Intersections of 2-$(v,4,1)$ Directed Designs

E.S. Mahmoodian and Nasrin Soltankhah
Department of Mathematical Sciences
Sharif University of Technology
P.O. Box 11365–9415
Tehran, I.R. Iran
emahmood@irearn.bitnet
soltan@irearn.bitnet

ABSTRACT. The intersection problem for a pair of transitive triple systems (or 2-$(v,3,1)$ directed designs) is solved by Lindner and Wallis and independently by H.L. Fu in 1982–1983. In this paper we determine the intersection problem for a pair of 2-$(v,4,1)$ directed designs.

1 Introduction

Let $0 < t \leq k \leq v$ and $\lambda > 0$ be integers, and V be a set of v elements. Each ordered k-tuple of distinct elements of V is called a *block*. In this note by an n-tuple of V, we mean an ordred n-subset of V. With these meanings of 'block' and 'n-tuple', which we shall use throughout this paper, we make the following definition. A t-(v,k,λ) *directed design* (or simply a t-(v,k,λ)DD) is a pair $(V,\mathcal{B})$, where V is a v-set, and $\mathcal{B}$ is a collection of blocks, such that each t-tuple of V appears in precisely λ blocks. Note that a t-tuple is said to appear in a k-tuple, if its components are contained in that block as a set, and they appear with the same order. For example the 4-tuple *abcd* contains the ordered pairs *ab*, *ac*, *ad*, *bc*, *bd*, and *cd*.

The problem of determining the possible number of common blocks between two designs with the same parameters is studied extensively. For a recent survey on this problem see Billington [1]. Lindner and Wallis [8] and independently H.L. Fu [5] settled the spectrum of possible intersection sizes for 2-$(v,3,1)$DDs (transitive triple systems) for all admissible v. Also the spectrum of possible intersection sizes for ordinary designs $S(2,3,v)$ and $S(2,4,v)$ is settled by Lindner and Rosa [7] and by Colbourn, Hoffman

JCMCC 20 (1996), pp. 225-236

and Lindner [4] respectively. In this paper, we solve the intersection problem for 2-$(v,4,1)$DDs. The existence problem of 2-$(v,4,\lambda)$DDs has been solved in [9]. The necessary and sufficient condition for the existence of a 2-$(v,4,1)$DD is $v \equiv 1 \pmod 3$.

The number of blocks in a 2-$(v,4,1)$DD is equal to $b_v = \frac{v(v-1)}{6}$. Let $J_D(v) = \{0,1,\ldots,b_v-2,b_v\}$, and let $I_D(v)$ denote the set of all possible integers m, such that there exist two 2-$(v,4,1)$DDs with exactly m common blocks. It is clear that $I_D(v) \subseteq J_D(v)$. We prove the following:

Main Theorem. For each $v \equiv 1 \pmod 3$, $v \neq 7$, $I_D(v) = J_D(v)$ and $I_D(7) = \{0,1,7\}$.

For the rest of this section we state some definitions which are needed in the sequel.

Let $K = \{k_1,\ldots,k_l\}$ be a set of numbers. A 2-(v,K,λ) *design* is a v-set V and a collection of k_i-subsets also called blocks, such that every 2-subset of V appears exactly λ times in the blocks. These designs are also called pairwise balanced designs (PBD).

We define a group divisible design as in Hanani [6]. Let V be a v-set such that $V = \cup_{i=1}^{t} G_i$, $G_i \cap G_j = \emptyset$, $|G_i| \in M$ for all i. G_i's are called *groups*. A *group divisible design*, $GD(k,\lambda,M;v)$, is a collection of k-subsets of a v-set V also called blocks such that each block intersects each group in at most one element and a pair of elements of V from different groups occurs in exactly λ blocks.

A *directed group divisible design* $DGD(k,\lambda,M;v)$ (or simply a DGD) is a group divisible design GD in which every block is ordered and each ordered pair formed from elements of different groups occurs in the same number of blocks. If $M = \{m\}$ then we simply write $DGD(k,\lambda,m;v)$.

A (v,k,t) *directed trade* (or simply a (v,k,t)DT) of volume s consists of two disjoint collections T' and T'', each of s blocks, such that each t-tuple occurs in the same number of blocks T' as of T''. Such a DT is usually denoted by $T = T' - T''$.

Let D be a t-(v,k,λ)DD and $T = T' - T''$ be a (v,k,t)DT. If D contains the collection of blocks of T'', then by substituting the blocks of T' for the blocks of T'' in the design, we obtain a new t-(v,k,λ)DD which is denoted by $D + T$. This method of "trade off" is used frequently in this paper.

2 Some small cases

In this section we discuss some small cases needed for general constructions.

$\mathbf{I_D(4) = J_D(4)}$

Let D_1 and D_2 be two 2-$(4,4,1)$DD on the set $\{0,1,2,3\}$, given below. D_1: 0123, 3210 ; D_2: 1023, 3201. We have $|D_1 \cap D_1| = 2$, $|D_1 \cap D_2| = 0$. □

$\{0,1,7\} \subseteq \mathbf{I_D}(7)$

Let D_1 be a 2-$(7,4,1)$DD with the base block (6 0 3 5) mod 7, and D_2 with the base block (5 3 0 6) mod 7, and let α be a permutation given by $\alpha = (01425)$ on the set $\{0,1,\ldots,6\}$. We have $|D_1 \cap D_1| = 7$, $|D_1 \cap D_2| = 0$, and $|D_1 \cap D_2\alpha| = 1$. □

$\mathbf{I_D}(10) = \mathbf{J_D}(10)$

Let D_1 be the following 2-$(10,4,1)$DD, on the set $\{0,1,\ldots,9\}$.

1324	2156	3517	4189	8610	7901	2938	
4207	0582	6972	0436	5309	7683	9654	8745.

Now we list some small $(10,4,2)$DTs:

Directed Trade	Blocks removed	Blocks added
T	8610 7901	8601 7910
T_*	7901 6972	9701 6792
T_1	2156 3517	2516 3157
T_2	2938 7683	2983 7638
T_3	1324 4207	1342 2407
T_4	8745 9654	8754 9645

Let $D_2 = D_1 + T$ and $D_3 = D_1 + T_*$. We have $|D_1 \cap D_2| = 13$, $|D_2 \cap D_3| = 12$, and for $i = 1,2,3,4$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_1| = 15 - (2i+2);$$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_3| = 15 - (2i+3).$$

For the following permutations on the elements of each block of D_3 we have

Permutation α	Intersection number of D_1 and $D_3\alpha$
(0123456789)	0
(03985267)	1
(047)	2
(0423)(15)(67)(89)	3

This results in $I_D(10) = J_D(10)$. □

$\mathbf{I_D}(13) = \mathbf{J_D}(13)$

Let D_1 be the following 2-(13, 4, 1)DD on the set $\{0, 1, \ldots, 9, a, b, c\}$.

123*a*	456*a*	789*a*		147*b*	258*b*	369*b*
*b*321	*b*654	*b*987		*c*741	*c*852	*c*963
159*c*	267*c*	348*c*		3570	2490	1680
0951	0762	0843		*a*753	*a*942	*a*861
			*abc*0			
			0*cba*			

Now we list some small (13, 4, 2)DTs:

Directed Trade	Blocks removed	Blocks added
T	789*a* *b*987	879*a* *b*978
T_*	789*a* *a*942	78*a*9 9*a*42
T_1	123*a* *b*321	213*a* *b*312
T_2	456*a* *b*654	465*a* *b*564
T_3	147*b* *c*741	417*b* *c*714
T_4	258*b* *c*852	285*b* *c*582
T_5	369*b* *c*963	396*b* *c*693
T_6	159*c* 0951	519*c* 0915
T_7	267*c* 0762	627*c* 0726
T_8	348*c* 0843	438*c* 0834
T_9	3570 *a*753	3750 *a*573
T_{10}	1680 *a*861	6180 *a*816
T_{11}	*abc*0 0*cba*	*bac*0 0*cab*
T_{12}	2490 *a*942	4290 *a*924

Let $D_2 = D_1 + T$, $D_3 = D_1 + T_*$. We have $|D_1 \cap D_2| = 24$, $|D_2 \cap D_3| = 23$, and

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_1| = 26 - (2i + 2) \quad i = 1, \ldots, 12;$$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_3| = 26 - (2i + 3) \quad i = 1, \ldots, 11.$$

This results in $I_D(13) = J_D(13)$. □

3 Recursive Constructions

We introduce two constructions, which will be applied in constructing designs with required intersection sizes.

Construction 1. If there exists a group divisible design $(G, \mathcal{B})$ of order v with block size 4 and groups each of size congruent to 0 (mod 3), then there exists a 2-$(2v + 1, 4, 1)$DD.

Proof. Let $(G, \mathcal{B})$ be a group divisible design on the element set V. We form a 2-$(2v+1,4,1)$DD on the element set $V \times Z_2 \cup \{\infty\}$ as follows. For each block $b \in \mathcal{B}$, say $b = \{x, y, z, w\}$, we form a $DGD(4,1,2;8)$ on $b \times Z_2$, such that its groups are $\{x\} \times Z_2, \{y\} \times Z_2, \{z\} \times Z_2, \{w\} \times Z_2$. This DGD exists, as we will see later on. Now for each group g of G, we substitute a 2-$(2|g|+1,4,1)$DD on $(g \times Z_2) \cup \{\infty\}$. □

Construction 2. If there exists a group divisible design $(G, \mathcal{B})$ of order v with block size 4 and groups of size 2 and 5, then there exists a 2-$(2v,4,1)$DD.

Proof. Let $(G, \mathcal{B})$ be such a GD on the element set V. We form a 2-$(2v,4,1)$DD on the element set $V \times Z_2$. For each block $b \in \mathcal{B}$, say $b = \{x, y, z, w\}$ we place a $DGD(4,1,2;8)$ on $b \times Z_2$, such that its groups are $\{x\} \times Z_2$, $\{y\} \times Z_2$, $\{z\} \times Z_2$, $\{w\} \times Z_2$. For each group $g \in G$ we place a 2-$(10,4,1)$DD if $|g| = 5$, and we place a 2-$(4,4,1)$DD if $|g| = 2$ on $g \times Z_2$.□

In applying constructions 1 and 2 we need some GDs and DGDs. We may use a $GD(4,1,\{3,6\};v)$, a $GD(4,1,\{2,5\};v)$, and a $DGD(4,1,2;8)$.

A $GD(4,1,\{3,6\};v)$ exists for all $v \equiv 0 \pmod 3$, $v \neq 9, 18$. For $v \equiv 0, 3 \pmod{12}$, they may be obtained by omitting an element from a 2-$(v+1,4,1)$ design. For $v \equiv 6, 9 \pmod{12}$ they may be obtained by taking an element of the block of size 7 in a 2-$(v+1,\{4,7^*\},1)$ design with only one block of size 7 [2], and omitting this element from all the blocks which contain it.

A $GD(4,1,\{2,5^*\};v)$ with one group of size 5 exists for all $v \equiv 5 \pmod 6$, $v \neq 11, 17$ by [2], and a $GD(4,1,2;v)$ exists for all $v \equiv 2 \pmod 6$, $v \neq 8$ by [3]. Then it can be deduced that a $GD(4,1,\{2,5\};v)$ exists for all $v \equiv 2, 5 \pmod 6$, $v \neq 8, 11, 17$.

We can construct a $DGD(4,1,2;8)$ on the set $\{0, 1, \ldots, 7\}$ as follows:
groups: 12, 34, 56, 07.
blocks: 5103, 4016, 3175, 6714, 2360, 5247, 0425, 7632.

Consider all $DGD(4,1,2;8)$s with the same groups. Let $I_G(8)$ be the set of all possible integers m, such that there exist two such DGDs with exactly m common blocks.

$\mathbf{I_G(8) = \{0,1,2,3,4,5,6,8\}}$

Consider the $DGD(4,1,2;8)$ constructed above, and let D_1 be its blocks. Now we list some small directed trades:

Directed Trade	Blocks removed	Blocks added
T	5103 4016	5013 4106
T_*	4016 0425	0416 4025
T_1	3175 6714	3715 6174
T_2	2360 7632	2630 7362
T_3	5247 0425	2547 0452

Let $D_2 = D_1 + T$ and $D_3 = D_1 + T_*$. We have $|D_1 \cap D_2| = 6$, $|D_2 \cap D_3| = 5$, and

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_3| = 8 - (2i + 3) \quad i = 1, 2\ ;$$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_1| = 8 - (2i + 2) \quad i = 1, 2, 3.$$

This results in $I_G(8) = \{0, 1, 2, 3, 4, 5, 6, 8\}$. □

Lemma 1. Let $(G, \mathcal{B})$ be a group divisible design of order v with b blocks, each of size 4, and $r + s$ groups, r of size 3 and s of size 6. For $1 \leq i \leq b$, let $a_i \in I_G(8)$. For $1 \leq i \leq r$, let $c_i \in I_D(7)$; for $1 \leq i \leq s$, let $d_i \in I_D(13)$. Then there exist two 2-$(2v + 1, 4, 1)$DDs intersecting in precisely

$$\sum_{i=1}^{b} a_i + \sum_{i=1}^{r} c_i + \sum_{i=1}^{s} d_i$$

blocks.

Proof. Using construction 1, take two copies of the same group divisible design $(G, \mathcal{B})$ and construct on them two 2-$(2v + 1, 4, 1)$DDs. Corresponding to each of the blocks $B_1, \ldots, B_b$, place on $B_i \times Z_2$ in the two systems $DGD(4, 1, 2; 8)$s having the same groups, and a_i blocks in common. Corresponding to groups G_i of size 3, place 2-$(7, 4, 1)$DDs with c_i blocks in common, and for groups H_i of size 6, place 2-$(13, 4, 1)$DDs with d_i blocks in common. □

For the $2v$ construction we also have a similar lemma.

Lemma 2. Let $(G, \mathcal{B})$ be a group divisible design of order v with b blocks, each of size 4, and $r + s$ groups, r of size 2 and s of size 5. For $1 \leq i \leq b$, let $a_i \in I_G(8)$. For $1 \leq i \leq r$, let $c_i \in I_D(4)$; for $1 \leq i \leq s$, let $d_i \in I_D(10)$. Then there exist two 2-$(2v, 4, 1)$DDs intersecting in precisely

$$\sum_{i=1}^{b} a_i + \sum_{i=1}^{r} c_i + \sum_{i=1}^{s} d_i$$

blocks.

4 Applying recursions

In this section, we prove the following main theorems.

Theorem 1. For $v \equiv 1$ or 7 (mod 12), $v \neq 7, 19, 37$, $I_D(v) = J_D(v)$.

Proof. There are four possibilities for v: $v = 2(12k)+1$, $v = 2(12k + 3) + 1$, $v = 2(12k + 6) + 1$ or $v = 2(12k + 9) + 1$. Now we may apply construction

1. All the required DGDs and GDs exist. By Lemma 1 and the fact that $\{0,1,7\} \subseteq I_D(7)$, $I_D(13) = J_D(13)$ and $I_G(8) = \{0,1,2,3,4,5,6,8\}$, we deduce $I_D(v) = J_D(v)$ for $v \equiv 1$ or 7 (mod 12), $v \neq 7, 19, 37$. □

Theorem 2. For $v \equiv 4$ or 10 (mod 12), $v \neq 16, 22, 34$ $I_D(v) = J_D(v)$.

Proof. In this case, either v is $v = 2(6k+2)$ or $v = 2(6k+5)$. We may apply construction 2. By Lemma 2 and the fact that $I_D(4) = J_D(4)$, $I_D(10) = J_D(10)$ and $I_G(8) = \{0,1,2,3,4,5,6,8\}$, we deduce $I_D(v) = J_D(v)$ for $v \equiv 4$ or 10 (mod 12), $v \neq 16, 22, 34$. □

5 Remaining small cases

Six small orders $\{7, 16, 19, 22, 34, 37\}$ remain. In this section we handle these small cases.

$\mathbf{I_D(7) = \{0,1,7\}}$

For $v = 7$ we show that there exist only two non-isomorphic directed designs of this order. Using this result we obtain the intersection numbers. The existence of exactly two non-isomorphic 2-(7,4,1)DD is shown in the following three steps.

(i) For a given 2-(7,4,1)DD on the set of elements $\{0,1,\dots,6\}$, we may consider a matrix of size 7×4, whose rows are the blocks of this design. Let x be an element and x_i be the number of appearances of x in the i-th column $(1 \leq i \leq 4)$ of the matrix. We count the number of all ordered pairs such as xy and yx respectively, for a constant x. We have

$$3x_1 + 2x_2 + x_3 = 6 \textit{ and } x_2 + 2x_3 + 3x_4 = 6$$

respectively. From these equations it follows that $0 \leq x_1 \leq 2$ and $0 \leq x_4 \leq 2$. Since the 2-(7,4,2) design obtained from a 2-(7,4,1)DD is symmetric every two blocks have two elements in common. Thus $x_1 = 2$ or $x_4 = 2$ is impossible. Thus $0 \leq x_1 \leq 1$ and $0 \leq x_4 \leq 1$. We solve these two equations for $x_1 = 0$ and for $x_1 = 1$.

(1) : $x_1 = 0,\ x_2 = 2,\ x_3 = 2,\ x_4 = 0$
(2) : $x_1 = 0,\ x_2 = 3,\ x_3 = 0,\ x_4 = 1$
(3) : $x_1 = 1,\ x_2 = 0,\ x_3 = 3,\ x_4 = 0$
(4) : $x_1 = 1,\ x_2 = 1,\ x_3 = 1,\ x_4 = 1$

Clearly for each fixed column (i) we have

$$\sum_{0 \leq x \leq 6} x_i = 7$$

Let a_j be the number of elements with frequencies as in solution (j) above, ($j = 1,2,3,4$). Then for the first and fourth columns we have.

$$0 \times a_1 + 0 \times a_2 + a_3 + a_4 = 7$$
$$0 \times a_1 + 1 \times a_2 + 0 \times a_3 + 1 \times a_4 = 7$$

By solving these equations along with $a_1 + a_2 + a_3 + a_4 = 7$, we obtain $a_1 = a_2 = a_3 = 0$ and $a_4 = 7$. Thus for each element x, $0 \leq x \leq 6$ we have

$$x_1 = x_2 = x_3 = x_4 = 1.$$

(*ii*) By the above result and by an easy argument one may show that, if there exists a block of the form $xyzw$, then no two adjacent elements in this block, say xy, can be adjacent in any other block.

(*iii*) Now if 1234 is a block in a 2-$(7,4,1)$DD, and if the second element of the second block is 1, then in column 1 of this block we must have 3 or 4.

If it is 3, then a unique 2-$(7,4,1)$DD, may be constructed as follows:

D_1 : 1234 3156 2610 0541 5302 6425 4063

If that element is 4, then a unique 2-$(7,4,1)$DD, may be constructed as follows:

D_2 : 1234 4156 5310 2061 6403 0542 3625

For any permutation $\alpha \in S_7$ we have $|D_1\alpha \cap D_2| = 0$ or 1. Thus D_1 and D_2 are non-isomorphic. And for any permutation α we have $|D_1\alpha \cap D_1| = 0, 1$ or 7 and $|D_2\alpha \cap D_2| = 0, 1$ or 7. Therefore we deduce $I_D(7) = \{0, 1, 7\}$. □

Note. One may produce two non-isomorphic cyclic 2-$(7,4,1)$DDs with base blocks (5 3 0 6) and (6 0 3 5) mod 7 respectively. These designs are isomorphic to D_1 and D_2 respectively.

$\mathbf{I_D(16) = J_D(16)}$

Let D_1 be the following 2-$(16,4,1)$DD, on the set $\{0, 1, \ldots, 9, a, b, c, d, e, f\}$.

1248	2359	346*a*	457*b*	568*c*	679*d*	78*ae*	*ea*21	*fb*32	*c*431
*d*542	*e*653	*f*764	89*bf*	19*ac*	2*abd*	3*bce*	4*cdf*	*cb*95	*dca*6
*edb*7	*fec*8	*fd*91	15*de*	26*ef*	137*f*	16*b*0	27*c*0	38*d*0	*ba*84
8751	9862	*a*973	0*b*61	0*c*72	0*d*83	49*e*0	5*af*0	0*e*94	0*fa*5

Now we list some small $(16,4,2)$DTs:

Directed Trade	Blocks removed	Blocks added
T	1248 ea21	2148 ea12
T_*	1248 ba84	1284 ba48
T_1	2359 fb32	3259 fb23
T_2	346a c431	436a c341
T_3	457b d542	547b d452
T_4	568c e653	658c e563
T_5	679d f764	769d f674
T_6	89bf cb95	8b9f c9b5
T_7	19ac dca6	19ca dac6
T_8	2abd edb7	2adb ebd7
T_9	3bce fec8	3bec fce8
T_{10}	4cdf fd91	4cfd df91
T_{11}	15de 8751	51de 8715
T_{12}	26ef 9862	62ef 9826
T_{13}	137f a973	173f a937
T_{14}	16b0 0b61	61b0 0b16
T_{15}	27c0 0c72	72c0 0c27
T_{16}	38d0 0d83	83d0 0d38
T_{17}	49e0 0e94	490e e094
T_{18}	5af0 0fa5	a5f0 0f5a
T_{19}	78ae ba84	7a8e b8a4

Let $D_2 = D_1 + T$ and $D_3 = D_1 + T_*$. We have $|D_1 \cap D_2| = 38$, $|D_2 \cap D_3| = 37$, and

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_3| = 40 - (2i+3) \quad i = 1, \ldots, 18;$$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_1| = 40 - (2i+2) \quad i = 1, \ldots, 19.$$

This results in $I_D(16) = J_D(16)$. □

$\mathbf{I_D(19) = J_D(19)}$

Let D_1 be a 2-(19,4,1)DD on the set $\{0, 1, \ldots, 18\}$, with base blocks (0 3 12 1), (12 0 4 18), (17 3 0 13) mod 19 ([9]). Now we list some small (19,4,2)DTs:

$T:$ $T' = \{0\,3\,12\,1,\ 17\,3\,0\,13\}$; $T'' = \{3\,0\,12\,1,\ 17\,0\,3\,13\}$.

T_* $T'_* = \{17\,3\,0\,13,\ 11\,18\,3\,17\}$; $T''_* = \{3\,17\,0\,13,\ 11\,18\,17\,3\}$.

$T_i:$ $T'_i = \{17+i\ 3+i\ 0+i\ 13+i,\ 0+i\ 3+i\ 12+i\ 1+i\}$;
$T''_i = \{17+i\ 0+i\ 3+i\ 13+i,\ 3+i\ 0+i\ 12+i\ 1+i\}$ $i = 1,\dots,18$.

$T_{l+19}:$ $T'_{l+19} = \{3+l\ 0+l\ 12+l\ 1+l,\ 12+l\ 0+l\ 4+l\ 18+l\}$;
$T''_{l+19} = \{3+l\ 12+l\ 0+l\ 1+l,\ 0+l\ 12+l\ 4+l\ 18+l\}$ $l = 0,\dots,17$.

Let $D_2 = D_1 + T$ and $D_3 = D_1 + T_*$. We have $|D_1 \cap D_2| = 55$, $|D_2 \cap D_3| = 54$, and

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_3| = 57 - (2i+3) \quad i = 1,\dots,18;$$

$$|(D_2 + \sum_{j=1}^{i} T_j) \cap D_1| = 57 - (2i+2) \quad i = 1,\dots,18;$$

$$|(D_2 + \sum_{j=1}^{l} T_{j+19}) \cap D_1| = 57 - (l+40) \quad l = 0,\dots,17.$$

This results in $I_D(19) = J_D(19)$. □

$\mathbf{I_D(22) = J_D(22)}$

In [2] it is shown that there exists a 2-$(22, \{4, 7^*\}, 1)$ design. If we replace the block of size 7 by a 2-$(7,4,1)$DD and put a 2-$(4,4,1)$DD on each block of size 4, then we obtain a 2-$(22,4,1)$DD. From the fact that $\{0,1,7\} \subseteq I_D(7)$ and $I_D(4) = J_D(4)$, we deduce $J_D(22) - \{b-5, b-3\} \subseteq I_D(22)$, where b is the number of blocks of the design. For the remaining intersection numbers, we may use a recursive construction as follows.

We have $22 = 3 \times 7 + 1$. We construct a 2-$(7,4,1)$DD on the set $A = \{1,\dots,7\}$ and we take a Kirkman triple system of order 15 on the set $B = \{8,\dots,22\}$. Let $P_1,\dots,P_7$ be parallel classes of this system. The 4-tuples of D_1, the desired 2-$(22,4,1)$DD on the set $A \cup B$ are:

(i) the 4-tuples of 2-$(7,4,1)$DD;

(ii) the 4-tuples $xyzi$, $(i+1)zyx$ mod 7 such that $\{x,y,z\}$ is a triple in P_i, $i = 1,\dots,7$.

Note that with any prior order on the triples of P_i's the resulting design is a 2-$(22,4,1)$DD.

Now we introduce some directed trades on D_1. If $\{x,y,z\}$, $\{a,b,c\} \in P_1$ and $\{x',y',z\} \in P_2$, then D_1 may be constructed so that the following blocks belong to D_1:

$$xyz1,\ abc1,\ 2zyx,\ 2cba,\ x'y'z2,\ 3zy'x'$$

Now take these small $(22,4,2)$DTs:

Directed Trade	Blocks removed	Blocks added
T	$xyz1\ 2zyx\ x'y'z2$	$yxz1\ z2xy\ x'y'2z$
T_1	$abc1\ 2cba$	$bac1\ 2cab$

Let $D_2 = D_1 + T$. We have $|D_1 \cap D_2| = b-3$, $|D_1 \cap (D_2 + T_1)| = b-5$. Therefore $I_D(22) = J_D(22)$. □

$\mathbf{I_D(34) = J_D(34)}$

By an argument similar to above, using the 2-$(34, \{4, 7^*\}, 1)$ design constructed in [2], it can be shown that $J_D(34) - \{b-5, b-3\} \subseteq I_D(34)$. For the remaining two values, we construct a 2-$(34,4,1)$DD as in [9]. Take a 2-$(11,5,1)$DD with the base block $(3\ 5\ 1\ 4\ 9)$ mod 11. On each block $b = xyzuw$ of this directed design we form a 2-$(16,4,1)$ design on the set $b \times Z_3 \cup \{\infty\}$, such that it contains the quadruples $\{x\} \times Z_3 \cup \{\infty\}$, $\{y\} \times Z_3 \cup \{\infty\}$, $\{z\} \times Z_3 \cup \{\infty\}$, $\{u\} \times Z_3 \cup \{\infty\}$, $\{w\} \times Z_3 \cup \{\infty\}$ and we put an order on the quadruples of this design, such that its quadruples have the order induced by block b. By this method we may construct a 2-$(34,4,1)$DD, D_1: such that the following blocks belong to D_1,

$$(3,1)(5,1)(1,2)(4,1),\ (5,1)(3,1)(6,1)(11,2),$$
$$(3,2)(5,2)(1,3)(4,2),\ (5,2)(3,2)(6,2)(11,3),\ (6,1)(8,2)(4,1)(1,2).$$

Now take these small $(34,4,2)$DTs.

$$T: \quad T' = \{(3,1)(5,1)(1,2)(4,1),\ (5,1)(3,1)(6,1)(11,2),\ (6,1)(8,2)(4,1)(1,2)\};$$
$$T'' = \{(5,1)(3,1)(4,1)(1,2),\ (3,1)(5,1)(6,1)(11,2),\ (6,1)(8,2)(1,2)(4,1)\}.$$

$$T_1: \quad T_1' = \{(3,2)(5,2)(1,3)(4,2),\ (5,2)(3,2)(6,2)(11,3)\};$$
$$T_1'' = \{(5,2)(3,2)(1,3)(4,2),\ (3,2)(5,2)(6,2)(11,3)\}.$$

Let $D_2 = D_1 + T$. We have $|D_1 \cap D_2| = b-3$, $|D_1 \cap (D_2 + T_1)| = b-5$. Therefore $I_D(34) = J_D(34)$. □

$\mathbf{I_D(37) = J_D(37)}$

To construct 2-$(37,4,1)$DD's we use a general recursive construction described below.

Construction 3. If there exists a 2-$(v,4,1)$ design, then there exists a 2-$(3v-2,4,1)$DD.

Proof. Let D be a 2-$(v,4,1)$ design on the set $\{1,\dots,v-1\} \cup \{\infty\}$. If a block $b \in D$ contains ∞, say $b = \{x,y,z,\infty\}$, then we replace b by a 2-$(10,4,1)$DD on the set $(\{x,y,z\} \times Z_3) \cup \{\infty\}$. If b does not contain ∞, say $b = \{x,y,z,w\}$, then we replace b by a $GD(4,1,3;12)$ on the set $b \times Z_3$, such that its groups are $\{x\} \times Z_3$, $\{y\} \times Z_3$, $\{z\} \times Z_3$, $\{w\} \times Z_3$ and on each block of this $GD(4,1,3;12)$ we form a 2-$(4,4,1)$DD. □

Since $37 = 3 \times 13 - 2$, we may use construction 3 for 2-$(37,4,1)$DD. Since $I_D(4) = J_D(4)$ and $I_D(10) = J_D(10)$, therefore we can deduce $I_D(37) = J_D(37)$. □

References

[1] E.J. Billington, The intersection problem for combinatorial designs, *Congressus Numerantium* (to appear).

[2] A.E. Brouwer, Optimal packings of K_4's into a K_n, *J. Combin. Theory,* Series A**26** (1979), 278–297.

[3] A.E. Brouwer, H. Hanani and A. Schrijver, Group divisible designs with block size four, *Discrete Math.* **20** (1977), 1–10.

[4] C.J. Colbourn, D.G. Hoffman and C.C. Lindner, Intersections of $S(2,4,v)$ designs, *Ars Combin.* **33** (1992), 97–111.

[5] H.L. Fu, Directed triple systems having a prescribed number of triples in common, *Tamkang J. Math.* **14** (1983), 85–90.

[6] H. Hanani, Balanced incomplete block designs and related designs, *Discrete Math.* **11** (1975), 255–369.

[7] C.C. Lindner and A. Rosa, Steiner triple systems having a prescribed number of triples in common, *Canad. J. Math.* **27** (1975), 1166–1175. Corrigendum: *Canad. J. Math.* **30** (1978), 890.

[8] C.C. Lindner and W.D. Wallis, Embeddings and prescribed intersection of transitive triple systems, *Annals of Discrete Math.* **15** (1982), 265–272.

[9] D.J. Street and J.R. Seberry, All DBIBDs with block size four exist, *Utilitas Math.* **18** (1980), 27–34.