Bounds on an Independent Distance Domination Parameter

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ABSTRACT. Let $m \geq 1$ be an integer and let G be a graph of order n. A set $\mathcal{D}$ of vertices of G is a m-dominating set of G if every vertex of $V(G) - \mathcal{D}$ is within distance m from some vertex of $\mathcal{D}$. An independent set of vertices of G is a set of vertices of G whose elements are pairwise nonadjacent. The minimum cardinality among all independent m-dominating sets of G is called the independent m-domination number and is denoted by $id(m, G)$. We show that if G is a connected graph of order $n \geq m + 1$, then $id(m, G) \leq (n + m + 1 - 2\sqrt{n}) / m$, and this bound is sharp.

1 Introduction

In this paper, we use fairly standard graph theoretic terminology and notation. For example, for a connected graph G, the distance $d(u, v)$ between two vertices u and v is the length of a shortest u-v path. If S is a set of vertices of G and v is a vertex of G, then the *distance from* v *to* S, denoted by $d_G(v, S)$ or simply $d(v, S)$, is the shortest distance from v to a vertex of S.

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Let $G = (V, E)$ be a graph with vertex set V and edge set E. If X and Y are subsets of V, X *dominates* Y if and only if each vertex of $Y - X$ is adjacent to some vertex of X. In particular, if X dominates V, then X is called a *dominating set* of G. An independent set of vertices is a set of vertices whose elements are pairwise nonadjacent. The fact that every maximal independent set of vertices in a graph is also a minimal dominating set motivated Cockayne and Hedetniemi [12] in 1974 to initiate the study of '*independent domination*' in graphs. A dominating set of vertices in a graph that is also an independent set is called an *independent dominating* set. The minimum cardinality among all independent dominating sets of a graph G is called the *independent domination number* of G and is denoted by $i(G)$. The parameter $i(G)$ has received considerably attention in the literature (see, for instance, [1, 2, 6, 15, 29]). For an excellent bibliography on dominating and independent dominating sets we refer the reader to [23].

In this paper, we extend the definition of independent dominating sets in graphs. Let $m \geq 1$ be an integer and let $G = (V, E)$ be a graph. In [24], if X and Y are subsets of V, then the set X is said to *m-dominate* Y if and only if each vertex of $Y - X$ is within distance m from some vertex of X. In particular, if X m-dominates V, then X is defined to be an *m-dominating set* of G. An m-dominating set of vertices in a graph that is also an independent set we call an *independent m-dominating* set. The *independent m-domination number* $id(m, G)$ of G is the minimum cardinality among all independent m-dominating sets of G. Thus $\mathcal{D}$ is an independent 1-dominating set of G if and only if $\mathcal{D}$ is an independent dominating set of G. Hence $id(1, G) = i(G)$.

We show that if G is a connected graph of order $n \geq m + 1$, then $id(m, G) \leq (n + m + 1 - 2\sqrt{n}) / m$, and this bound is sharp.

Results on the concept of m-domination in graphs have been presented by, among others, Bascó and Tuza [3, 4], Beineke and Henning [5], Bondy and Fan [7] , Chang [8], Chang and Nemhauser [9, 10, 11], Fraisse [15], Fricke, Hedetniemi, and Henning [16, 17], Fricke, Henning, Oellermann, and Swart [18], Hattingh and Henning [21, 22], Henning, Oellermann, and Swart [24, 25, 26, 27, 28], Meir and Moon [30], Mo and Williams [31], Slater [32], Topp and Volkmann [33], and Xin He and Yesha [34].

2 Known results

Let v be a vertex of a graph G, and let m be a positive integer. Then the set of all vertices of G different from v and at distance at most m from v in G is defined in [15] as the *m-neighbourhood* of v in G and is denoted by $N_m(v)$. We begin by stating the following result from [26], which will prove useful to us later.

Theorem A *For $m \geq 1$, if G is a connected graph of order at least $m+1$, then there exists a minimum m-dominating set $\mathcal{D}$ of G such that for each $v \in \mathcal{D}$, there exists a vertex $w \in V(G) - \mathcal{D}$ at distance exactly m from v such that $N_m(w) \cap \mathcal{D} = \{v\}$.*

Next we mention known upper bounds on $id(m, G)$ for a connected graph G. The following result is due to Gimbel and Vestergaard [20].

Theorem B *If G is a connected graph of order $n \geq 2$, then*

$$i(G) \leq n + 2 - 2\sqrt{n},$$

and this bound is sharp.

That the bound given in Theorem B is sharp, may be seen by considering the graph G obtained from a complete graph on $k+1$ vertices by attaching to each of its vertices k disjoint paths of length 1. Then $n = (k+1)^2$ and $i(G) = k^2 + 1$, so $i(G) = n + 2 - 2\sqrt{n}$.

Since $i(G) = id(1, G)$, Theorem B gives a sharp upper bound on $id(1, G)$ for a connected graph G. For $m \geq 2$, Beineke and Henning [5] established the following upper bound on $id(m, G)$ for a connected graph G.

Theorem C *For $m \geq 2$, if G is a connected graph of order $n \geq m$, then*

$$id(m, G) \leq \frac{n}{m},$$

and this bound is asymptotically best possible.

That the bound given in Theorem C is in a sense best possible, may be seen by considering the connected graph G constructed as follows: For k and b very large integers, let G be obtained from a complete graph on b vertices by attaching to each of its vertices k disjoint paths of length m. (The graph G is shown in Figure 1.) Then $id(m, G) = (b-1)k + 1$ and $n = |V(G)| = b(mk+1)$, so

$$\frac{id(m, G)}{n} = \frac{bk - k + 1}{bmk + b} = \frac{1 - \frac{1}{b} + \frac{1}{bk}}{m + \frac{1}{k}} \overset{b,k \to \infty}{\longrightarrow} \frac{1}{m}.$$

If we restrict our attention to trees, then Beineke and Henning [5] established the following upper bound on $id(m, T)$ for small values of m.

Theorem D *For $m \in \{1, 2, 3\}$, if T is a tree of order $n \geq m+1$, then $id(m, T) \leq n/(m+1)$, and this bound is sharp.*

That the bound given in Theorem D is sharp, may be seen by considering a tree T_m of order n obtained from a path on b vertices by attaching a path of length m to each vertex of the path. Then $id(m, T_m) = b = n/(m+1)$.

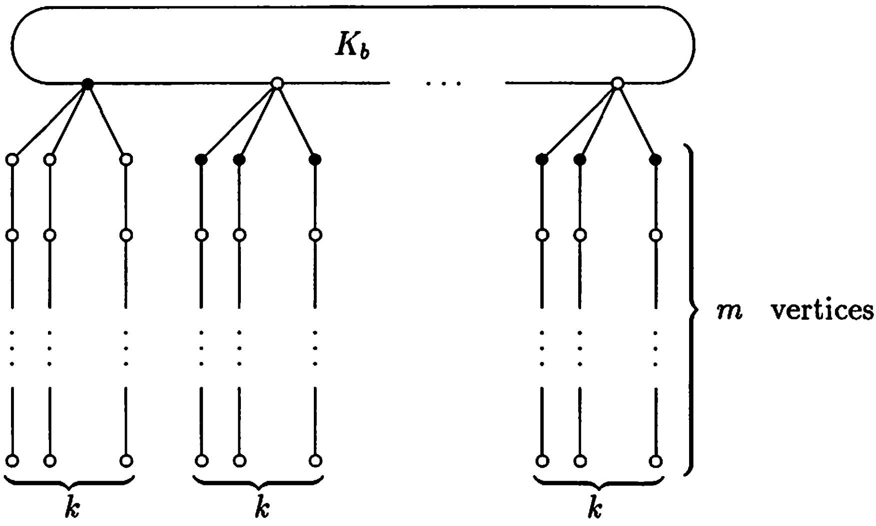

Figure 1: The graph G.

3 Complexity issues

In this section, we consider the decision problem corresponding to the problem of computing $id(m, G)$ for any fixed integer $m \geq 1$. We show that from a computational point of view the problem of finding $id(m, G)$ appears to be very difficult.

The following independent dominating set problem is known to be NP-complete (see Garey and Johnson [19]), and remains NP-complete for the class of bipartite graphs, as shown by Corneil and Perl [14].

PROBLEM: Independent dominating set ($IDOM$)
INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.
QUESTION: Is $i(G) \leq k$ (that is, is there a vertex set $S \subseteq V$ such that S is an independent dominating set with $|S| \leq k$)?

We will demonstrate a polynomial time reduction of the problem $IDOM$ to show that the following problem is also NP-complete.

PROBLEM: Independent m-dominating set ($ImDOM$)
INSTANCE: A graph $H = (V, E)$ and a positive integer $j \leq |V|$.
QUESTION: Is $id(m, H) \leq j$ (that is, is there a vertex set $S \subseteq V$ such that S is an independent m-dominating set with $|S| \leq k$)?

Theorem 1 *Problem $ImDOM$ is NP-complete, even when restricted to bipartite graphs.*

Proof: It is obvious that $ImDOM$ is a member of NP since we can, in polynomial time, guess at a subset of vertices, verify that its cardinality is at most j, and then verify that it is an independent m-dominating set. To see that $ImDOM$ is NP-complete it is next shown that a polynomial time algorithm for $ImDOM$ could be used to solve $IDOM$ in polynomial time.

Starting with an instance $G = (V, E)$ and $k \leq |V| = n$ and $|E| = q$ for problem $IDOM$, we can construct the graph H from G by attaching to each vertex of G a path of length $m-1$. Thus in forming H from G we have added $n(m-1)$ new vertices and $n(m-1)$ new edges. That is, $|V(H)| = nm$ and $|E(G)| = q + n(m-1)$, so the graph H can be constructed from G in time polynomial in n. We note that if G is bipartite, then so too is H.

Lemma 1 $id(m, H) = i(G)$.

Proof: Let I be an independent dominating set of G of cardinality $i(G)$. Then I is an independent m-dominating set of H, so $id(m, H) \leq |I| = i(G)$. On the other hand, let $\mathcal{D}$ be an independent m-dominating set of H of cardinality $id(m, H)$. The minimality of $\mathcal{D}$ implies that $\mathcal{D}$ contains at most one vertex from each path of length $m-1$ added to G in forming H. Let $v \in V(G)$ and consider the path of length $m-1$ attached to v in forming the graph H. If $\mathcal{D}$ contains a vertex w that belongs to this path, then neither v nor any vertex adjacent to v in G belongs to $\mathcal{D}$, for otherwise we may remove w from $\mathcal{D}$ to produce an independent m-dominating set of H of cardinality less that $|\mathcal{D}|$. Replacing the vertex w in $\mathcal{D}$ with the vertex v produces a new independent m-dominating set of H of cardinality $|\mathcal{D}|$. Hence we may assume that each vertex of $\mathcal{D}$ belongs to G. But then $\mathcal{D}$ forms an independent dominating set of G, so $i(G) \leq |\mathcal{D}| = id(m, H)$. Consequently, $id(m, H) = i(G)$. □

Lemma 1 implies that if we let $j = k$, then $i(G) \leq k$ if and only if $id(m, H) \leq j$. This completes the proof of Theorem 1. □

4 Bounds on $id(m, G)$ for a connected graph G

Since the problem of computing $id(m, G)$ appears to be a difficult one, it is desirable to find good upper bounds on this parameter. In this section, we prove the following result, which improves on that of Theorem C and generalizes that of Theorem B.

Theorem 2 *For $m \geq 1$, if $G = (V, E)$ is a connected graph of order $n \geq m+1$, then*

$$id(m, G) \leq \frac{n + m + 1 - 2\sqrt{n}}{m}$$

and this bound is sharp.

Proof: Let $\mathcal{D} = \{v_1, \ldots, v_b\}$ be a minimum m-dominating set of G that satisfies the statement of Theorem A. We introduce the following notation. For $i = 1, \ldots, b$, let

$W_i = \{w \in V - \mathcal{D} \mid d(v_i, w) = m \text{ and } N_m(w) \cap \mathcal{D} = \{v_i\}\}$,

$X_i = \{x \in V \mid x \text{ belongs to a } v_i\text{-}w \text{ path of length } m \text{ for some } w \in W_i\}$, and

$U_i = \{u \in V \mid u \text{ is the vertex adjacent to } v_i \text{ on some } v_i\text{-}w \text{ path of length } m \text{ for some } w \in W_i\}$.

By our choice of $\mathcal{D}$, we know that $W_i \neq \emptyset$ for all i. Hence $|X_i| \geq m + 1$ and $v_i \in X_i$ for all i.

Claim 1 *$X_i \cap X_j = \emptyset$ for $1 \leq i < j \leq b$.*

Proof: Suppose $x \in X_i \cap X_j$ for some i and j with $1 \leq i < j \leq b$. Then there exists a vertex w_i (w_j) in W_i (W_j, respectively) such that a v_i-w_i path (v_j-w_j path, respectively) of length m contains the vertex x. But then at least one of w_i and w_j is within distance m from both v_i and v_j, which produces a contradiction. □

By Claim 1, and since $\mathcal{D}$ m-dominates V, we can partition V into sets $V_1, \ldots, V_b$, where each V_i induces a *connected* graph of radius at most m, and where $X_i \subseteq V_i$ and v_i m-dominates V_i. Let S be the set produced by the following algorithm.

Algorithm 1 :

Begin

1. $S \leftarrow \emptyset$, $I \leftarrow \{1, \ldots, b\}$ *and* $i \leftarrow 1$.

2. $S \leftarrow S \cup \{v_i\}$ *and* $I \leftarrow I - \{i\}$.

3. ***For*** $j \in I$ ***do***

    If $v_i v_j \in E$ ***then***

    3.1. ***For*** $u \in U_j$ ***do***

        If ($d(u, w) = m-1$ *for some* $w \in W_j$ *satisfying* $d(w, S) > m$) ***then*** $S \leftarrow S \cup \{u\}$.

    End for

    3.2. $I \leftarrow I - \{j\}$.

End for

4. *If $I = \emptyset$, then continue; otherwise, let $i' \in I$, set $i \leftarrow i'$, and return to Step 2.*

5. *If S m-dominates V, then stop; otherwise, continue.*

6. $T \leftarrow \{ t \in V \mid d(t, S) > m \}$.

7. ***For** $t \in T$ **do***

   7.1. ***If** $t \in V_j$ **then***

   $u_t \leftarrow$ (*the vertex adjacent to v_j on some v_j-t path of length m in $\langle V_j \rangle$*)

   7.2. ***If** $d(t, S) > m$ **then** $S \leftarrow S \cup \{u_t\}$.*

   End for

End

We prove that the set S produced by Algorithm 1 is an independent m-dominating set of G of cardinality at most $(n + m + 1 - 2\sqrt{n})/m$. We begin with two claims.

Claim 2 *If $v_i v_j \in E$ in Step 3 of Algorithm 1, then when j is removed from I in Step 3.2, the set S m-dominates W_j.*

Proof: In Step 3 of Algorithm 1, if $v_i v_j \in E$, then we proceed systematically through the vertices of U_j, placing a vertex in S only if it is at distance $m - 1$ from a vertex of W_j which is not m-dominated by a vertex already in S. Suppose that after the completion of Step 3, $d(w_j, S) > m$ for some $w_j \in W_j$. Consider a v_j-w_j path of length m, and let u_j be the vertex adjacent to v_j on this path. Then $u_j \in U_j$. Since $d(u_j, w_j) = m - 1$ and $d(w_j, S) > m$, the vertex u_j would have been added to S in Step 3 of Algorithm 1, producing a contradiction. □

Claim 3 *If $t \in T$, then $d(t, \mathcal{D}) = m$.*

Proof: Suppose $d(t, v_j) \leq m - 1$ for some $j\,(1 \leq j \leq b)$. Let S be the set constructed when the set T in Step 6 of the algorithm is defined. Since $d(t, S) > m$, $v_j \notin S$. Thus, by the way in which the set S is constructed, there is a vertex v_i of $S \cap \mathcal{D}$ adjacent to v_j. But then $d(t, v_i) \leq m$, contradicting the fact that $d(t, S) > m$. □

If $t \in T$, then $t \in V_j$ for some $j\,(1 \leq j \leq b)$. Since v_j m-dominates V_j, and $\langle V_j \rangle$ is connected, $d(v_j, t) \leq m$ in $\langle V_j \rangle$. However, by Claim 3, $d(v_j, t) \geq m$. Consequently, $d(v_j, t) = m$ in $\langle V_j \rangle$. Hence there exists a v_j-t path of length m in $\langle V_j \rangle$, and therefore the vertex u_t described in Step 7.2.

of the algorithm does indeed exist. It is now evident from the way in which the set S is constructed, that S is an m-dominating set of G. We show next that S is also an independent set of G.

Claim 4 *The set S produced by Algorithm* 1 *is an independent set.*

Proof: If $v_i \in S$, then, by the way in which the set S is constructed, no vertex v_j of $\mathcal{D}$ adjacent to v_i belongs to S. We show next that immediately before a vertex $u \in U_j$ is placed in the set S in Step 3 of the algorithm, $d(u, S) > 1$. We know that before $u \in U_j$ is placed in S, there exists a vertex $w \in W_j$ satisfying $d(w, S) > m$ and $d(u, w) = m - 1$. Hence before $u \in U_j$ is placed in S, it is adjacent to no vertex of S, for otherwise $d(w, S) \leq m$. Furthermore, immediately before a vertex u_t is placed in S in Step 7.2. of the algorithm, we know that $d(t, S) > m$. However $d(u_t, t) = m - 1$, implying that $d(u_t, S) > 1$ before u_t is placed in S for otherwise $d(t, S) \leq m$. Thus whenever a vertex is added to S at any stage of the algorithm, it is adjacent to no other vertex of S. □

It remains for us to show that $|S| \leq (n + m + 1 - 2\sqrt{n})/m$. For $i = 1, \ldots, b$, let $|V_i| = n_i$. By the Pigeonhole Principle, at least one of the sets V_i contains at least n/b vertices. Relabeling the sets if necessary, we may assume that $n_1 \geq n/b$. For $i = 1, \ldots, b$, let $S_i = S \cap V_i$. We show that, for each $i = 1, \ldots, b$,

$$|S_i| \leq \frac{n_i - 1}{m}. \tag{1}$$

Since $|X_i| \geq m + 1$, we know that $n_i = |V_i| \geq |X_i| \geq m + 1$ for all i. For each $v_i \in S$, since v_i m-dominates V_i, it is evident from the way in which the set S is constructed that v_i is the only vertex of V_i in S. Hence equation (1) holds for all i for which $v_i \in S$. The only values of i for which equation (1) is in doubt, are those integers j for which $v_j \notin S$.

If $v_j \notin S$, then, by the way in which the set S is constructed, there is a vertex v_i of $S \cap \mathcal{D}$ adjacent to v_j. After the completion of Step 3 of Algorithm 1, if $S \cap U_j \neq \emptyset$, then let $u_{j,1}, \ldots, u_{j,r_j}$ be the order in which the vertices of U_j were placed in S. For each k with $1 \leq k \leq r_j$, immediately before $u_{j,k}$ was placed into the set S in Step 3 of the algorithm, we know that there exists a $w_{j,k} \in W_j$ satisfying $d(w_{j,k}, S) > m$ and $d(u_{j,k}, w_{j,k}) = m-1$. For $k = 1, \ldots, r_j$, let $Q_{j,k}$ be a $u_{j,k}$-$w_{j,k}$ path of length $m - 1$.

Claim 5 *If $S \cap U_j \neq \emptyset$, then*

$$V(Q_{j,k}) \cap V(Q_{j,\ell}) = \emptyset \text{ for } 1 \leq k < \ell \leq r_j.$$

Proof: Suppose $x \in V(Q_{j,k}) \cap V(Q_{j,\ell})$ for some k and ℓ with $1 \le k < \ell \le r_j$. Since $k < \ell$, it follows from the ordering on the vertices of $S \cap U_j$ that $d(u_{j,k}, w_{j,\ell}) > m$. If the length of the $u_{j,\ell}$-x section of $Q_{j,\ell}$ is at least the length of the $u_{j,k}$-x section of $Q_{j,k}$, then the $u_{j,k}$-$w_{j,\ell}$ path obtained by following the $u_{j,k}$-x section of $Q_{j,k}$, and then proceeding along the x-$w_{j,\ell}$ section of $Q_{j,\ell}$ has length at most that of $Q_{j,\ell}$. That is to say, $d(u_{j,k}, w_{j,\ell}) \le m - 1$, which produces a contradiction. On the other hand, if the length of the $u_{j,\ell}$-x section of $Q_{j,\ell}$ is less than that of the $u_{j,k}$-x section of $Q_{j,k}$, then $d(u_{j,\ell}, w_{j,k}) < m-1$. Since $v_j u_{j,\ell} \in E$, it follows that $d(v_j, w_{j,\ell}) < m$, which once again produces a contradiction. □

If $S \cap U_j \neq \emptyset$, then $\cup_{k=1}^{r_j} V(Q_{j,k}) \subseteq X_j \subseteq V_j$. Hence we have the following result.

Claim 6 *If $S \cap U_j \neq \emptyset$ and $S \cap U_c \neq \emptyset$ for $c \neq j$, then*

$$V(Q_{j,k}) \cap V(Q_{c,\ell}) = \emptyset \text{ for } 1 \le k \le r_j \text{ and } 1 \le \ell \le r_c.$$

If $T \neq \emptyset$, then let $u_{t_1}, \ldots, u_{t_s}$ be the order in which the vertices of T were placed in S in Step 7.2. of the algorithm. If $t_i \in T$, then $t_i \in V_j$ for some j $(1 \le j \le b)$. It follows then from Claim 3 that $d(v_j, t_i) = m$ and $d(u_{t_i}, t_i) = m - 1$. For $i = 1, \ldots, s$, let P_i be a u_{t_i}-t_i path of length $m - 1$ in $\langle V_j \rangle$.

Claim 7 *$V(P_k) \cap V(P_\ell) = \emptyset$ for $1 \le k < \ell \le s$.*

Proof: Suppose $x \in V(P_k) \cap V(P_\ell)$ for some k and ℓ with $1 \le k < \ell \le s$. Suppose $t_k \in V_i$ and $t_\ell \in V_j$ where $1 \le i \le j \le b$. Then $V(P_k) \subseteq V_i$ and $V(P_\ell) \subseteq V_j$, implying necessarily that $i = j$. Since $k < \ell$, it follows from the ordering on the vertices of $S \cap T$ that $d(u_{t_k}, t_\ell) > m$. If the length of the u_{t_ℓ}-x section of P_ℓ is at least the length of the u_{t_k}-x section of P_k, then the u_{t_k}-t_ℓ path obtained by following the u_{t_k}-x section of P_k, and then proceeding along the x-t_ℓ section of P_ℓ has length at most that of P_ℓ. That is to say, $d(u_{t_k}, t_\ell) \le m - 1$, which produces a contradiction. On the other hand, if the length of the u_{t_ℓ}-x section of P_ℓ is less than that of the u_{t_k}-x section of P_k, then $d(u_{t_\ell}, t_k) < m - 1$, implying that $d(v_j, t_k) < m$, once again producing a contradiction. □

Claim 8 *If $S \cap U_j \neq \emptyset$, then*

$$V(Q_{j,k}) \cap V(P_\ell) = \emptyset \text{ for } 1 \le k \le r_j \text{ and } 1 \le \ell \le s.$$

Proof: Suppose $x \in V(Q_{j,k}) \cap V(P_\ell)$ for some k and ℓ with $1 \le k \le r_j$ and $1 \le \ell \le s$. Suppose $t_k \in V_i$ $(1 \le i \le b)$. Then $V(P_\ell) \subseteq V_i$. Since $V(Q_{j,k}) \subseteq V_j$, it follows that $i = j$. By the way in which the set S is constructed, the vertex $u_{j,k}$ was placed in S before the vertex u_{t_ℓ}. We know, therefore, that $d(u_{j,k}, t_\ell) > m$. Hence the length of the $u_{j,k}$-x section of $Q_{j,k}$ exceeds that of the u_{t_ℓ}-x section of P_ℓ, for otherwise $d(u_{j,k}, t_\ell) \leq m-1$. But this implies that the u_{t_ℓ}-$w_{j,k}$ path obtained by following the u_{t_ℓ}-x section of P_ℓ, and then proceeding along the x-$w_{j,k}$ section of $Q_{j,k}$ has length less than that of $Q_{j,k}$. Thus, $d(u_{t_\ell}, w_{j,k}) < m-1$. Since $v_j u_{t_\ell} \in E$, this implies that $d(v_j, w_{j,k}) < m$, which produces a contradiction. □

We now consider the sets S_j for which $v_j \notin S_j$. It follows from Claims 5, 6, 7, and 8, that there is a collection of $|S_j|$ vertex disjoint paths of length $m-1$ that belong to $\langle V_j \rangle$ and do not contain the vertex v_j. Thus $n_j = |V_j| \geq |\{v_j\}| + m|S_j|$, so $|S_j| \leq (n_j - 1)/m$. Hence equation (1) is true for all $i = 1, \ldots, b$. Since S is an independent m-dominating set of G, we now have

$$\begin{aligned}
id(m,G) \leq |S| &= |S_1| + \textstyle\sum_{i=2}^{b} |S_i| \\
&\leq |\{v_1\}| + \textstyle\sum_{i=2}^{b} (n_i - 1)/m \qquad \text{(by equation (1))} \\
&= 1 + ((n - n_1) - (b-1))/m \\
&\leq 1 + \left(n - \tfrac{n}{b} - b + 1\right)/m \qquad \text{(since } n_1 \geq \tfrac{n}{b}\text{)} \\
&= \tfrac{1}{m}\left(m + n + 1 - \tfrac{n}{b} - b\right).
\end{aligned}$$

The last expression is maximized with $b = \sqrt{n}$. Thus

$$id(m,G) \leq |S| \leq \frac{1}{m}\left(m + n + 1 - 2\sqrt{n}\right).$$

That this upper bound on $id(m,G)$ is sharp may be seen by considering the graph G shown in Figure 1 with $b = mk+1$. Then $id(m,G) = 1 + (b-1)k = 1 + mk^2$ and $n = |V(G)| = b(mk+1) = (mk+1)^2$. Thus

$$\frac{1}{m}\left(m + n + 1 - 2\sqrt{n}\right) = 1 + mk^2 = id(m,G).$$

This completes the proof of the theorem. □

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