A note on the strong 2-cover conjecture for graphs without K_5 minors

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ABSTRACT. In [J. of Combinatorial Theory (B), 40(1986), 229–230], Fleischner proved that if G is a 2-edge-connected planar graph and if $\mathcal{C}_0 = \{C_1, \cdots, C_k\}$ is a collection of edge-disjoint cycles of G, then G has a cycle double cover $\mathcal{C}$ that contains $\mathcal{C}_0$. In this note, we show that this holds also for graphs that do not have a subgraph contractible to K_5.

Our terminology follows that of Bondy and Murty [1]. For the definitions of cycle covers and cycle decompositions, see [6]. The strong 2-Cover Conjecture asserts that given a cycle C in a 2-edge-connected graph G, there exists a cycle 2-cover $\mathcal{C}$ with $C \in \mathcal{C}$. In [3], Fleischner proved the following:

Theorem A (Fleischner [3]). *Let G be a 2-edge-connected planar graph and let $\mathcal{C}_0 = \{C_1, \cdots, C_k\}$ be a set of edge-disjoint cycles of G. Then there exists a cycle 2-cover $\mathcal{C}$ of G such that $\mathcal{C}_0$ is a subfamily of $\mathcal{C}$.* □

In this note we shall generalize Theorem A to the following:

Theorem 1. *Let G be a 2-edge-connected graph that does not have a subgraph contractible to K_5, and let $\mathcal{C}_0 = \{C_1, \cdots, C_k\}$ be a set of edge-disjoint cycles of G. Then there exists a cycle 2-cover $\mathcal{C}$ of G such that $\mathcal{C}_0$ is a subfamily of $\mathcal{C}$.*

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Our proof of Theorem 1 is based on the following result, which generalizes a theorem in [5].

Theorem 2. *Let G be a 2-edge-connected graph that does not have a subgraph contractible to K_5, and let G' be an eulerian supergraph of G obtained from G by duplicating every edge of G at most once. Then there exists a cycle decomposition $\mathcal{D}$ of G' such that each element of $\mathcal{D}$ corresponds to a cycle of G.*

Proof of Theorem 1: We follow the idea of Fleischner [2]. Let $X = \bigcup_{C \in \mathcal{C}_0} E(C)$, and let G' be the eulerian supergraph of G obtained from G by duplicating every edge in $E(G) - X$ exactly once. By Theorem 2, G' has a cycle decomposition $\mathcal{D}$ such that each $C \in \mathcal{D}$ can be viewed as a cycle in G. Thus $\mathcal{D}$ and $\mathcal{C}_0$ together will form a cycle 2-cover of G that has $\mathcal{C}_0$ as a subfamily. □

In order to prove Theorem 2, we need more terms. Let G be a graph. For a vertex $v \in V(G)$, let $P(v)$ denote a partition of the set of edges incident with v in G. An element of $P(v)$ is called a *forbidden part at v*. Let $\mathbf{P}=\bigcup_{v \in V(G)} P(v)$, and call $\mathbf{P}$ a *set of forbidden parts* of G. A graph G with an associated set of forbidden parts $\mathbf{P}$ is denoted by $(G, \mathbf{P})$.

A cycle decomposition $\mathcal{D}$ of $(G, \mathbf{P})$ is *good* with respect to $\mathbf{P}$ if for every cycle $C \in \mathcal{D}$ and for any $P \in \mathbf{P}$, $|E(C) \cap P| \leq 1$. An edge cut of $(G, \mathbf{P})$ is *bad* if there is some part $P \in \mathbf{P}$ such that $2|P \cap T| > |T|$. The following theorem was first proved by Fleischner and Frank [4] for planar graphs and was recently generalized by Zhang [7] to its current form:

Theorem B (Zhang [7]). *Let G be an eulerian graph containing no subgraph contractible to K_5 and let $\mathbf{P}$ be a set of parts of G without bad cuts. Then $(G, \mathbf{P})$ has a good cycle decomposition with respect to $\mathbf{P}$.* □

Proof of Theorem 2: Let $X = E(G') - E(G)$. For each $v \in V(G) = V(G')$, let E_v denote the edges incident with v in G'. We define $P(v)$ as follows: if $e \notin X$ and $e \in E_v$, then $\{e\}$ is a part in $P(v)$; if $e \in E_v \cap X$, then e must be a duplicate of an edge e' incident with v in G, and we define $\{e, e'\}$ to be a part in $P(v)$. Having defined $P(v)$ in the above way for every vertex $v \in V(G)$, we obtain a set of forbidden parts $\mathbf{P}$ of G'. With this definition of $\mathbf{P}$, one can easily see that a cycle decomposition $\mathcal{D}$ of $(G', \mathbf{P})$ is good with respect to $\mathbf{P}$ if and only if every cycle $C \in \mathcal{D}$ corresponds to a cycle in G. We shall first show that $(G', \mathbf{P})$ has no bad cuts.

By contradiction, we assume that there is a bad cut T and so there is some forbidden part $P \in \mathbf{P}$ such that $2|P \cap T| > |T|$. Since G' is eulerian, $|T|$ is even. Since G is connected, $|T| \geq 2$. By the definition of $\mathbf{P}$, $|P| \leq 2$ and so we have $4 \geq 2|P \cap T| > |T| \geq 2$. It follows that $|P \cap T| = 2 = |T|$. However, this forces that T consists of an edge $e' \in G$ and an edge $e \in X$

which is a duplicate of e', and so G has a cut-edge e', contrary to the assumption that G is 2-edge-connected.

Thus (G', $\mathbf{P}$) has no bad cuts. Since G has no subgraph contractible to K_5, G' has no such subgraph either. Thus by Theorem B, G' must have a good cycle decomposition $\mathcal{D}$ with respect to $\mathbf{P}$. By the definition of $\mathbf{P}$, each element of this $\mathcal{D}$ corresponds to a cycle in G. This proves Theorem 2. □

To conclude this note, we indicate that the Petersen graph P_{10}, which can indeed be contracted to a K_5, does not have this property when $|\mathcal{C}_0| = 2$. In fact, let C_1, C_2 be the two 5-cycles obtained from P_{10} by deleting a perfect matching of P_{10}. Let $\mathcal{C}_0 = \{C_1, C_2\}$. Then any cycle 2-cover of P_{10} that contains $\mathcal{C}_0$ as a subfamily would yield a cycle cover of P_{10} of length at most 20, which was proved impossible in [2].

References

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