Vertex splitting, parity subgraphs and circuit covers

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ABSTRACT. Let G be a 2-edge-connected graph and v be a vertex of G and $F \subset F' \subset E(v)$ such that $1 \leq |F|$ and $|F| + 2 = |F'| \leq d(v) - 1$. Then there is a subset F^* such that $F \subset F^* \subset F'$ (here, $|F^*| = |F| + 1$), and the graph obtained from G by splitting the edges of F^* away from v remains 2-edge-connected unless v is a cut-vertex of G. This generalizes a very useful Vertex-Splitting Lemma of Fleischner. Let $\mathcal{C}$ be a circuit cover of a bridge-less graph G. The depth of $\mathcal{C}$ is the smallest integer k such that every vertex of G is contained in at most k circuits of $\mathcal{C}$. It is conjectured by L. Pyber that every bridge-less graph G have a circuit cover $\mathcal{C}$ such that the depth of $\mathcal{C}$ is at most $\Delta(G)$. In this paper, we prove that (i). every bridge-less graph G has a circuit cover $\mathcal{C}$ such that the depth of $\mathcal{C}$ is at most $\Delta(G)+2$ and (ii). if a bridge-less graph G admits a nowhere-zero 4-flow or contains no subdivision of the Petersen graph, then G has a circuit cover $\mathcal{C}$ such that the depth of $\mathcal{C}$ is at most $2\lceil\Delta(G)/3\rceil$.

1 Introduction

Definition 1.1 *A* circuit *of a graph is a connected 2-regular subgraph, while a* cycle *of a graph is a subgraph with even degree at every vertex. A bridge b of a graph G is an edge of G that is not contained in any circuit of G.*

*Partial support provided by the National Science Foundation under Grant DMS-9306379

All other standard graph-theoretic terms that are used in this paper can be found for instance in [3].

Definition 1.2 *Let G be a bridge-less graph. A family C of circuits of G is called a* circuit cover *of G if each edge of G is contained in some circuit of C.*

Definition 1.3 *Let C be a circuit cover of a graph G and $v \in V(G)$. The* depth of the circuit cover C at the vertex v *is the number of circuits of C containing v and is denoted by $cd_C(v)$. The* depth of the circuit cover C *is* $\max\{cd_C(v) : v \in V(G)\}$ *and is denoted by $cd_C(G)$.*

The following conjecture was presented by László Pyber at the Julius Petersen Graph Theory Conference, Denmark, 1990.

Conjecture 1.1 *(Pyber, [2]) Each bridge-less graph G with maximum degree $\Delta(G)$ has a circuit cover C such that*

$$cd_C(G) \leq \Delta(G).$$

In this paper, we prove that

Theorem 1.1 *Every bridge-less graph G has a circuit cover C such that the depth of C is at most $\Delta(G) + 2$.*

Theorem 1.2 *If a bridge-less graph G admits a nowhere-zero 4-flow or contains no subdivision of the Petersen graph, then G has a circuit cover C such that the depth of C is at most $2\lceil \Delta(G)/3 \rceil$.*

Let H be a subgraph of G. The set of edges of H incident with v is denoted by $E_H(v)$. The number of components of a graph G is denoted by $\omega(G)$.

Definition 1.4 *Let G be a graph and v be a vertex of G and $F \subset E(v)$. The graph $G_{[v;F]}$ obtained from G by splitting the edges of F away from v, (that is, adding a new vertex v' and changing the end v of the edges of F to be v'. See figure 1). Note that the new vertex created in the splitting F away from v is always denoted by v' in the contest.*

Theorem 1.3 *(Fleischner [4], or see [7, 8]) Let G be a connected and bridge-less graph and $v \in V(G)$ (with $d(v) \geq 4$) and $e_0, e_1, e_2 \in E_G(v)$. In the case of v being a cut-vertex, choose e_0, e_2 from distinct blocks of G. Then either $G_{[v;\{e_0,e_1\}]}$ or $G_{[v;\{e_0,e_2\}]}$ is connected and bridge-less.*

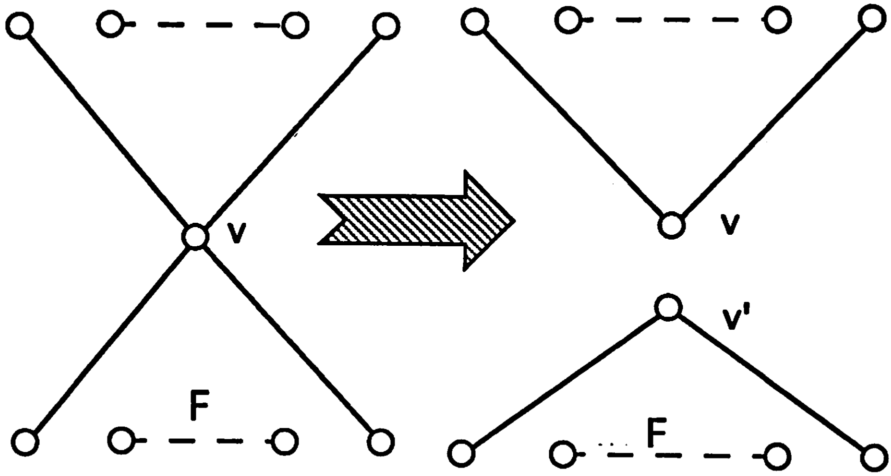

Figure 1: G and $G_{[v;F]}$: splitting F away from v

Theorem 1.3 is the well-known vertex-splitting lemma which has been used by various authors in the studies of compatible decompositions, integer flows, cycle covers and graph colorings. (For instance, [4], [5], [6], [7], [8], [9], [12], [13], [14], [16], [19], [20], etc.) The following theorem is an analogy of Theorem 1.3 which can be used for splitting a graph to meet certain degree requirement.

Theorem 1.4 *Let G be a connected and bridge-less graph and v be a vertex of G and $F \subset F' \subset E(v)$ such that $1 \leq |F|$ and $|F| + 2 = |F'| \leq d(v) - 1$. Then there is a subset F^* such that $F \subset F^* \subset F'$ (here, $|F^*| = |F|+1$), and $G_{[v;F^*]}$ is connected and bridge-less, unless $G_{[v;F]}$ or $G_{[v;F']}$ is disconnected.*

Theorem 1.4 and Theorem 1.3 have the same property that the vertex-splitting operation preserves the embedding property of graphs.

2 Vertex splitting

A few lemmas are needed before the proof of Theorem 1.4.

Lemma 2.1 *Let G be a bridge-less graph and v be a vertex of G and $F \subset E(v)$ such that $2 \leq |F| \leq d(v) - 2$. Then $G_{[v;F]}$ is bridge-less if and only* *if either v' and v are in two distinct components or there are a pair of edge-disjoint paths in $G_{[v;F]}$ joining v' and v.*

Proof. If $G_{[v;F]}$ has a bridge b, then there must be a path P of $G_{[v;F]}$ joining v', v and containing b since G is bridge-less and each circuit of G containing b is broken into a path because of the splitting of the vertex v. Thus, v', v are in the same component of $G_{[v;F]}$ and we assume that there are a pair of edge-disjoint paths S_1, S_2 joining v', v in $G_{[v;F]}$. Without loss of generality, let S_1 be a path not containing the edge b. Thus, the symmetric difference of P and S_1 yields a cycle of $G_{[v;F]}$ containing b. This contradicts that b is a bridge of $G_{[v;F]}$. The another direction of the lemma is trivial. □

Lemma 2.2 *Let G be a connected, bridge-less graph and v be a vertex of G and $F \subset E(v)$ such that $2 \leq |F| \leq d_G(v) - 2$. If the graph $G' = G_{[v;F]}$ has a bridge $e \in E_{G'}(v')$ (or $e \in E_{G'}(v)$), then either $G_{[v;F \setminus \{e\}]}$ or $G_{[v;F \cup \{e\}]}$ is not connected.*

Proof. Without loss of generality, we suppose that G is connected. Assume that G' has a bridge $e \in E_{G'}(v')$. The bridge $e = v'x$ must be contained in a path connecting v and v' since G is bridge-less and connected. Hence G' is connected and $G' \setminus \{e\}$ is disconnected with x contained in the component containing v but not v'. Therefore $G_{[v;F \setminus \{e\}]}$ is disconnected. □

Proof of Theorem 1.4. Let $F' \setminus F = \{e_1, e_2\}$. Let $H_j = G_{[v;F \cup \{e_j\}]}$ for $j = 1, 2$. Assume that both H_1 and H_2 have bridges and both $G_{[v;F]}$ and $G_{[v;F']}$ are connected.

By Lemma 2.2, neither e_1 nor e_2 is a bridge of H_j ($j = 1, 2$) since both $G_{[v;F]}$ and $G_{[v;F']}$ are connected. By Lemma 2.1, H_1 does not have a pair of edge-disjoint paths joining v and v'. Hence, e_1, e_2 are in two distinct blocks, say K_1 and K_2, of H_1. Since neither e_1 nor e_2 is a bridge of H_1, K_1, K_2 are non-trivial blocks. Let C_i be a circuit of K_i containing e_i for $i = 1, 2$. In the graph $H_2 = G_{[v;F \cup \{e_2\}]}$, the circuits C_1, C_2 are broken into two edge-disjoint paths connecting v and v'. By Lemma 2.1, H_2 is bridge-less. □

Now, we are to deduce Theorem 1.3 from Theorem 1.4.

Proof of Theorem 1.3. It is obvious if v is not a cut-vertex of G. Assume that v is a cut-vertex of G. Let $F = \{e_0\}$ and $F' = \{e_0, e_1, e_2\}$. If $G_{[v;F]}$ is disconnected then e_0 is a bridge of G. This contradicts that G is bridge-less. Now, by Theorem 1.4, we only need to show that $G_{[v,F']}$ is connected. Note that each non-trivial block of G contains at least two edges and G is bridge-less, for $i = 0, 2$ let f_i ($f_i \neq e_i$) be an edge of $E_G(v)$ contained in the block of G containing e_i. Since e_0, e_2 are chosen from distinct blocks of G and $|\{e_0, e_2, f_0, f_2\}| = 4$ and $|F'| = 3$, $E_{G_{[v;F']}}(v)$ contains at least one of

$\{f_0, f_2\}$ and hence, v', v are in the same component of $G_{[v;F']}$. Therefore, $G_{[v;F']}$ is connected and Theorem 1.3 follows. □

Remarks.

Similar to Theorem 1.3, the splitting operation induced in Theorem 1.4 also preserve the embedding property of G. Let the edges incident with a vertex v be arranged on a surface in the order as $e_1, \cdots, e_d$. If one chooses $F' = \{e_i, e_{i+1}, \cdots, e_{j-1}, e_j\}$ and $F = \{e_{i+1}, \cdots, e_{j-1}\}$, then the graph $G_{[v,F^*]}$ preserves the embedding property on the same surface where F^* is one of $\{e_i, e_{i+1}, \cdots, e_{j-1}\}$ and $\{e_{i+1}, \cdots, e_j\}$.

Note that the vertex splitting operations in Theorem 1.3 and its generalization, Theorem 1.4 preserve the 2-edge-connectivity of graphs. While for higher edge-connectivity, Nash-Williams has the following theorem. (Note, the following theorem does not preserve the embedding property of graphs)

Theorem 2.3 *(Nash-Williams [22]) Let k be an even integer and G be a k-connected graph and $v \in V(G)$. Let a be an integer such that $k \le a \le d(v) - k$. Then there is an edge subset $F \subset E(v)$ such that $|F| = a$ and $G_{[v;F]}$ is k-edge-connected.*

3 Parity subgraphs

Definition 3.1 *Let G and H be two graphs. The graph H is said to be obtained from G by a* vertex splitting π *if $V(H)$ has a partition $\{U_i : i = 1, \cdots, n\}$ such that G can be obtained from H by identifying every U_i ($i = 1, \cdots, n$) to a single vertex v_i. If a graph H is obtained from G by a vertex splitting π, then we write $H = \pi(G)$ and $U_i = \pi(v_i)$ for each $v_i \in V(G)$. (Here, we simply consider $E(G) = E(\pi(G))$).*

Definition 3.2 *A graph G is* quasi-cubic *if the degree of each vertex of G is either three or two.*

By recursively applying Theorem 1.4 to each vertex of a graph with degree at least four (with $a, b = 3$, or 2), we have the following corollary,

Lemma 3.1 *Let G be a bridge-less graph. Then G has a vertex-splitting π such that*
(1). $\pi(G)$ is bridge-less;
(2). $\pi(G)$ is quasi-cubic;
(3). For each vertex $v \in V(G)$ with degree $d_G(v) \ge 2$, $\pi(v)$ is a set of $t(v)$ degree-three-vertices and $s(v)$ degree-two-vertices where

$$t(v) = \begin{cases} d_G(v)/3 & \text{if } d_G(v) \equiv 0 \bmod 3 \\ (d_G(v)-2)/3 & \text{if } d_G(v) \equiv 2 \bmod 3 \\ (d_G(v)-4)/3 & \text{if } d_G(v) \equiv 1 \bmod 3, \end{cases}$$

and

$$s(v) = \begin{cases} 0 & \text{if } d_G(v) \equiv 0 \bmod 3 \\ 1 & \text{if } d_G(v) \equiv 2 \bmod 3 \\ 2 & \text{if } d_G(v) \equiv 1 \bmod 3. \end{cases}$$

Definition 3.3 *A subgraph P of a graph G is a parity subgraph of G if $d_P(v) \equiv d_G(v)$, (mod 2) for each vertex $v \in V(G)$.*

Theorem 3.2 *Let G be a graph. Then G has a parity subgraph P such that for each vertex $v \in V(G)$*

$$d_P(v) \le \begin{cases} d_G(v)/3 & \text{if } d_G(v) \equiv 0 \bmod 3 \\ (d_G(v)-2)/3+2 & \text{if } d_G(v) \equiv 2 \bmod 3 \\ (d_G(v)-4)/3+4 & \text{if } d_G(v) \equiv 1 \bmod 3. \end{cases}$$

Proof. By Lemma 3.1, let π be a vertex splitting of G such that $\pi(G)$ is described in Lemma 3.1. The underlying graph of $\pi(G)$ is cubic and bridge-less and therefore has a perfect matching M, (Petersen Theorem [17], or see [3] p79). The subgraph Q of $\pi(G)$ induced by the edges of M is a parity subgraph of $\pi(G)$ such that

$$d_Q(v) = \begin{cases} 1 & \text{if } d_{\pi(G)}(v) = 3 \\ 0 \text{ or } 2 & \text{if } d_{\pi(G)}(v) = 2. \end{cases}$$

Thus the subgraph P of G induced by the edges of $E(Q)$ is a parity subgraph of G satisfying the description of the theorem. □

The properties of parity subgraphs and their relations with the problems of integer flow, circuit cover can be found in the papers [15, 25, 26]. Theorem 3.2 will be applied in next section.

4 Depth of circuit cover

4.1 A general upper bound

The following theorem (the 8-Flow Theorem) is one of the fundamental results that we will use in this paper.

Lemma 4.1 *(Jaeger [11], or see [13]) Every bridge-less graph G has a cycle cover consisting of at most three cycles.*

Note that a cycle is a union of edge-disjoint circuits. As an immediate corollary of Lemma 4.1, the following result was originally observed by Pyber ([2]).

Theorem 4.2 *Each bridge-less graph G with maximum degree $\Delta(G)$ has a circuit cover C such that*

$$cd_C(G) \le \frac{3}{2}\Delta(G).$$

However, for a quasi-cubic graph, Theorem 4.2 can be improved

Lemma 4.3 *Each bridge-less quasi-cubic graph G has a circuit cover C such that*

$$cd_C(G) \le 3.$$

Proof. For a quasi-cubic graph, each cycle of G is the union of vertex-disjoint circuits. Thus, by Lemma 4.1, for a 3-cycle cover of G, each vertex is contained in at most three circuits. □

An immediate corollary of Lemma 3.1 and Lemma 4.3 is the following result which generalizes Theorem 4.2.

Theorem 4.4 *Let G be a bridge-less graph. Then G has a circuit cover C such that for each vertex $v \in V(G)$*

$$cd_C(v) \le \begin{cases} d_G(v) & \text{if } d_G(v) \equiv 0 \bmod 3 \\ d_G(v) - 2 + 3 = d_G(v) + 1 & \text{if } d_G(v) \equiv 2 \bmod 3 \\ d_G(v) - 4 + 6 = d_G(v) + 2 & \text{if } d_G(v) \equiv 1 \bmod 3 \end{cases}$$

The following result and Theorem 1.1 are immediate corollaries of Theorem 4.4.

Corollary 4.5 *Let G be a bridge-less graph with the maximum degree $\Delta(G)$. Then G has a circuit cover C such that*

$$cd_C(G) \le \begin{cases} \Delta(G) & \text{if } \Delta(G) \equiv 0 \bmod 3 \\ \Delta(G) + 1 & \text{if } \Delta(G) \equiv 2 \bmod 3 \\ \Delta(G) + 2 & \text{if } \Delta(G) \equiv 1 \bmod 3 \end{cases}$$

4.2 A better bound for certain families of graphs

It was observed by Pyber [2] that the famous Circuit Double Cover Conjecture (Conjecture 5.1 in next section, by Szekeres [21], Seymour [18]) implies Conjecture 1.1. Since we have known ([12], [1]) that a graph admitting a nowhere-zero 4-flow or containing no subdivision of the Petersen graph has a circuit double cover, Conjecture 1.1 holds for those graphs. That is,

Theorem 4.6 *If a bridge-less graph G admits a nowhere-zero 4-flow or contains no subdivision of the Petersen graph, then G has a circuit cover C such that $cd_C(G) \le \Delta(G)$.*

This result is to be generalized in this section. The following lemmas are fundamental in this section.

Lemma 4.7 *(Jaeger [11], or see [13]) Every 4-edge-connected graph G admits a nowhere-zero 4-flow.*

Lemma 4.8 *(Zhang [24], or see [26, 10]) Let G be a graph and P be a parity subgraph of G. If G admits a nowhere-zero 4-flow, then G has a circuit cover $\mathcal{C}$ such that each edge $e \in E(P)$ is contained in precisely two circuits of $\mathcal{C}$ and each edge $e \in E(G) \setminus E(P)$ is contained in precisely one circuit of $\mathcal{C}$.*

Lemma 4.9 *(Alspach, Goddyn and Zhang [1], or see [10, 25]) Let G be a bridge-less graph and P be a parity subgraph of G. If G contains no subdivision of the Petersen graph, then G has a circuit cover $\mathcal{C}$ such that each edge $e \in E(P)$ is contained in precisely two circuits of $\mathcal{C}$ and each edge $e \in E(G) \setminus E(P)$ is contained in precisely one circuit of $\mathcal{C}$.*

(Please refer to [13] or [23] for the definition and properties of integer flows, and refer to [15] and [26] for the relations of parity subgraphs and integer flows, circuit covers.)

Now we are ready to use Theorem 3.2 and Lemmas 4.7, 4.8, 4.9 to obtain the following theorems.

Theorem 4.10 *Let G be a bridge-less graph. If G admits a nowhere-zero 4-flow or contains no subdivision of the Petersen graph or is 4-edge-connected, then G has a circuit cover $\mathcal{C}$ such that*

$$cd_{\mathcal{C}}(v) \leq \begin{cases} \frac{2}{3}d_G(v) & \text{if } d_G(v) \equiv 0 \bmod 3 \\ \frac{2}{3}(d_G(v)+1) & \text{if } d_G(v) \equiv 2 \bmod 3 \\ \frac{2}{3}(d_G(v)+2) & \text{if } d_G(v) \equiv 1 \bmod 3 \end{cases}$$

Proof. Let P be a parity subgraph of G described in Theorem 3.2. Let $\mathcal{C}$ be a circuit cover of G covering each edge of P twice and each edge of $E(G) \setminus E(P)$ once. Since a circuit is a connected 2-regular subgraph, for each vertex $v \in V(G)$, the number of circuits of $\mathcal{C}$ containing v is

$$\frac{1}{2}[2d_P(v) + (d_G(v) - d_P(v)] = \frac{1}{2}(d_G(v) + d_P(v))$$

$$\leq \begin{cases} \frac{1}{2}(d_G(v) + \frac{1}{3}d_G(v)) = \frac{2}{3}d_G(v) & \text{if } d_G(v) \equiv 0 \bmod 3 \\ \frac{1}{2}[d_G(v) + \frac{1}{3}(d_G(v)-2)+2] = \frac{2}{3}(d_G(v)+1) & \text{if } d_G(v) \equiv 2 \bmod 3 \\ \frac{1}{2}[d_G(v) + \frac{1}{3}(d_G(v)-4)+4] = \frac{2}{3}(d_G(v)+2) & \text{if } d_G(v) \equiv 1 \bmod 3 \end{cases}$$

□

An immediate corollary of Theorem 4.10 is Theorem 1.2.

5 Edge-depth of circuit cover

Definition 5.1 *Let C be a circuit cover of a graph G and $e \in E(G)$. The* edge-depth of the circuit cover C at the edge e *is the number of circuits of C containing e and is denoted by $ced_{\mathcal{C}}(e)$. The* edge-depth of the circuit cover C *is* $\max\{ced_{\mathcal{C}}(e) : e \in E(G)\}$ *and is denoted by $ced_{\mathcal{C}}(G)$.*

The following conjecture is an equivalent version of the famous Circuit Double Cover Conjecture. The problem of circuit double cover has been extensively studied in recent years. (See surveys, [10, 12, 13, 25], etc.)

Conjecture 5.1 *(Szekeres [21], Seymour [18]) Every bridge-less graph G has a circuit cover C such that $ced_{\mathcal{C}}(G) \leq 2$.*

Note, the 8-flow theorem (Lemma 4.1) implies that every bridge-less graph has a circuit cover with the edge-depth at most three.

Definition 5.2 *Let G be a bridge-less graph. A circuit cover C is* shortest *if the total length of circuits of C is shortest among all circuit covers of G.*

There are many articles in the topic of shortest circuit cover. (See survey, [10], etc.)

Conjecture 5.2 *(Zhang [25]) Every bridge-less graph G has a shortest circuit cover C with $ced_{\mathcal{C}}(G) \leq 2$.*

Conjecture 5.2 implies Circuit Double Cover Conjecture (Conjecture 5.1). Conjecture 5.2 holds for certain families of graphs, for example, the graphs admitting nowhere-zero 4-flow (Lemma 4.8) and the bridge-less graphs containing no subdivision of the Petersen graph (Lemma 4.9). On the other hand, we do not even know if there exists a constant upper bound for the edge-depths of shortest circuit covers for all bridge-less graphs. The following is a weak version of Conjecture 5.2.

Conjecture 5.3 *There is an integer K such that every bridge-less graph G has a shortest circuit cover C with $ced_{\mathcal{C}}(G) \leq K$.*

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