The Greatest Common Divisor Index of a Graph

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ABSTRACT

A graph H is G-decomposable if H can be decomposed into subgraphs, each of which is isomorphic to G. A graph G is a greatest common divisor of two graphs G_1 and G_2 if G is a graph of maximum size such that both G_1 and G_2 are G-decomposable. The greatest common divisor index of a graph G of size $q \geq 1$ is the greatest positive integer n for which there exist graphs G_1 and G_2, both of size at least nq, such that G is the unique greatest common divisor of G_1 and G_2. If no such integer n exists, the greatest common divisor index of G is infinite. Several graphs are shown to have infinite greatest common divisor index, including matchings, stars, small paths, and the cycle C_4. It is shown for an edge-transitive graph F of order p with vertex independence number less than $p/2$ that if G is an F-decomposable graph of sufficiently large size, then G is also $(F-e) \cup K_2$-decomposable. From this it follows that each such edge-transitive graph has finite index. In particular, all complete graphs of order at least 3 are shown to have greatest common divisor index 1 and the greatest common divisor index of the odd cycle C_{2k+1} lies between k and $4k^2-2k-1$. The graphs $K_p - e$, $p \geq 3$, have infinite or finite index depending on the value of p; in particular, $K_p - e$ has infinite index if $p \leq 5$ and index 1 if $p \geq 6$.

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1. Introduction

A nonempty graph H is *decomposable* into the subgraphs $G_1, G_2, \ldots, G_n$ of H if no graph G_i $(1 \le i \le n)$ has isolated vertices and $E(H)$ can be partitioned into $E(G_1), E(G_2), \ldots, E(G_n)$. If $G_i \cong G$ for each integer i $(1 \le i \le n)$, then H is *G-decomposable*, in which case we say G *divides* H and write $G \mid H$. In general, we follow [4] for graph theory notation and terminology.

Let G_1 and G_2 be two nonempty graphs. In [1] a graph G without isolated vertices is defined to be a *greatest common divisor* of G_1 and G_2 if G is a graph of maximum size such that $G \mid G_1$ and $G \mid G_2$. Since K_2 divides every nonempty graph, it is evident that every two nonempty graphs have a greatest common divisor. For the graphs G_1 and G_2 of Figure 1, their unique greatest common divisor G is shown.

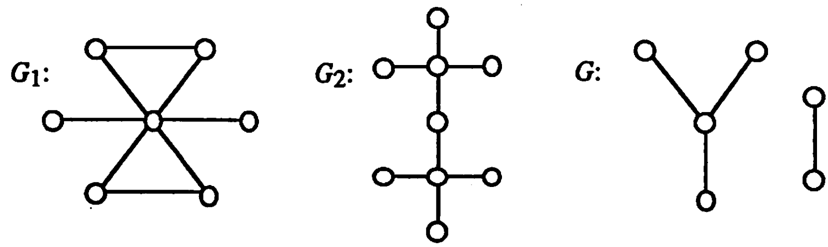

Figure 1

Although the two graphs G_1 and G_2 of Figure 1 have a unique greatest common divisor, this is, by no means, always the case. Indeed it was shown in [2] that for every positive integer n, there exist graphs G_1 and G_2 having exactly n greatest common divisors. We denote the set of greatest common divisors of G_1 and G_2 by $\mathrm{GCD}(G_1, G_2)$ and write $\mathrm{GCD}(G_1, G_2) = G$ if the greatest common divisor is uniquely G. Greatest common divisors of graphs were investigated in detail in [6].

In this paper we consider pairs of graphs with a prescribed unique greatest common divisor, but with an added condition. Suppose that G_1 and G_2 are graphs of sizes q_1 and q_2, respectively, and G is a graph of size q. If G is a greatest common divisor of G_1 and G_2, then $q \mid q_1$ and $q \mid q_2$; indeed, $q \mid \gcd(q_1, q_2)$. Even though G is a greatest common divisor of G_1

and G_2, the size q of G need not equal $\gcd(q_1, q_2)$. In fact, the graphs G_1 and G_2 of Figure 1 have size 8, while G has size 4.

A basic question concerning greatest common divisors is the following: For a given graph G (without isolated vertices), do there exist graphs G_1 and G_2 such that G is a greatest common divisor of G_1 and G_2? This question surely has an affirmative answer since we may take $G_1 \cong G$ and $G_2 \cong G$, or even take $G_1 \cong G$ and choose G_2 to be any graph for which $G \mid G_2$. In these cases, not only is G a greatest common divisor of G_1 and G_2, it is the unique greatest common divisor of G_1 and G_2.

On the basis of these observations, our revised question becomes: For a given graph G (without isolated vertices), do there exist graphs G_1 and G_2, neither of which is isomorphic to G, such that G is a greatest common divisor of G_1 and G_2? This new question also has an affirmative answer since we may take $G_1 \cong 2G$ (two disjoint copies of G) and $G_2 \cong 3G$. Clearly $G \mid G_1$ and $G \mid G_2$. If G has size q, then G_1 has size $2q$ and G_2 has size $3q$. Since $\gcd(2q, 3q) = q$, the greatest possible size of a greatest common divisor of G_1 and G_2 is q. However, G has size q and, consequently, G is a greatest common divisor of G_1 and G_2. Certainly in the definitions of G_1 and G_2, we may replace the integers 2 and 3 by any two relatively prime integers. This response to our new question is not completely satisfactory, however. For example, suppose that $G \cong P_3$ (a path of order 3). Let $G_1 \cong 2G$ and $G_2 \cong 3G$ (see Figure 2). As we observed earlier, G is a greatest common divisor of G_1 and G_2. However, G is not the only greatest common divisor of G_1 and G_2. So too is $G' \cong 2K_2$ (see Figure 2). Are there two graphs G_1 and G_2, neither of which is isomorphic to $G \cong P_3$, such that G is the *unique* greatest common divisor of G_1 and G_2? The answer is yes, for if we take $H_1 \cong K_{1,4}$ and $H_2 \cong K_{1,6}$, then $G \cong P_3$ is the unique greatest common divisor of H_1 and H_2. Can graphs G_1 and G_2 of even larger size be found with this property? Here too the answer is yes, as can be seen by taking $G_1 \cong K_{1,2n}$ and $G_2 \cong K_{1,2n+2}$ for arbitrarily large positive integers n. This leads us to the main concept of this paper.

For a graph G of size $q \geq 1$, we define the *greatest common divisor index* (or simply the *index*) $i(G)$ of G as the greatest positive integer n for which there exist graphs G_1 and G_2, both of size at least nq, such that $\mathrm{GCD}(G_1, G_2) = G$. If no such integer n exists, then we define this index to be ∞. Hence, a graph G has infinite index if and only if there exist graphs of arbitrarily large size having G as their unique greatest common divisor. If G is a graph such that $\mathrm{GCD}(G_1, G_2) = G$ implies that G_1 or G_2 is isomorphic to G, then G has index 1. Certainly, every graph (without isolated vertices) has an index.

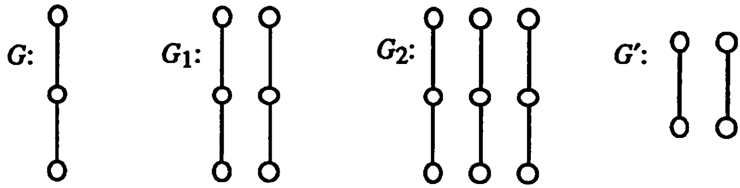

Figure 2

2. Graphs With Infinite Greatest Common Divisor Index

We now give several examples of graphs having infinite index. We first show that the index of all matchings (disjoint copies of K_2) is infinite.

Theorem 1 For every positive integer n,

$$i(nK_2) = \infty.$$

Proof For positive integers a and b, $\mathrm{GCD}(aK_2, bK_2) = \gcd(a, b)K_2$. Hence for distinct primes p_1 and p_2, $\mathrm{GCD}(p_1nK_2, p_2nK_2) = nK_2$. Since p_1 and p_2 can be chosen arbitrarily large, $i(nK_2) = \infty$. ❑

A similar proof gives the following result concerning stars.

Theorem 2 For every positive integer n,

$$i(K_{1,n}) = \infty.$$

These results can be extended as follows.

Theorem 3 For all positive integers $m, n_1, n_2, \dots, n_k$ $(k \geq 1)$,

$$i(mK_2 \cup K_{1,n_1} \cup K_{1,n_2} \cup \dots \cup K_{1,n_k}) = \infty.$$

Proof Let $G \equiv mK_2 \cup K_{1,n_1} \cup K_{1,n_2} \cup \dots \cup K_{1,n_k}$. The result follows as before by considering distinct primes p_1 and p_2 and observing that G is the unique greatest common divisor of the graphs $G_1 \equiv p_1 mK_2 \cup$

$K(1, p_1n_1) \cup K(1, p_1n_2) \cup \ldots \cup K(1, p_1n_k)$ and $G_2 \cong p_2mK_2 \cup K(1, p_2n_1) \cup K(1, p_2n_2) \cup \ldots \cup K(1, p_2n_k)$. ❑

By Theorems 1 and 2, the index of the paths P_2 and P_3 of orders 2 and 3, respectively, is ∞. We show that this is the case for P_4 and P_5 as well. For this purpose, the following result from [3] will be useful.

Lemma A Let H be a graph containing an edge adjacent to every other edge of H. If H is G-decomposable for some graph G, then G is connected.

Theorem 4 $i(P_4) = \infty$.

Proof Let p_1 and p_2 be distinct primes. For the graphs $G_1 \cong p_1P_4$ and G_2 shown in Figure 3, with $k = p_2$, we show that $\mathrm{GCD}(G_1, G_2) = P_4$.

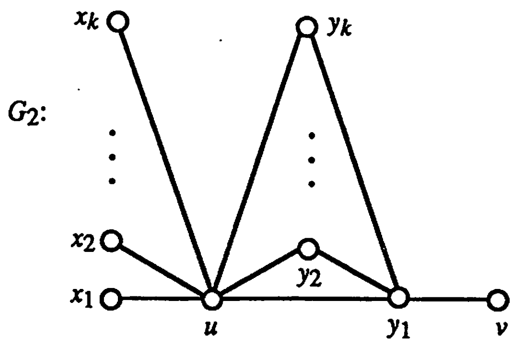

Figure 3

Since p_1 and p_2 are distinct primes, $\gcd(3p_1, 3p_2) = 3$. The divisors of size 3 in G_1 are $P_4, P_3 \cup K_2$, and $3K_2$. However, by Lemma A, the graph G_2 is neither $(P_3 \cup K_2)$-decomposable nor $3K_2$-decomposable since the edge uy_1 is adjacent to all other edges of G_2. Now G_2 is P_4-decomposable into the path x_1, u, y_1, v and the $k-1$ paths x_i, u, y_i, y_1 $(2 \le i \le k)$. Therefore, the path P_4 is the only divisor of size 3 of the graph G_2. Hence, $\mathrm{GCD}(G_1, G_2) = P_4$. Since p_1 and p_2 may be chosen to be arbitrarily large, $i(P_4) = \infty$. ❑

In order to facilitate showing the existance of other graphs having infinite index, we present two lemmas. We write $\beta_1(G)$ for the edge independence number of a graph G. Clearly, if G is a subgraph of H, then $\beta_1(G) \le \beta_1(H)$.

Lemma 5 Let G and H be graphs without isolated vertices such that $G \mid H$ and $q(H) / q(G) = k$. Let $e_1, e_2, \dots, e_r$ be r edges of H and define $H' = H - \{e_1, e_2, \dots, e_r\}$. If $r < k$, then $\beta_1(G) \le \beta_1(H')$.

Proof In any G–decomposition of H, the edges $e_1, e_2, \dots, e_r$ are contained in at most r copies of G. Hence G is a subgraph of H' and the result follows. ❑

Lemma 6 Let G and H be graphs without isolated vertices such that H is G–decomposable. Let x and y be adjacent vertices of H and define $H'' = H - x - y$. If G has s components, then H'' contains $s - 1$ components of G as a subgraph.

Proof In a G–decomosition of H, assume that G_1 is the copy of G containing the edge $e = xy$ and F_1 is the component of G_1 containing e. Since no edge of the remaining components of G_1 is incident with x or y, the graph H'' contains $s - 1$ components of G as a subgraph. ❑

Theorem 7 $i(P_5) = \infty$.

Proof Let p_1 and p_2 be distinct primes, where $p_2 = 2k + 1$ for some positive integer k. Let $G \cong p_1 P_5$ and let G_2 be the graph shown in Figure 4. We show that $\text{GCD}(G_1, G_2) = P_5$.

Observe that the size of a greatest common divisor of G_1 and G_2 is at most $\gcd(4p_1, 4p_2) = 4$. The graph G_2 is P_5-decomposable into p_2 paths, namely u_i, x, v_i, y, u_{i+1} for $i = 1, 2, \dots, 2k - 1$, together with the paths $u_{2k}, x, v_{2k}, y, u_1$ and z, x, y, w_1, w_2. Since G_1 is P_5-decomposable, P_5 is a greatest common divisor of G_1 and G_2.

Assume that H is a greatest common divisor of G_1 and G_2 different from P_5. Since $H \mid G_1$ and H has size 4, it follows that H is one of $P_4 \cup K_2$, $2P_3$, $P_3 \cup 2K_2$, and $4K_2$. Let $G' = G_2 - w_1 w_2$ and $G'' = G_2 - x - y$. Since $\beta_1(G') = 2$, it follows by Lemma 5 that $\beta_1(H) \le 2$. Therefore, H cannot be $P_4 \cup K_2$, $P_3 \cup 2K_2$, or $4K_2$. Also, since no component of G'' contains more than one edge, it follows by Lemma 6 that

H cannot be $2P_3$. Hence $\mathrm{GCD}(G_1, G_2) = P_5$. Since p_1 and p_2 can be chosen arbitrarily large, $i(P_5) = \infty$. ❑

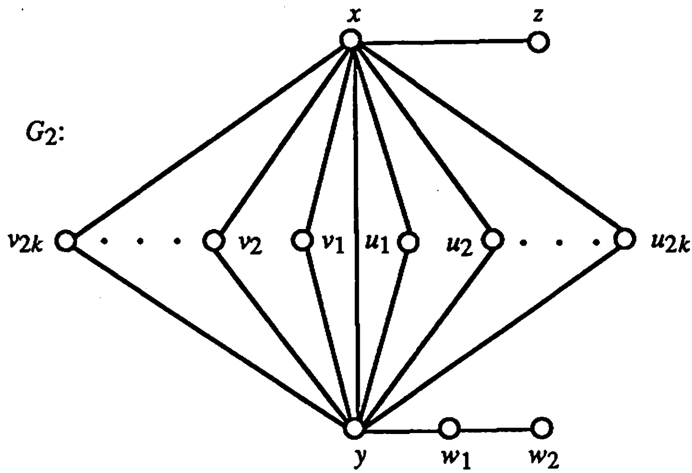

Figure 4

The index of a path P_n for $n \geq 6$ is not known and it appears to be difficult to determine. Next, we show that C_4 has infinite index.

Theorem 8 $i(C_4) = \infty$.

Proof Let p_1 and p_2 be distinct primes. Let $G_1 \cong p_1 C_4$ and let G_2 be the graph shown in Figure 5, where the vertices of G_2 are labeled as indicated with $k = p_2 - 1$. In other words, G_2 is obtained by identifying a vertex of degree $2k$ in $K_{2,2k}$ with the vertex u of the cycle C: v, u, w, z, v and identifying the other vertex of degree $2k$ with the vertex v of C.

The size of a greatest common divisor of G_1 and G_2 is at most $\gcd(4p_1, 4p_2) = 4$. We show that $\mathrm{GCD}(G_1, G_2) = C_4$. For $i = 1, 2, \ldots, k$, define H_i to be the 4-cycle u, x_i, v, y_i, u and let H_{k+1} be the 4-cycle u, v, z, w, u. Then G_2 is decomposable into the 4-cycles H_i $(1 \leq i \leq k + 1)$. Now since G_1 is C_4-decomposable, it follows that C_4 is a greatest common divisor of G_1 and G_2.

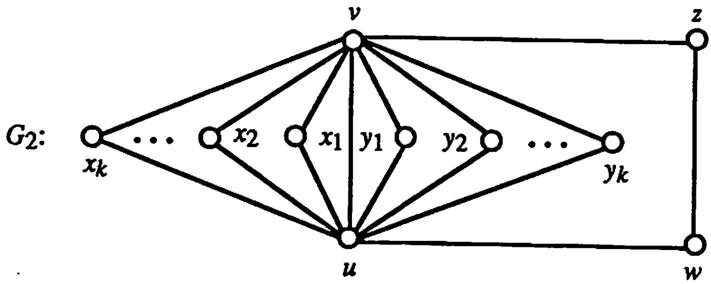

Figure 5

Assume that H is a greatest common divisor of G_1 and G_2 different from C_4. Since $H \mid G_1$ and H has size 4, it follows that H is one of $P_4 \cup K_2$, $2P_3$, $P_3 \cup 2K_2$, and $4K_2$. Let $G' = G_2 - wz$ and $G'' = G_2 - u - v$. Since $\beta_1(G') = 2$, we have $\beta_1(H) \leq 2$ by Lemma 5. Consequently, H cannot be $P_4 \cup K_2$, $P_3 \cup 2K_2$, or $4K_2$. Since no component of G'' contains more than one edge, it follows by Lemma 6 that H cannot be $2P_3$. Hence $\mathrm{GCD}(G_1, G_2) = C_4$. Since p_1 and p_2 can be chosen arbitrarily large, $i(C_4) = \infty$. ❑

The index of the cycle C_5 and other larger odd cycles will be considered in the next section. We conclude this section by showing that the index of the graph $K_4 - e$, obtained by removing an arbitrary edge from K_4, is infinite.

Theorem 9 $i(K_4 - e) = \infty$.

Proof Define $G \equiv K_4 - e$, and let p_1 and p_2 be distinct primes. Furthermore, let $G_1 \equiv p_1 G$, and let G_2 be the graph of size $5p_2$ shown in Figure 6. Thus, both G_1 and G_2 are G–decomposable.

Since p_1 and p_2 are distinct primes, G is a greatest common divisor of G_1 and G_2. Assume that H is a greatest common divisor of G_1 and G_2 different from G. Since $H \mid G_1$, it follows that H is disconnected. Since G_2 is decomposable into p_2 copies of H and $\deg u = 2p_2 + 1$, it follows that $\Delta(H) \geq 3$. Since $\Delta(G_1) = 3$, we have that $\Delta(H) = 3$. Hence, the only possibilities for H are the graphs H_1, H_2, and H_3 of Figure 7.

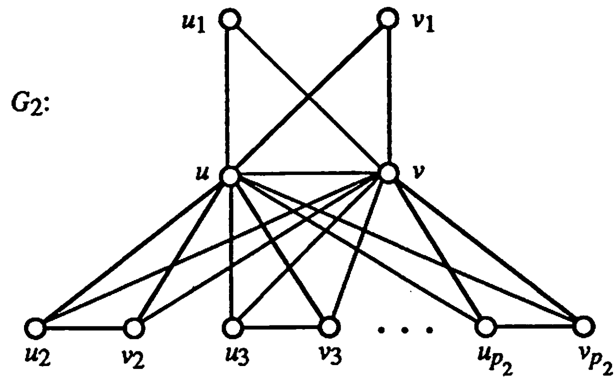

Figure 6

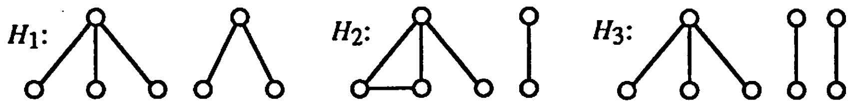

Figure 7

Let $G' = G_2 - \{u_2v_2, u_3v_3, \dots, u_{p_2}v_{p_2}\}$ and $G'' = G_2 - u - v$. Since no component of G'' contains more than one edge, H cannot be H_1 by Lemma 6. Since $\beta_1(G') = 2$, it follows that $\beta_1(H) \le 2$ by Lemma 5. Consequently, H is neither H_2 nor H_3. Thus no such graph H exists, and G is the unique greatest common divisor of G_1 and G_2. Since p_1 and p_2 can be chosen arbitrarily large, $i(G) = \infty$. ❑

3. Graphs With Finite Greatest Common Divisor Index

A graph F is *edge-transitive* if the edge automorphism group of F is transitive, i.e., every edge of F can be mapped into any other edge of F by some edge automorphism. Hence the graph $F - e$ obtained by deleting an arbitrary edge e from F is well-defined. Each complete graph of order at least 2 is edge-transitive, so $K_p - e$ is well-defined for $p \ge 2$. In particular, $K_3 - e \cong P_3$.

The following theorem will be useful in showing the existence of graphs of finite index. The (vertex) independence number of a graph G will be denoted by $\beta(G)$.

Theorem 10 Let F be an edge–transitive graph of order p and size q with $\beta(F) < p/2$. If G is an F–decomposable graph of sufficiently large size, then G is $(F-e) \cup K_2$–decomposable as well. In particular, if G is an F–decomposable graph of size mq, where

$$m \geq 2[\binom{p}{2} - q + 1], \tag{1}$$

then G is also $(F-e) \cup K_2$–decomposable.

Proof Let G be an F–decomposable graph with decomposition $\mathcal{D}$ into m copies of F, where m satisfies inequality (1). For subgraphs F' and F'' in $\mathcal{D}$, we define $\{F', F''\}$ to be a *minor pair* if F' and F'' have fewer than $p/2$ vertices in common; otherwise, $\{F', F''\}$ is a *major pair*.

Suppose that $\{F', F''\}$ is a major pair and $W = V(F') \cap V(F'')$. Since $|W| \geq p/2$, the set W is neither independent in F' nor in F''. Hence, there is an edge e' of F' both of whose incident vertices belong to F'', and an edge e'' of F'' both of whose incident vertices belong to F'. Now, let F' be a fixed subgraph in $\mathcal{D}$. Since there is at least one edge e'' of F'' joining two vertices of F' (and so these vertices are not adjacent in F'), there are at most $\binom{p}{2} - q$ such graphs F''.

Next we construct a graph H of order m whose vertices are the subgraphs in $\mathcal{D}$. We join two vertices F' and F'' of H by an edge if $\{F', F''\}$ is a minor pair. From what we observed earlier, the minimum degree $\delta(H)$ of H is at least $m - 1 - \binom{p}{2} + q$.
By inequality (1),

$$2\delta(H) \geq 2m - 2[\binom{p}{2} - q + 1] \geq m$$

or $\delta(H) \geq m/2$. By a theorem of Dirac [5], H is hamiltonian. Let F_1, $F_2, \ldots, F_m, F_1$ be a hamiltonian cycle of H. Consequently, the m pairs $\{F_i, F_{i+1}\}$, $1 \leq i \leq m$, where $F_{m+1} = F_1$, are minor pairs. For $i = 1, 2, \ldots,$ m, let $U_i = V(F_i) - V(F_{i+1})$. Since $|U_i| \geq p/2$, the set U_i is not independent in F_i. Hence there exists an edge e_i of F_i that is incident with no vertex of F_{i+1}. Therefore, the graph obtained by adding to $F_{i+1} - e_{i+1}$ the edge e_i and its incident vertices is isomorphic to $(F-e) \cup K_2$ and so G is $(F-e) \cup K_2$–decomposable. ❑

We now present two corollaries of Theorem 10.

Corollary 11 For every integer $p \geq 3$, $i(K_p) = 1$.

Proof Since $\beta(K_p) = 1$, it follows by Theorem 10 that $i(K_p) \leq 1$. Since $i(G) \geq 1$ for every graph G, $i(K_p) = 1$. ❑

Corollary 12 There exist graphs of arbitrarily large but finite index. In particular, for every integer $k \geq 2$,

$$k \leq i(C_{2k+1}) \leq 4k^2 - 2k - 1.$$

Proof Consider the cycle C_{2k+1}, $k \geq 2$, of length $2k + 1$. Let $G_1 = kC_{2k+1}$ and $G_2 = K_{2k+1}$. Then C_{2k+1} is a common divisor of G_1 and G_2. Necessarily every other common divisor of G_1 and G_2 having size at least $2k + 1$ is disconnected and has order exceeding $2k + 1$, which is impossible since G_2 has order $2k + 1$. Thus C_{2k+1} is the unique greatest common divisor of G_1 and G_2 and so $i(C_{2k+1}) \geq k$.

Since $\beta(C_{2k+1}) = k$, it follows, by Theorem 10, that $i(C_{2k+1})$ is finite and

$$i(C_{2k+1}) \leq 4k^2 - 2k - 1. \quad ❑$$

For the cycle C_5, we can say a bit more.

Corollary 13 $3 \leq i(C_5) \leq 11$.

Proof By Corollary 12, it follows that $2 \leq i(C_5) \leq 11$. Let $G_1 = 3C_5$ and $G_2 = K_5 \cup C_5$. Since G_1 and G_2 are C_5–decomposable and have order 15, the graph C_5 is a greatest common divisor of G_1 and G_2. Indeed, C_5 is the unique greatest common divisor of G_1 and G_2. ❑

Whether the even cycles as well have finite index is not known. Next we give a class of graphs, none of which is edge–transitive, that have finite index, indeed index 1.

Theorem 14 For every integer $p \geq 6$, $i(K_p - e) = 1$.

Proof It suffices to show that if G is a $(K_p - e)$–decomposable graph containing at least two copies of $K_p - e$, then G too is F–decomposable, where $F \cong \overline{(p-4)K_1 \cup 2K_2} \cup K_2$, that is, F is the graph obtained be deleting two independent edges from the component of order p in $K_p \cup K_2$. Thus, let $\mathcal{D}$ be a decomposition of G into the copies $F_1, F_2, \ldots, F_k$ $(k \geq 2)$ of $K_p - e$. Necessarily, $|V(F_i) \cap V(F_j)| \leq 2$ for $1 \leq i, j \leq k$ and $i \neq j$. Hence, for every integer i $(1 \leq i \leq k)$, there is an edge e_i in F_i neither of whose incident vertices belongs to F_{i+1}. Thus $F_i - e_i \cong \overline{(p-4)K_1 \cup 2K_2}$ and so the graphs obtained by adding e_i and its incident vertices to $F_{i+1} - e_{i+1}$ for $i = 1, 2, \ldots, k$ (where $F_{k+1} - e_{k+1} = F_1 - e_1$) produces an F-decomposition of G. □

4. Concluding Remarks

By Theorem 14, $i(K_p - e) = 1$ for $p \geq 6$ and by Theorems 2 and 9, $i(K_3 - e) = \infty$ and $i(K_4 - e) = \infty$. Hence only $i(K_5 - e)$ remains to be determined. We now show that $i(K_5 - e) = \infty$.

Theorem 15 $i(K_5 - e) = \infty$.

Proof Let p_1 and p_2 be distinct primes. Define $G_1 \cong p_1(K_5 - e)$ and let G_2 be the graph shown in Figure 8, where $t = p_2 - 1$.

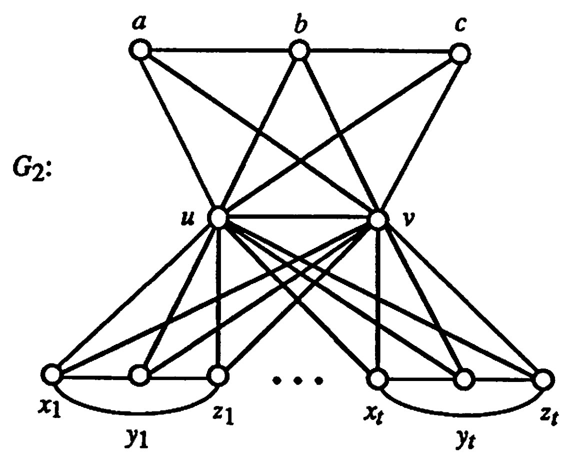

Figure 8

Certainly, $K_5 - e$ is a greatest common divisor of G_1 and G_2. Suppose that H is a greatest common divisor of G_1 and G_2 such that $H \not\cong K_5 - e$. Thus H has size 9. Since $H \mid G_1$, it follows that $\Delta(H) \le 4$, H is disconnected, and the order of each component of H is at most 5. Since G_2 is decomposable into p_2 copies of H and $\deg u = 3p_2 + 1$, we must have $\Delta(H) = 4$. Let H' be a component of H containing a vertex w of degree 4 and let $H'' = H - V(H')$. Since $G_2 - u - v \cong P_3 \cup tK_3$, it follows by Lemma 6 that each component of H'' is a subgraph of K_3. Since $\Delta(H') = 4$, it also follows that each copy of H' contains u or v and hence uv is an edge of a copy of H'.

Suppose that H has no vertex other than w having degree at least 3. In a given H-decomposition of G_2, let r be the number of copies of H in which w occurs at u or v. Then in each of these copies, a vertex of H of degree at most 2 occurs at the remaining vertex in $\{u, v\}$, and in $p_2 - r$ copies of H, there are vertices of H of degree at most 2 at u and v. Thus, counting the number of edges incident with u or v, we obtain

$$6p_2 + 1 \le 4r + 2r + 4(p_2 - r) = 4p_2 + 2r \le 6p_2,$$

a contradiction. Hence H contains a vertex d of degree at least 3 and, necessarily, $d \in V(H')$ and d is adjacent to w.

Let $G = G_2 - \{a, b, c\} - uv$ and $G' = \langle\{a, b, c, u, v\}\rangle \cong K_5 - e$. Consider an arbitrary, but fixed H-decomposition of G_2 into the copies H_1, $H_2, \ldots, H_{p_2}$ of H, where $H_i' \cong H'$ and $H_i'' \cong H''$ are subgraphs of H_i and where w_i (respectively d_i) $\in V(H_i)$ corresponds to the vertex w (respectively d) of H for each $i \in \{1, 2, \ldots, p_2\}$.

Suppose that H' is 2-connected. Since the order of H' is 5 and each 2-connected subgraph of $G_2 - u$ or $G_2 - v$ has order at most 4, it follows that $\{u, v\} \subseteq V(H_i')$ for each $i \in \{1, 2, \ldots, p_2\}$. Thus $E(H_i'') \subseteq E(G_2 - u - v)$ for each i, and each edge of G_2 incident with u or v lies in H_i' for some i. Since $|E(G_2 - u - v)| < 3p_2$, it follows that $|E(H'')| \le 2$ and so $|E(H')| \ge 7$. Without loss of generality, say $ua \in E(H_1')$. Since $\delta(H') \ge 2$, $\{av, ac\} \cap E(H_1') \ne \emptyset$. But if $av \notin E(H_1')$, then a is an endvertex of the copy of H' which contains av, which is a contradiction. Hence $\{ua, av\} \subseteq E(H_1')$ and $\{uc, cv\} \subseteq E(H_i')$ for some $i \in \{1, 2, \ldots, p_2\}$. A similar argument shows that $\{ub, bv\} \subseteq E(H_j')$ for some $j \in \{1, 2, \ldots, p_2\}$.

If $uv \notin E(H_1')$, then the facts that H' is 2-connected and $\Delta(H') = 4$ show that $i = j = 1$ and all edges incident with b are in H_1'; hence $H' \cong W_5$, the wheel of order 5. Since $\deg u = \deg v = 3p_2 + 1$, there is a subgraph H_k' (respectively H_l') with a vertex of degree 4 at u (respectively v). Since H' is 2-connected of order 5, the only possibility is $k = l$, say $k = l = 2$. Then u and v have degree 4 in H_2', so $H_2' \ncong W_5$, a contradiction. Hence $uv \in E(H_1')$.

If $i \neq 1$, that is, if $\{uc, cv\} \subseteq E(H_i')$ for $i \neq 1$, then $uv \notin E(H_i')$ and the fact that H' is 2-connected shows that no vertex has degree 4 in H_i', a contradiction. Hence again $i = j = 1$ so that H' contains $\overline{K_3} + K_2$ as spanning subgraph. Now for each $i \in \{2, 3, \dots, p_2\}$, w_i occurs at one of the vertices x_j, y_j, z_j for some $j \in \{1, 2, \dots, t\}$. Let $d \neq w$ satisfy $\deg_{H'} d = 4$ and say, without loss of generality, that w_2 occurs at x_2 and d_2 occurs at y_2. Since $H \ncong K_5 - e$, at most one of the edges uz_2 and vz_2 is in H_2', say $uz_2 \notin E(H_2')$. Then $uz_2 \notin E(H_i')$ for any i and, since $\overline{K_3} + K_2$ is a subgraph of H', $uz_2 \notin E(H_i')$ for any i (even if $vz_2 \notin E(H_2')$), a contradiction.

Therefore, H' is not 2-connected and thus contains a cutvertex which, necessarily, is w. Since $\deg_{H'} d \geq 3$ and $d \neq w$, it follows that H' consists of an endvertex adjacent to w, together with a 2-connected subgraph Q where $Q \cong K_4 - e$ or $Q \cong K_4$. Let $Q_i \cong Q$ be a subgraph of H_i' for each $i \in \{1, 2, \dots, p_2\}$ and say $Q_1, Q_2, \dots, Q_p$ are contained in G while $Q_{p+1}, Q_{p+2}, \dots, Q_{p_2}$ are not. For each $j \in \{1, 2, \dots, t\}$, let $W_j = \{x_j, y_j, z_j\}$ and $W = \bigcup_{j=1}^{t} W_j$. Now, for each $i \in \{1, 2, \dots, p\}$, Q_i contains at least one edge of $\langle W_j \rangle$ for some $j \in \{1, 2, \dots, t\}$ and no vertices in W_k, $k \neq j$, while H_i' contains at least two edges of $\langle W_j \rangle$. Also, if $1 \neq i$, then H_1' contains no edges and Q_1 no vertices from $\langle W_j \rangle$. Hence $p \leq t$ and there is a one-to-one correspondence between the subgraphs $Q_i, i \in \{1, 2, \dots, p\}$, and those sets W_j which contain vertices of these subgraphs. Without loss of generality, say $V(Q_i) \cap W_i \neq \emptyset$ for each $i \in \{1, 2, \dots, p\}$.

Suppose that $Q \cong K_4$. Then $q(H'') = 2$ and $E(\langle W_i \rangle) \subseteq E(Q_i)$ for each $i \in \{1, 2, \dots, p\}$. Note that each subgraph Q_i that does not contain both u and v lies entirely in G or in $G' - uv$, while the copy Q_r of Q containing both u and v (and thus uv), lies entirely in G' or in $G + uv$. However, the fact that $Q \cong K_4$ shows that each subgraph Q_i in G' must contain uv. It follows that $p = t$, $r = p_2$ and, since no edges of $\langle W \rangle$ are available, Q_r is a subgraph of G'. But then no edges disjoint from $E(Q_r)$ are available for H_r'', a contradiction.

Consequently, $Q \cong K_4 - e$, $q(H'') = 3$, and at most one edge of $\langle W_i \rangle$ is not an edge of H_i', $i \in \{1, 2, \ldots, p\}$. Let $uv \in E(H_r')$. Then $E(H_r'') \subseteq E(\langle W \rangle) \cup \{ab, bc\}$ since no edge of H_r'' is incident with u or v. Hence at most $t - 1$ edges not incident with u or v are available for H_i'', $i \in \{1, 2, \ldots, p_2\} - \{r\}$, so that $2t + 1$ edges of the H_i'' are incident with u or v. But this means that at least one subgraph H_i'' has three edges incident with either u or v, but not both (since each subgraph H_i' contains u or v). Thus $\Delta(H'') = 3$, which is impossible since each component of H'' is a subgraph of K_3. ❑

We can now summarize the results concerning $i(K_p - e)$, $p \geq 3$, as follows.

Corollary 16 The index of $K_p - e$ is

$$i(K_p - e) = \begin{cases} \infty & 3 \leq p \leq 5 \\ 1 & p \geq 6. \end{cases}$$

What characteristic a graph has that gives it a finite (or infinite) index would be interesting to know. However, for the present at least, this question remains unanswered.

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