On finite bases for some PBD-closed sets

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ABSTRACT. Let $H_a^1 = \{v\colon v \geq a, v \equiv 0,1 \pmod a\}$. It is well known that such sets are PBD-closed. Finite bases are found for these sets for $a = 6$, 7 and 8. At the same time we improve the result of Mullin in [6] about finite bases of $H^a = \{v\colon v \geq a+1, v \equiv 1 \pmod a\}$ for $a = 5$ and 6.

1 Introduction

The terminology and notation on this paper follow from that of [6,8]. Let

$$H^a = \{v\colon v \geq a+1, v \equiv 1 \pmod a\}$$
$$H_a^1 = \{v\colon v \geq a, v \equiv 0,1 \pmod a\}.$$

It is well known that such sets are PBD-closed. Finite bases have been found for H^a when $2 \leq a \leq 7$ (see [6,8]), and for H_a^1 when $2 \leq a \leq 5$ (see [8]).

In this paper, finite bases are found for H_a^1 for $a = 6$, 7 and 8. At the same time, we improve the result of Mullin in [6] about finite bases of H^a for $a = 5$ and 6.

For our proof, we record the well-known observation below. An incomplete TD (or ITD) $TD(k,n)-TD(k,m)$ is a quadruple $(X, \mathbf{G}, \mathbf{H}, \mathbf{A})$, which satisfies the following properties:

(1) X is a set of cardinality kn,

(2) $\mathbf{G} = \{G_i\colon 1 \leq i \leq k\}$ is a partition of X into k groups of size n,

(3) $\mathbf{H} = \{H_i\colon 1 \leq i \leq k\}$, where each $G_i \supset H_i$, and $|H_i| = m$, $1 \leq i \leq k$,

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(4) $\mathbf{A}$ is a set of $n^2 - m^2$ blocks of size k, each of which intersects each group in a point,

(5) every pair of points $\{x, y\}$ from distinct groups, such that at least one of x, y is in $U_{1 \le i \le k}(G_i - H_i)$, occurs in a unique block of $\mathbf{A}$.

If $m = 0$ in a $TD(k,n) - TD(k,m)$, the design becomes a $TD(k,n)$. It is well-known that a $TD(k,n)$ is equivalent to $k-2$ mutually orthogonal Latin squares (MOLS) of order n. Denote by $N(n)$ the maximum number of MOLS of ordr n. For a list of lower bounds on $N(n)$, $N \le 10,000$, we refer the reader to Brouwer [1].

An incomplete PBD (or IPBD) is a triple $(X, Y, \mathbf{A})$, where X is a set of points, $X \supset Y$, and $\mathbf{A}$ is a set of blocks which satisfies the properties:

(1) for any $A \in \mathbf{A}$, $|A \cap Y| \le 1$,

(2) any two points x, z, not both in Y, occur in a unique block.

We say that $(X, Y, \mathbf{A})$ is a (v, w, K)-IPBD if $|X| = v$, $|Y| = w$, and $|A| \in K$ for every $A \in \mathbf{A}$. Denote $IB_w(K) = \{v$: a (v, w, K)-IPBD exists $\}$.

The following construction is referred to as the Singular inderect product (SIP) (see [6]).

Lemma 1.1. *Suppose K is a set of positive integers and $u \in K$, suppose v, w and a are integers such that $0 \le a \le w \le v$, and suppose that the following designs exist:*

(1) a $TD(u, v-a) - TD(u, w-a)$,

(2) a (v, w, K)-IPBD, and

(3) a $(u(w-a)+a, K)$-PBD.

Then $u(v-a)+a \in IB_u(K) \cap IB_{u(w-a)+a}(K)$. Hence, in particular, $u(v-a)+a \in B(K)$.

If we let $w = a$ in the SIP, we obtain the Singular direct product.

Lemma 1.2. *Suppose K is a set of positive integers and $u \in K$. Suppose v and w are non-negative integers such that $w \le v$, and there exists a $TD(u, v-w)$, a (v, w, K)-IPBD, and a (w, K)-PBD. Then $u(v-w)+w \in IB_u(K) \cap IB_v(k) \cap IB_w(K)$. Hence, in particular, $u(v-w)+w \in B(K)$.*

If we furthewr specialize this construction by letting $w = 0$, we obtain the Direct product.

Lemma 1.3. *Suppose K is a set of positive integers and $u, v \in K$. If there exists a $TD(u, v)$, then $uv \in IB_u(K) \cap IB_v(K)$. Hence, in particular, $uv \in B(K)$.*

We also need the following construction(see [2]).

Lemma 1.4. *Suppose K is a set of positive integers and u, $u+1 \in K$. Suppose v and w are non-negative integers such that $w \leq v$, and there exists a $TD(u+1,v)$. Then $v+s \in B_s(K)$ and $w+s \in B(K)$ implies $uv+w+s \in B(K)$.*

For convenience, we let

$$V^a = \{v\colon v \text{ is inessential in } H^a\}$$
$$V_a = \{v\colon v \text{ is inessential in } H_a^1\}.$$

The following constructions are clear.

Lemma 1.5. *Suppose that there exists a $B(a+1,v)$, then $\{v, v-1, v-a, v-a-1\} \subset V_a$.*

Lemma 1.6. *Suppose that there exists a $RB(a,v)$, then $v+(v-1)/(a-1) \in V^a$ if $(v-1)/(a-1) \in H^a$, and $v+r \in V_a$ if $r \in H_a^1$ and $0 \leq r \leq (v-1)/(a-1)$.*

In order to apply SIP, we need incomplete transversal designs. We use constructions given in [2] to produce them.

Lemma 1.7. *Suppose there exists a $TD(k,m)$, a $TD(k,m+1)$, a $TD(k+1,t)$, and $0 \leq u \leq t$. Then there exists a $TD(k,mt+u) - TD(k,u)$. Moreover, if a $TD(k,u)$ exists, then*

(1) a $TD(k,mt+u) - TD(k,m)$ exists if $u \neq t$,

(2) a $TD(k,mt+u) - TD(k,t)$ exists if $m > k-2$.

Lemma 1.8. *Suppose there exist a $TD(k,m)$, a $TD(k,m+1)$, a $TD(k,m+2)$, a $TD(k+2,t)$, a $TD(k,u)$ and $0 \leq u$, $v \leq t$. Then there exists a $TD(k,mt+u+v) - TD(k,v)$. Moreover, if a $TD(k,v)$ exists and $u \neq t$, $v \neq t$, then a $TD(k,mt+u+v) - TD(k,m)$ exists.*

Lemma 1.9. *Suppose there exists a $TD(k,m)$, a $TD(k,m+1)$, a $TD(k,m+2)$, a $TD(k+u+1,t)$, a $TD(k,v)$ and $0 \leq v < t$. Then there exists a $TD(k,mt+u+v) - TD(k,m+u)$.*

For $TD(k,v)$ we know (see [2])

Lemma 1.10.

(1) There exists a $TD(7,v)$ if $v \geq 63$,

(2) There exists a $TD(8,v)$ if $v \geq 77$,

(3) There exists a $TD(9,v)$ if $v \geq 781$.

For our purpose we also need the following examples.

From [1] we have

Example 1.11: There exists a $TD(6,10) - TD(6,2)$.

From [5] we have

Example 1.12:

(1) $\{65, 85\} \subset RB(5)$,

(2) $\{31, 66, 76, 96\} \subset B(6)$,

(3) $\{126, 156, 186\} \subset RB(6)$,

(4) $\{91, 169\} \subset B(7)$,

(5) $57 \in B(8)$,

(6) $\{120, 232, 288\} \subset RB(8)$.

From [3,4] we have

Example 1.13:

(1) $\{246, 306, 366, 546, 606\} \subset RB(6)$,

(2) $\{379, 421\} \subset B(7)$,

(3) $344 \in RB(8)$.

From [7] we have

Example 1.14: If q is a prime power, then $q^3 + q^2 - q + 1 \in B(q+1, q^2 - q + 1, q^2 + 2)$.

2 Finite bases of H^a

For H^5 and H^6 the following results were devised by Mullin in [6].

Lemma 2.1. *The 52 values in Table 1 are a basis for H^5.*

6	11	16	21	26	36	41	46	51	56
61	71	86	101	116	131.	141	146	161	166
191	196	201	206	221	226	231	236	251	261
266	281	286	291	296	311	316	321	326	351
356	376	386	401	416	436	441	446	476	491
591	596								

Table 1

Lemma 2.2. *The 98 values in Table 2 are a basis for H^6.*

7	13	19	25	31	37	43	55	61	67
73	79	97	103	109	115	121	127	139	145
157	163	181	193	199	205	211	223	229	235
241	253	265	271	277	283	289	295	307	313
319	331	349	355	361	367	373	379	391	397
409	415	421	439	445	451	457	487	493	499
643	649	655	661	667	685	691	697	709	727
733	739	745	751	781	787	811	1063	1069	1231
1237	1243	1249	1255	1315	1321	1327	1543	1549	1567
1579	1585	1783	1789	1795	1801	1819	1831		

Table 2

We shall show, in this section, that the integers underlined in Table 1 and Table 2 are also in V^5 and V^6, respectively.

Theorem 2.3. *The 40 values not underlined in Table 1 are a basis for H^5.*

Proof: We need only discuss those values underlined in Table 1.

Putting $q = 5$ in Example 1.14 we have $146 \in B(6,21,26)$ and then $146 \in V^5$.

Apply Lemma 1.2 with the following expressions:

$$281 = 11 \times 25 + 6$$
$$401 = 16 \times 25 + 1.$$

We have $281 \in B(6,11)$, $401 \in B(16,26)$ and $281, 401 \in V^5$.

The remaining values shown in Table 3, are all done by the SIP. The required incomplete TDs are all constructed by Lemmas1.7, 1.8 and 1.9.

equation	w	$IPBD$	ITD	$6(w-a)+a$
376=6(66-4)+4	11	66=11×6	62=8×7+1+5	46
386=6(66-2)+2	11	66=11×6	64=9×7+1	56
416=6(81-14)+14	16	81=65+16	67=9×7+3+1	26
		65 ∈ RB(5)	TD(6,10) − TD(6,2)	
436=6(76-4)+4	6	76 ∈ B(6)	72=9×7+7+2	16
441=6(76-3)+3	6	76 ∈ B(6)	73=9×7+7+3	21
446=6(76-2)+2	6	76 ∈ B(6)	74=9×7+7+4	26
491=6(91-11)+11	16	91=15×6+1	80=16×5	41
591=6(106-9)+9	21	106=85+21	97=12×8+1	81
		85 ∈ RB(5)		
596=6(106-8)+8	21	106=85+21	98=13×7+7	86

Table 3

Theorem 2.4. *The 81 values not underlined in Table 2 are a basis for H^6.*

Proof: We need only discuss those values underlined in Table 2.

Apply Lemma 1.5 with the condidtion $\{379, 421\} \subset B(7)$ (see Example 1.13), we have $\{379, 421\} \subset V^6$.

Apply Lemma 1.6 with the condidtion $\{186, 246, 306, 366, 546, 606\} \subset RB(6)$ (see Examples 1.12 and 1.13), we have $\{223, 295, 367, 439, 655, 727\} \subset V^6$.

The remaining values shown in Table 4, are all done by the SIP. The required incomplete TDs are all constructed by Lemmas 1.7, 1.8 and 1.9.

equation	w	$IPBD$	ITD	$7(w-a)+a$
307=7(49-6)+6	7	49=7×7	TD(6,43)−TD(6,1)	13
1543=7(223-3)+3	7	223=186+37 186 ∈ RB(6)	220=29×7+13+4	31
1549=7(223-2)+2	7	223=186+37	221=29×7+13+5	37
1783=7(295-47)+47	49	295=246+49 246 ∈ RB(6)	248=31×7+29+2	61
1789=7(295-46)+46	49	295=246+49	249=31×7+29+3	67
1795=7(295-45)+45	49	295=246+49	250=31×7+29+4	73
1801=7(295-44)+44	49	295=246+49	251=31×7+29+5	79
1819=7(295-41)+41	49	295=246+49	254=31×7+29+8	97
1831=7(295-39)+39	49	295=246+49	256=31×7+29+10	109

Table 4

3 Finite bases of H_a^1

For H_6^1 we have

Lemma 3.1. $\{31, 66, 96, 138\} \subset V_6$.

Proof: $138 \in V_6$ comes from Lemma 1.6 and $126 \in RB(6)$, others v from Lemma 1.5 and $v \in B(6)$.

Theorem 3.2. *The 17 values given in Table 5 are a basis for H_6^1.*

6	7	12	13	18	19	24	25	30	36
37	54	55	60	61	97	102			

Table 5

Proof: First apply Lemma 1.4 with $u = 6$, $s = 0$ or 1, and $v \in \{12\} \cup \{v$ a prime power : $v \equiv 0, 1, 5 \pmod 6$ and $7 \leq v \leq 49\}$. We know the result is true for $r \leq 343$ except for the v values in Lemma 3.1. Then apply Lemma

1.4 with $u = 6$, $s = 0$, $v \equiv 1 \pmod 6$ and $v \geq 55$. We know the result is true for $r > 343$. For the existence of the required $TD(u, v)$ see Lemma 1.10 and [1]. The proof is complete.

For H_7^1 we have

Lemma 3.3. $\{91, 169, 176, 183\} \subset V_7$.

Proof: $91, 169 \in V_7$ comes from $91, 169 \in B(7)$. Applying Lemma 1.2 with the parameter $183 = 14 \times 13 + 1$ implies $183 \in B(14)$ and $183 \in V_7$. Deleting one block from a $TD(8, 23)$ implies $176 \in B(7, 8, 22)$ and $176 \in V_7$.

Lemma 3.4. $\{112, 113, 119, 120, 280, 281, 287\} \subset V_7$.

Proof: Apply Lemma 1.5 with $\{120, 288\} \subset B(8)$.

Theorem 3.5. *The 47 values given in Table 6 are a basis for* H_7^1.

7	8	14	15	21	22	28	29	35	36
42	43	70	71	77	78	84	85	106	126
127	133	134	140	141	147	148	154	155	161
162	168	175	182	189	238	239	245	246	252
253	259	260	266	267	273	274			

Table 6

Proof: First apply Lemma 1.4 with $u = 7$, $s = 0$ or 1, and $v \in \{v \text{ a prime power} : v \equiv 0, 1, 6 \pmod 7 \text{ and } 7 \leq v \leq 49\} \cup \{50\}$. We know the result is true for $r \leq 400$ except for the v values in Lemmas 3.3 and 3.4. Then apply Lemma 1.4 with $u = 7$, $s = 0$, and $v \equiv 1 \pmod 7$ and $v \geq 57$. We know the result is true for $r > 400$. For the existence of the required $TD(u, v)$ see Lemma 1.10 and [1]. The Proof is complete.

For H_8^1 we have

Lemma 3.6. $\{57, 120, 121, 232, 233, 240, 241, 248, 296, 297, 304, 305, 312, 313, 320, 321, 376\} \subset V_8$.

Proof: $57 \in V_8$ comes from $57 \in B(8)$, others from Lemma 1.6 and $\{120, 232, 288, 344\} \subset RB(8)$.

Theorem 3.7. *The 28 values given in Table 7 are a basis for* H_8^1.

8	9	16	17	24	25	32	33	40	41
48	49	56	88	89	96	97	104	105	112
113	160	161	168	169	176	177	184		

Table 7

Proof: First apply Lemma 1.4 with $u = 8$, $s = 0$ or 1, and $v \in \{v$ a prime power : $v \equiv 0, 1, 7 \pmod 8$ and $8 \leq v \leq 64\} \cup \{56, 57\}$. Weknow the result is true for $r \leq 577$ except for the values of v in Lemma 3.6. Then apply Lemma 1.4 with $u = 8$, $s = 0$ and $v \equiv 1 \pmod 8$ and $v \geq 65$. We know the result is true for $r > 577$. For the existence of the requires $TD(u, v)$ see Lemma 1.10 and [1]. The Proof is complete.

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