On Locally Semi-Complete Digraphs

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ABSTRACT. Several theorems about hamiltonian, pan-cyclic and other properties of locally semi-complete digraphs are obtained in this paper.

1 Introduction

A digraph D consists of a pair $V(D)$, $E(D)$, where $V(D)$ is a finite set of vertices and $E(D)$ is a set of ordered vertex pairs xy, called *arcs*. All digraphs considered in this paper have no loop and parallel arcs in the same direction (parallel arcs in opposite direction are allowed). If xy is an arc of digraph D, then we say that x dominates y and we will use the notation $x \to y$, or simply, xy to denote this. If A and B are subsets of $V(D)$, such that there is a complete connection between A and B and all arcs between vertices in A and vertices in B are directed toward B, we say that A dominates B and use the notation $A \to B$ to denote this fact. For any subset A of $V(D) \cup E(D)$, $D - A$ denotes the subdigraph obtained by deleting all vertices of A and their incident arcs and then deleting the arcs of A still present. The subgraph induced by a vertex set A of D is defined as $D - (V(D) - A)$ and is denoted by $D(A)$. We may write $x \in D$ instead of $x \in V(D)$ or $x \in E(D)$, but the meaning will always be clear. A digraph is connected if its underlying graph is connected.

A dipath is a digraph with vertex set $\{x_1, x_2, \dots, x_n\}$ and arc set $\{x_1x_2, x_2x_3, \dots, x_{n-1}x_n\}$, such that all the vertices and arcs shown are distinct. We call such a dipath an (x_1, x_n)-path and denote it by $x_1 \to x_2, x_2 \to$

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$x_3, \ldots, x_{n-1} \rightarrow x_n$, or simply $x_1 x_2 \ldots x_n$. In the following an (x, y)-path always means a dipath from x to y. A directed cycle is defined as a dipath, the only difference being that the endvertices are the same.

We let $N^+(v)$ (respectively, $N^-(v)$) denote the set of vertices u in $V(D)$ for which $v \rightarrow u$ (respectively, $u \rightarrow v$). Sometimes we shall call $N^+(v)$ (respectively, $N^-(v)$) the outset of v (respectively, the inset of v).

A digraph $D = (V(D), E(D))$ is called a *semi-complete* digraph if for each pair of distinct vertices x, $y \in V(D)$, either xy or yx belongs to $E(D)$ (it could be both in D). Obviously, each tournament is a semi-complete digraph.

A digraph $D = (V, A)$ is called *local semi-complete* digraph if for each $v \in V(D)$, the induced subgraphs $D(N^-(v))$ and $D(N^+(v))$ are both semi-complete. (This definition was introduced by Bang-Jensen [1])

A digraph is *strongly connected* if for each pair of vertices x and y, there is a path from x to y and a path from y to x.

2 Vertex pancyclic and non-complete cycles

It is well-known that every strongly connected tournament (or semi-complete digraph) has a Hamilton cycle and is vertex-pancyclic. A natural question is that whether or not a strongly connected, locally semi-complete digraph has a similar property. In [1] J. Bang-Jensen proved that a strongly connected, locally semi-complete digraph has a Hamilton cycle and other related results. Obviously, strongly connected, locally semi-complete digraphs do not generally have the property of being vertex-pancyclic. In this section and the next section, the property of being vertex-pancyclic will be further investigated.

Definition 2.1: A cycle C of a digraph D is called complete if the induced subgraph $D(V(C))$ is semi-complete.

It was found in [1] that a non-complete cycle in a locally semi-complete digraph is extendible. In this paper, we will give a further structural discussion about non-extendible complete cycles in Lemma 2.2, which includes Corollary 3.6 in [1], and will be very useful in the later discussion in this paper.

Lemma 2.2. *Let D be a locally semi-complete digraph and $C = v_1 \ldots v_r v_1$ be a cycle of D and u be a vertex adjacent to some vertex of C. Then either D has a cycle C' such that $V(C') = V(C) \cup \{u\}$ or the induced subgraph $D(V(C) \cup \{u\}$ is semi-complete, but not strongly-connected (that is, $D(V(C))$ is semi-complete and either $uv_i \in E(D)$ and $uv_i \notin E(D)$ for all $i = 1, \ldots, r$, or $v_u i \in E(D)$ and $uv_i \notin E(D)$ for all $i = 1, \ldots, r$).*

Proof: Assume that $D(V(C) \cup \{u\})$ has no Hamilton cycle. Without loss of generality, let u be dominated by a vertex v_μ of C. Since both $v_{\mu+1}$

and u are dominated by v_μ, both $v_{\mu+1}$ and u belong to a semi-complete subgraph of D. That is, there is at least one arc between $v_{\mu+1}$ and u. If $uv_{\mu+1}$ is an arc of D, then C can be extended by adding u between v_μ and $v_{\mu+1}$. Thus, we must have that

$$v_{\mu+1}u \text{ is an arc of } D \text{ if } v_\mu u \text{ is an arc of } D.$$

Obviously, we have that $v_i u$ is an arc of D for every $i = 1, \ldots, r$. Thus, $V(C) subseteq N^-(u)$ and therefore, $V(C) \cup \{u\}$ is contained in a semi-complete subgraph of D which is not strongly connected.

This proves Lemma 2.2. □

Corollary 2.3 (Bang-Jensen [1]). *If D is a strongly connected, locally semi- complete digraph of order n and a verse v is contained in a cycleC of length at least $\lceil \frac{n}{2} \rceil$, then v is contained in cycles of all possible lengths h for $|C| \leq h \leq n$.*

Proof: By Lemma 2.2, we may assume that C is complete and dominates (or is dominated by) every vertex adjacent to C. Since D is strongly connected, let P be a shortest path from a vertex dominated by C to a vertex dominating C. Since P is shortest, $V(P) \cap v(c) = \phi$ and $|V(P)| \leq |C|$. Thus replacing a segment Q of $C \setminus \{v\}$ such that $|V(Q)| = |V(P)| - 1$, we obtain a cycle containing v of length $|C| + 1$.

The proof is completed. □

Call a cycle C in a digraph extendible if C can be extended to a directed cycle of length $|C| + 1$. Since a non-complete cycle C in a connected locally semi-complete digraph is always extendible provided that C is not a hamiltonian cycle, we have the following corollary of Lemma 2.2.

Corollary 2.4. *A connected, locally semi-complete digraph D is strongly connected and has a hiamiltonian cycle if D contains a non-complete cycle.*

The next lemma is well-known (Moon's Theorem).

Lemma 2.5. *Every strongly connected tournament and semi-complete digraph is vertex-yancyclic.*

By Lemma 2.2, every non-complete cycle is extendible. How about a vertex which is not contained in any non-complete cycle? If this happens, the next proposition will give the answer.

Proposition 2.6. *Let D be a strongly connected locally semi-complete digraph and v be a vertex of D. If a vertex v is not contained in any non-complete cycle, then D is semi-complete.*

Proof: By Lemma 2.5, assume that D is not semi-complete. Let C be a longest cycle of D containing v. Since v is not contained in any non-complete cycle of D and D is not semi-complete, C is complete and C is not a hamiltonian. By Lemma 2.2, each vertex of $V(D)\setminus V(C)$ adjacent to some vertex of C dominates C or is dominated by C. Since D is strongly connected, let $P = x_1 \dots x_2$ be a shortest path in $D\setminus V(C)$ from a vertex dominated by C to a vertex dominating C. Thus C can be extended by adding P between a pair of consecutive vertices of C. This contradicts that C is a longest cycle containing v.

This proves Proposition 2.6. □

The following theorem is a corollary of Lemma 2.2, Corollary 2.3, Lemma 2.5, and Proposition 2.6.

Theorem 2.7. *Let D be a strongly-connected, locally semi-complete digraph of order n and v be a vertex of D. If v is contained in a cycle C of length k ($k < n$) and one of the following conditions holds, then v is contained in all cycles of lengths h, where $k \le h \le n$:*

(i) C is non-complete;

(ii) v is not contained in any non-complete cycle;

(iii) $k \ge \lceil \frac{n}{2} \rceil$.

3 Vertex-pancyclic and locally strongly connected digraph

Definition 3.1: A vertex v of a digraph is called locally strongly connected if the induced subgraph $D[N^+(v) \cup N^-(v) \cup \{v\}]$ is strongly connected. A digraph is locally strongly connected if every vertex of the digraph is locally strongly connected.

It is obvious that every strongly connected tournament D is locally strongly connected since $D[N^+(v) \cup N^-(v) \cup \{v\}] = D$. And it is well-known (Moon's Theorem) that a strongly connected tournament is vertex-pancyclic. A natural question is that whether or not a locally strongly connected, semi-complete digraph has a similar property. The following theorems give a positive answer.

Theorem 3.2. *Let D be a connected locally semi-complete digraph and v be a locally strongly connected vertex of D. Then v is contained in cycles of all possible lengths.*

Proof: Since $D_v = D[N^+(v) \cup N^-(v) \cup \{v\}]$ is strongly connected, there is an arc from $N^+(v)$ to $N^-(v)$. Thus v is contained in a cycle of length three. Let C be a cycle of D containing v and assume that C is not a Hamilton cycle and v is not contained in any cycle of length $|V(C)|+1$. Let w be a vertex of $D\setminus V(C)$ adjacent to some vertex of C. By Lemma 2.2, either C dominates the vertex w or w dominates C. Without loss of generality, assume that C dominates the vertex w. Since $w \in N^+(v)$ and the v is locally strongly connected, there is a dipath $P = x_1 \dots x_t$ from $w = x_1$ to some vertex $x_t \in N^-(v)$ in $D_v (= D[N^+(v) \cup N^-(v) \cup \{v\}])$.

Now let $s' = \min\{i\colon x_i \in V(P) \cap N^-(v)\}$. We claim that $\{x_1, \dots, x_{s'}\} \cap V(C) = \phi$. Assume that $V(P) \cap V(C) \neq \phi$ and let $s'' = \min\{i\colon x_{i+1} \in V(P) \cap V(C)\}$. Since $x_{s''}$ dominates some vertex of C, by Lemma 2.2, $v_{s''}$ dominates C. Hence $x_{s''} \in V(P) \cap N^-(v)$ and it proves our claim. Since $x_{s'-1} \in N^+(v) \setminus V(C)$ and $x_{s'} \in N^-(v) \setminus V(C)$, by Lemma 2.2 again, we have that C dominates $x_{s'-1}$ and is dominated by $x_{s'}$. Let $C = v_1 \dots v_r v_1$ where $v = v_1$. The cycle $v_1 x_{s'-1} x_{s'} v_3 \dots v_r v_1$ is a cycle of length $|V(C)|+1$.

This proves Theorem 3.2. □

Corollary 3.3. *If a connected locally semi-complete digraph D contains a locally strongly connected vertex, then D is pan-cyclic and D is strongly connected.*

Proposition 3.4. *Let C be a cycle of length r in a locally semi-complete digraph D. If there is a locally strongly connected vertex w adjacent to C, then there is a cycle C' of length $r+1$ in D. Further, for any vertex v of C, we can always find a cycle C' of length $r+1$ containing v and either adjacent to w or containing w.*

Proof: Let $C = v_1 \dots v_r v_1$ be a cycle of D. Let w be a locally strongly connected vertex of D which is adjacent to C. By contradiction, suppose that there is no cycle containing $V(C)$ with length $r+1$. Without loss of generality, assume that w dominates some vertex of C. Then $wv_j \in E(D)$ for $1 \leq i \leq r$ by Lemma 2.2. Since $D[\{w\} \cup N^+(w) \cup N^-(w)]$ is strongly connected, we can find a shortest path from $V(C)$ to $N^-(w)$, say $P_1 = v_1 w_1 \dots w_q$ in $D[\{w\} \cup N^-(w) \cup N^+(w)]$. Since P_1 is a shortest path, it follows that $\{w_1, \dots, w_q\} \subset V(D) - V(C)$ and $ww_i \in E(D)$ for $1 \leq i \leq q-1$.

Since w_q is the only vertex of $N^-(w)$ in P, and

$$\{w_1, \dots, w_{q-1}, v_1, \dots, v_r\} subseteq N^+(w),$$

$D\{w_1, \dots, w_{q-1}, v_1, \dots, v_r\}$ is semi-complete. Note that P_1 is shortest and $w_k v_h \in E(D)$ for $2 \leq k \leq q-1$. Then $v_1 w_1 w_2 v_3 \dots v_r v_1$ if $q > 2$ or $v_1 w_1 w_2 w v_4 \dots v_r v_1$ if $q = 2$ or $v_1 w_1 w v_3 \dots v_r v_1$ if $q = 1$ is an $(r+1)$-cycle where v_1 is any given vertex of C.

This proves Proposition 3.4. □

Theorem 3.5. *Let D be a connected locally semi-complete digraph and u be a vertex of D which is adjacent to a locally strongly connected vertex v of D. Then u is contained in cycles of length h, for each $h = 4, 5, \dots, n$.*

Proof: By Proposition 3.4 and Theorem 3.2, we only need to show that there is a cycle of length 3 or 4 containing u. Without loss of generality, assume that $uv \in E(D)$. Since $D[\{v\} \cup N^+(v) \cup N^-(v)]$ is strongly connected, we can find a shortest cycle containing uv in $D[\{v\} \cup N^+(v) \cup N^-(v)]$, say $C = uvu_1 \dots u_q u$. If $1 \leq q \leq 2$, we are done. Now assume that $q \geq 3$. Then $u_i v \in E(D)$ for $2 \leq i \leq q$ by the choice of C, and further more, $D[u_2, \dots, u_q, u]$ is semi-complete. Note that C is shortest, $uu_{q-1} \in E(D)$. Thus $uu_{q-1}u_q u$ must be a cycle of length 3.

This proves the Theorem 3.5. □

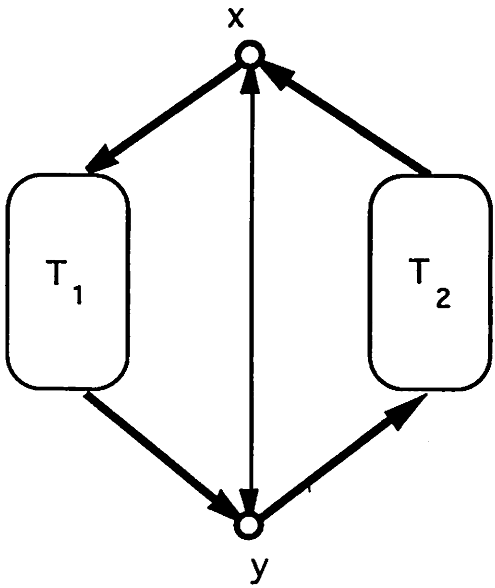

Figure 1

We should notice that a locally strongly connected, locally semi-complete digraph may not be semi-complete. (See figure 1). Thus, Theorem 3.2 implies Moon's Theorem, but not vice versa. The example D given in Figure 1 is constructed as follows: T_1, T_2 are two semi-complete digraphs and x, y are two vertices such that there are two opposite arcs between x, y, x dominates T_1 and is dominated by T_2, y dominates T_2 and is dominated by T_1.

References

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