$-subcube (R fixes the first 3 coordinate positions equal to 1). Clearly, if $x \in S - T$, then $x \in R$. Now consider the set X of vertices that have 110 in the first three coordinate positions respectively (X is also an $(n-3)$-subcube). Each vertex in X must be covered by a vertex in $S \cap R$. Thus $|S \cap R| \geq |X| = 2^{n-3}$ and $|R| = 2^{n-3}$ imply $R = R \cap S$. Thus, $S = T \cup R$. □

For a PDS we have a property similar to the celebrated *Sphere Packing Condition* for perfect codes:

Proposition 3.1 *Let $S \subseteq Q_n$ be a PDS with components $S_1, \ldots, S_m$ having dimensions $r_1, \ldots, r_m$ respectively. Then*

$$\sum_{i=1}^{m} 2^{r_i}(n - r_i + 1) = 2^n$$

Proof: The component S_i has 2^{r_i} vertices, each one has degree r_i in S_i, thus each one is adjacent to $n - r_i$ vertices not in S. Then the number of vertices of S_i and its "aura" (vertices at distance one apart from S_i) is $2^{r_i} + 2^{r_i}(n - r_i)$. Now, if we compute the sum of vertices for all the components and their "auras" we must obtain the total number of vertices in Q_n, that is, 2^n. □

Remark: If the uniformity conjecture is true we have $m(n-r+1) = 2^{n-r}$ where r is the dimension of each component. In this case $n - r$ must be

$2^t - 1$ for some non-negative integer t, which is a stronger condition than $n - r \equiv 1$ or 3 (mod 6). Also, we have that the number of components must be a power of 2. □

Theorem 3.1 *Let $S \subseteq Q_n$ be a proper PDS and $n \leq 8$.*

1. *If $n \leq 2$, then S is an $(n-1)$-subcube.*
2. *If $3 \leq n \leq 6$, then S is either an $(n-1)$-subcube or S is a pair of opposite $(n-3)$-subcubes.*
3. *If $n = 7$, then S is either a 6-subcube or a pair of opposite 4-subcubes or a perfect 1-error correcting code.*
4. *If $n = 8$, then S is either a 7-subcube or a pair of opposite 5-subcubes or a set of sixteen edges (that may be parallel or not).*

Proof: Let $S \subseteq Q_n$, $n \leq 8$, be a proper PDS with a component of S having dimension r. Theorem 2.2 gives that $n - r$ is congruent to 1 or 3 modulo 6. In addition, from Lemma 3.1, if S has an $(n-3)$-subcube, then S has exactly two components each one being an $(n-3)$-subcube. Assertions 1 and 2 follow directly from Theorem 2.2 and Lemma 3.1.

If $n = 7$, by Theorem 2.2 the possible dimensions of components of S are 6,4 and 0. The first yields an $(n-1)$-subcube, the second case yields a pair of $(n-3)$-subcubes, and the third case yields a PDS formed by sixteen 0-cubes, that is isolated vertices. They form a perfect 1-error correcting code. We know of the existence of such a code since 7 is one less than a power of 2, [3].

If $n = 8$, then by Theorem 2.2 the possible dimensions for components of S are 7,5 and 1. Thus S is either a 7-subcube or a pair of opposite 5-subcubes or a set of sixteen edges. This set of edges can be obtained by doubling a perfect 1-error correcting code in Q_7. However, we have seen examples with sixteen nonparallel edges, (the Felzenbaum and the Dejter-Weichsel PDS). □

4 Symmetry and Regular Edge-Partitions

4.1 Introduction

While n-cubes as well as the complements of linear perfect 1-error correcting codes in their respective $(2^r - 1)$-cubes are both vertex-transitive and edge-transitive graphs, it was found in [1] that the complement C_7 of the perfect code P_7 in Q_7 splits into two self-complementary cubic spanning subgraphs R_7 and B_7 which are edge-transitive but not vertex-transitive, and whose girth equals 10, (compare with the girth 6 of C_7 and the girth 4 of Q_7), where R_7 and B_7 stand for the colors red and blue with which we color edges.

Additional techniques may produce further splittings of distance-preserving regular subgraphs of hypercubes. As an example of this observation and a general contribution to this study of induced regular subgraphs of cubes, we present in Subsection 4.3 a splitting of the complement C_8 of the Dejter-Weichsel PDS in Q_8 into a 1-factor F_8 and two subgraphs R_8 and B_8 which are spanning and cubic, but not self-complementary, in Q_8. Neither C_8 nor R_8 nor B_8 are vertex-transitive not edge-transitive. The 1-factor F_8 is formed by those edges opposite to edges of the Dejter-Weichsel PDS in squares of Q_8.

4.2 R_7 as a Covering of the Heawood Graph

In order to continue, it is convenient to present an 8-covering graph of the Heawood graph. This covering graph coincides with R_7. In fact, the Heawood graph H is the bipartite graph of incidence of points and lines of the Fano plane. We may depict H as in [2] or as in Figure 1, where each vertex is labeled with a symbol of the form δi, where δ is a sign $\pm$ and i is a Fano point. The neighbors of a vertex labeled $-i$ are labeled with $+j$, where j varies among the three components of $\Phi(i)$. The neighbors of a vertex labeled $+i$ are labeled with $-j$, where j runs on the three components of $\Psi(i)$. The edges of H are labeled so that if e is an edge with endvertices labeled $-i$ and $+j$, then e is labeled with k so that ijk is a Fano line.

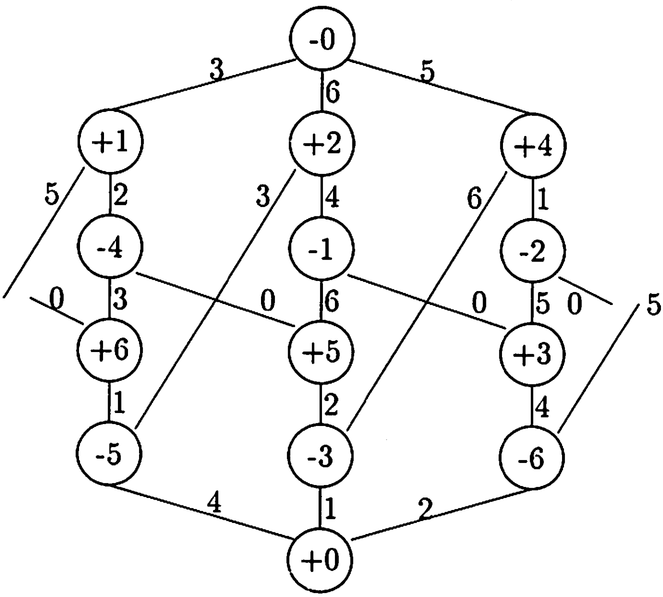

Figure 1: Heawood graph as a quotient graph of R_7

We want to define the projection graph map ρ corresponding to the claimed covering from R_7 onto H. In order to do this, let us denote the vertices of the perfect code in Q_7 with symbols of the form δk, where δ is a sign and $k = 0, 1, 2, 3, 4, 5, 6, 7$, so that $-7 = 0000000$; $+k$ is obtained from -7 by complementing the values at the j-th coordinates, where j runs on the components of $\Phi(k)$ and $-\delta k$ is obtained from δk by uniform complementation (of all coordinates). The inverse image through ρ of a vertex labeled δi in H is obtained from the codewords of the form δk by complementing their ith-coordinate.

How can we define the edges in R_7 so that ρ is well defined? Let e be an edge of H labeled with k. The endvertices of e are of the form $-i$ and $+j$. Then kij is a Fano line (or Steiner triple). For each of the even weight codewords, which are of the form $-t$ with $t = 0, 1, \ldots, 7$, there is exactly one 3-subcube q of Q_7 containing $-t$ and edges only in coordinate directions k,i and j. We trace in q the 3-path p departing from $-t$ and going respectively through edges in the directions i,k and j. Then we consider the middle edge of p as an edge in the inverse image of e via ρ. This provides the eight edges in this inverse image and defines ρ properly.

Both representations of H, as in [2] or as in Figure 1, are convenient to describe R_8 as a covering graph of H (where Figure 1 must indeed be bent so that edges partially represented on the left side have their continuation on the correspondingly labelled partially represented edges on the right side), but Figure 1 provides quickly that the longest path from any odd weight vertex v of R_7 has length equal to 8 and is realized by only one vertex at distance 8 from v, while the longest path from any even weight vertex u of R_7 has length 7 and is realized by exactly seven vertices at distance 7 from u. This shows immediately that R_7 is edge-transitive but not vertex-transitive. It can also be seen in Figure 1 that the girth of R_7 is equal to 10.

We remark that in each 3-subcube of Q_7 having a pair of opposite Hamming codeword vertices, in which case one of them is even weight and the other one is odd weight, the complement of this pair is a 6-cycle. We remark also that the complement C_7 of P_7 in Q_7 equals the edge-disjoint union of all the 6-cycles obtained this way, a totality of 56 6-cycles. These can be expressed by means of the seven Steiner triples (Fano lines) as in the definition of the Fano point-line correspondence Φ, and each one of these triples yields common coordinate directions for eight 3-subcubes of Q_7 as mentioned above, yielding the claimed 56 6-cycles. The spanning subgraph R_7 of C_7 happens to be the maximal edge-disjoint union of maximal independent edge-sets of these 56 6-cycles, selected by means of the cyclicity of our presentation of the Steiner triples. We will see subsequently a similar idea by presenting a partition of a 6-regular spanning subgraph of the complement of the Dejter-Weichsel PDS into two spanning 3-regular subgraphs.

4.3 Partitioning the Dejter-Weichsel PDS in Q_8

A convenient notation for the vertices of the Dejter-Weichsel PDS P_8 in Q_8 can be established as follows. Each one of them will be considered as an 8-tuple where the order of the coordinates will be taken as $1, 2, 3, 4, 5, 6, 0, 7$. Then a vertex of P_8 will be represented by a symbol of the form $\alpha\beta ab$, where α and β are signs and a and b are elements of $\{1, 2, 3, 4, 5, 6, 0, 7\}$ such that $a \neq b$, with the following meaning. Let $(-a, 0)$ be a vertex of Q_8, where $-a$ is a codeword of Q_7 as denoted above. Then a vertex of P_8 of the form $--ab$ is obtained from $(-a, 0)$ by complementation at the b-th coordinate. A vertex of the form $+-ab$ is obtained from $--ab$ by complementation at the a-th coordinate. A vertex of the form $\alpha + ab$ is obtained by uniform complementation of all the coordinates of $\alpha - ab$.

It is easy to recognize that C_8 has a 1-factor F_8 formed by all the edges parallel and opposite to edges of P_8 in squares of Q_8. The edges of P_8 are all the edges of the form $(\alpha\beta ab, \alpha'\beta ab)$, where $\alpha' = -\alpha$. Let $D_8 = C_8 - F_8$. Then D_8 is the edge-disjoint union of 6-cycles that appear in pairs related by uniform complementation. So we may as well give an edge-partition of the quotient graph X_8 of D_8 via the equivalence relation whose classes are of the form $\{a, a'\}$, where a' is as above. The image of a vertex $\alpha\beta ab$ in X_8 will be denoted αab. The partition of X_8 into 6-cycles is done with the notation:

$$ab.cde = (-ac, +bd, -ae, +bc, -ad, +be),$$

if $a \neq 7$ and with the notation

$$\delta a.cde = (\delta 7c, \delta ad, \delta 7e, \delta ac, \delta 7d, \delta ae),$$

where δ is a sign. Then a listing of the 6-cycles of X_8 is given by

$$02.136 \quad 04.265 \quad 01.453 \quad 06.157 \quad 05.237 \quad 03.467 \quad -0.412 \quad +0.365$$

and six other collections of other 6-cycles obtained from this one by all uniform translations modulo 7 of the intervening digits of the form $0, 1, 2, 3, 4, 5, 6$.

Now we will show how to partition the corresponding 6-cycles in D_8 obtained by lifting from these 6-cycles of X_8 into two independent sets of three edges each so that the resulting red and blue graphs R_8 and B_8 are 3-cubic. We may use the following notation to express these cycles:

$$\begin{array}{lll} 02.136 = x1z10z00, & 04.265 = 1x01zz00, & 01.453 = 11zxz000, \\ 06.157 = x101z00z, & 05.237 = 1xz1000z, & 03.467 = 110x0z0z, \\ -0.412 = xx0x0000, & +0.365 = 11x1xx10, & \end{array}$$

where for example cycle 02.136 in X_8 is covered by two 6-cycles of D_8, one of which is formed by the independent sets of edges shown in the following two columns:

$$x1110000 \quad x1010100$$

$$11z10100 \quad 01z10000$$

$$01010z00 \quad 11110z00$$

where x or z represent the coordinate that varies from 0 to 1 or viceversa, for each represented edge in question. Symbol x, respectively z, is used for those varying coordinate entries in edges for which the other entries that vary in the other edges of the independent set in question are unequally valued, respectively equally valued.

We agree to take as red edges those in the following list of columns corresponding to the explicited row of 6-cycles expressed above:

$x1010100$	$1x011000$	$111x0000$
$01x10000$	$10010x00$	$1100x000$
$11110x00$	$1101x100$	$11x11000$

$x1011000$	$1x110000$	$110x0100$
$1101x001$	$11x10001$	$11010x01$
$0101000x$	$1001000x$	$1100000x$

$010x0000$	$11x10110$
$x0010000$	$11011x10$
$1x000000$	$1111x010$

and obtained from these by rotating the coordinates $1,2,3,4,5,6,0$ while fixing the value (fixed or x) of the last 7-th coordinate, and also obtained from all these resulting edges by uniform complementation of all the coordinates (where x complements to x itself).

A description of the image of R_8 in X_8 follows. This may be taken as an atlas (collection) of seven chart subgraphs that we denote $R_{8,i}$, with i in $\{1,2,3,4,5,6,0\}$, where only $R_{8,0}$ is shown here, and the others may be obtained by uniform translation modulo 7 of $1,2,3,4,5,6,0$ and leaving symbol 7 itself fixed in all labels. Figure 2 depicts red chart $R_{8,0}$ (but the reader should bend this figure as indicated for Figure 1), where edges are labeled according to the coordinate direction along which they happen in Q_8; vertices are labeled, inside respective circles representing them, with their given notations in X_8 and some vertices have other labels j outside their representing circles, denoting to which other charts $R_{8,j}$ they also belong, so that the charts of this atlas may be used to recompose R_8.

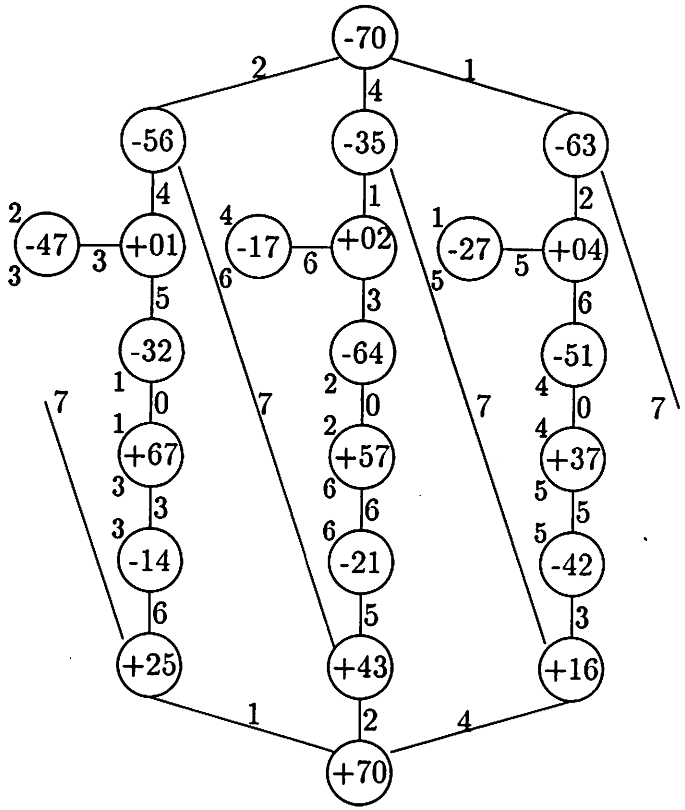

Figure 2: Red chart $R_{8,0}$ in X_8

A similar task is taken into account in Figure 3, where a corresponding blue chart $B_{8,i}$ is depicted, from which B_8 may be recomposed. From this information, we get that R_8 and B_8 are not isomorphic, so they are not self-complementary in the complement C_8 of P_8 in Q_8.

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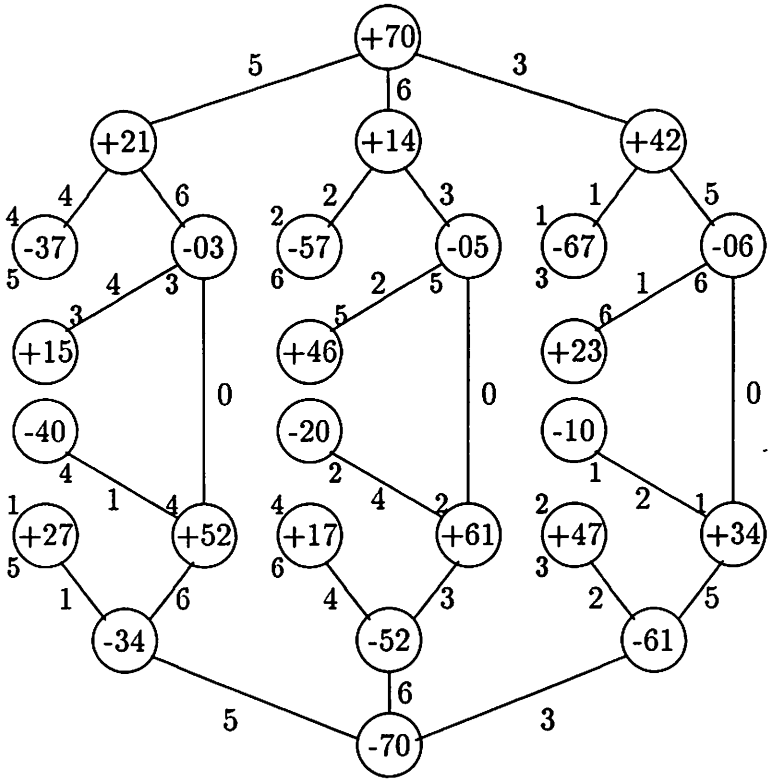

Figure 3: Blue chart $B_{8,0}$ in X_8

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