Cycle Covers of Planar 3-Connected Graphs

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ABSTRACT. We show that the edges of a planar 3-connected graph with n vertices can be covered by at most $[(n+1)/2]$ cycles. This proves a special case of a conjecture of Bondy that the edges of a 2-connected graph can be covered by at most $(2n-1)/3$ cycles.

1 Introduction

In [3] Lai and Lai prove that the edges of every plane triangulation with n vertices can be covered by at most $(2n-1)/3$ cycles. This proves a special case of a conjecture of Bondy [2] that the edges of every 2-connected graph with n vertices can be covered by at most $(2n-1)/3$ cycles. The plane triangulations are a subclass of the class of planar 3-connected graphs. We shall show that the edges of any planar 3-connected graph with n vertices can be covered by $[(n+1)/2]$ cycles, and that for even n this bound is sharp.

2 Definitions and Notation

All of our *graphs* are without loops and multiple edges. A *cycle* in a graph G is a simple closed curve made up of edges of G. We shall use the well-known theorem of Menger [4] that if x and y are two vertices of an n-connected graph then there are n paths from x to y meeting only at x and y.

In this paper we shall use edge shrinking and vertex splitting. We shall say that a graph G is obtained from a planar graph H by *shrinking* edge e if a graph isomorphic to G is obtained by contracting e to a point and coalescing double edges that bound any resulting 2-sided faces. If H and G are 3-connected and G is obtained from H by shrinking edge e we call e a *shrinkable edge*. The inverse of edge shrinking is *vertex splitting*. This is illustrated in Figure 1. When we split a vertex v, one or two edges meeting

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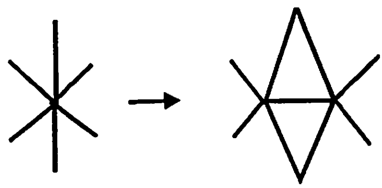
v may also be split (See Figure 1). This gives rise to three types of vertex splittings depending on how many edges are split. The edges that are split will be called the *splitting edges.*



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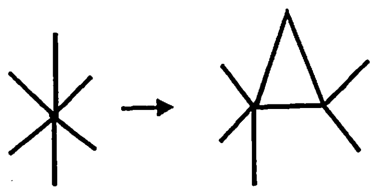

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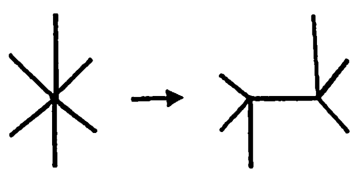


type III

Figure 1

We shall use the following theorem of Steinitz [5]:

Theorem. *The planar 3-connected graphs can be generated from T, the complete graph on four vertices, by vertex splitting.*

It follows that any planar 3-connected graph other than T has a shrinkable edge.

A graph G is called a *refinement* of a graph H provided a graph isomorphic to G can be obtained by adding vertices to edges of H (the set of vertices added is allowed to be empty).

An *n-wheel* is a graph consisting of a cycle with n-edges together with a vertex that is joined to each vertex of the cycle.

3 Cycle Coverings

Lemma 1. *If G is a planar 3-connected graph that is not T and e is an edge of G, then there is a shrinkable edge of G independent of (ie. not sharing a vertex with) e.*

Proof: The proof is very similar to the proof of a theorem by the author in [1]. We shall sketch the proof and the reader can consult [1] for details.

We begin by dualizing. We shall show that given an edge e of the dual G^* of G, there is a removable edge of G^* not sharing a common face with e. Let F and F' be the two faces of G containing e. If all paths from vertices of F, other than the vertices of e, return to F before meeting $F' - e$ then the vertices of e disconnect G. Thus we may find a path P joining $F - e$ and $F' - e$, giving us a subgraph of G (namely $F \cup F' \cup P$) that is a refinement H_1 of T in G^*.

Now we construct a sequence of subgraphs H_i such that each H_i is obtained from H_{i-1} by adding paths of edges that are in $G - H_{i-1}$, and such that each H_i is a refinement of a planar 3-connected graph. The paths to be added are chosen in the following manner. If H_{i-1} has a 2-valent vertex x, let P be a maximal path in H_{i-1} containing x such that all but the end vertices of P are 2-valent. If all paths in $G - H_{i-1}$ from 2-valent vertices of P return to P before meeting any other vertex of H_{i-1}, then G is not 3-connected, thus we can find a path from a 2-valent vertex of P to some other vertex of H_{i-1} not on P. We add this path to H_{i-1} creating H_i, which is also a refinement of a planar 3-connected graph. We continue in this way until we reach an H_i without 2-valent vertices. If $H_i = G$ then the last path added is the removable edge we seek. If H_i is not G, then we choose a vertex v of H_i that meets an edge e of $G - H_i$ and consider all paths in $G - H_i$ that begin with e. If there is such a path that ends at a vertex not joined to v, then we add that path creating H_{i+1}, and continue as above. If all such paths from all such vertices v end at vertices joined to v, we add one such path Q. We then consider all paths in $G - (H_i \cup Q)$ starting at a 2-valent vertex of Q (such a vertex must exist or G has a double edge). By the above reasoning one such path R must end at some vertex other than an end vertex of Q. Now we note that since all paths that we could add to H_i connect one vertex of an edge of H_i to the other vertex of that edge, $Q \cup R$ can be regarded as the union of three paths emanating from a 2-valent vertex of Q and ending at the vertices of a triangular face F'' of H_i. We add these three paths creating H_{i+1}. If $H_{i+1} = G$ then each of the three edges of F'' are removable. One of these edges is not on F or F''. Since we have shown that there is a removable edge not on any face containing e, duality gives us that there is a shrinkable edge not sharing a vertex with e. □

Theorem. *If G is a planar 3-connected graph with n vertices then the edges of G can be covered by at most $[(n+1)/2]$ cycles.*

Proof: Our proof is by induction on the number of vertices of G. The induction starts with T, the complete graph on four vertices, for which the theorem is obvious. Suppose now that G has $n > 4$ vertices. We choose a shrinkable edge $a'a''$ and shrink it to the vertex a producing a graph G'. Next by Lemma 1 we may choose a shrinkable edge $b'b''$ independent of an edge meeting a in G', and thus missing a in G' (provided G' is not T, a case we shall treat separately). We shrink $b'b''$ to a vertex b producing graph G''.

By induction G'' can be covered by at most $[(n-1)/2]$ cycles. We shall show that we can split vertices a and b to produce G, extend certain cycles that passed through a and b, and find a new cycle that contains the edges of G that are not covered by the existing and extended cycles from G''.

We begin by looking at the ways that we may extend cycles that pass through a. We shall first treat the case where both splitting are of type I. Figure2a shows the three ways that cycles may cover the two splitting edges meeting a. Figure 2b shows ways that we may extend these cycles. In each case there are edges that are not covered by the extended cycles. We shall call these the uncovered edges. For a given way of extending the cycles there is a corresponding set of uncovered edges (shown in Figure 2c as heavy edges). A given covering of the splitting edges together with a set of uncovered edges will be called an uncovered edge configuration (or UEC). In Figure 2c the label under the uncovered edges is the label for the corresponding UEC.

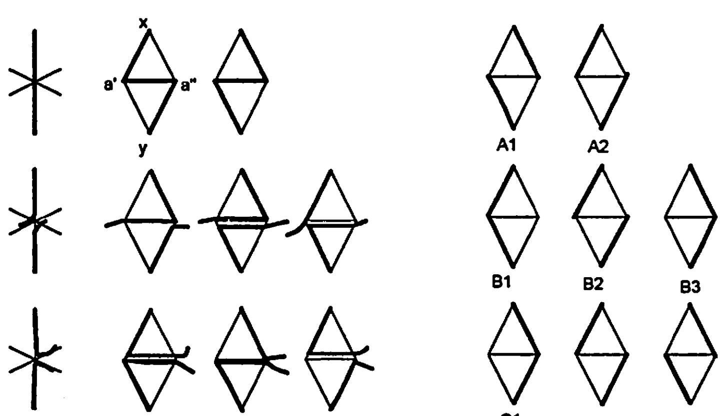


Figure 2a (the cycles)

Figure 2b (the cycle extensions)

Figure 2c (the uncovered edges)

We now add two vertices to G, producing a graph H. One new vertex, u, we will join to vertices x, y, (see Figure 2), and a point p between a' and a'' on edge $a'a''$ (the "point" p now becomes the vertex p). Let the other two vertices of the triangular faces containing b' and b'' be z and w. The second new vertex, v, we join to z, w, and a point q between b' and b'' on edge $b'b''$ (similarly, q becomes a vertex). It is easily verified that H is 3-connected. It follows that in H there are three paths P_1, P_2 and P_3 joining u and v and meeting only at u and v. We shall be interested only in the portions Q_1, Q_2 and Q_3, respectively, joining the vertex set $V_1 = \{a', a'', x, y\}$ to the vertex set $V_2 = \{b', b'', z, w\}$. It should be noted that some vertices of V_1 can be the same as vertices in V_2, in which case some of the Q_i's would just be those vertices. Although the sets V_1 and V_2 can have nonempty intersection we note that our choosing $b'b''$ to miss a prevents either of the triangles determined by vertices in V_1 from being the same faces as any triangles determined by the set V_2. Let K_1 be the subgraph of G consisting of the vertices in V_1 and the edges $a'a''$, $a'x$, $a'y$, $a''x$ and $a''y$. Let K_2 be the similarly constructed subgraph using vertices of V_2.

For each UEC we wish to construct a cycle consisting of two of the Q_i's, a path through K_1 covering the uncovered edges of that subgraph, and a path through K_2 covering the uncovered edges of that subgraph.

First we shall consider the case where K_1 and K_2 have no vertices in common. We observe that if two of the Q_i's meet K_1 at x and y, then for each type of UEC there is a path joining the two Q_i's which covers the uncovered edges of K_1. If two Q_i's meet two consecutive vertices on the boundary cycle of K_1, for example vertices x and a'', then for some but not all of the UEC's there is such a covering path. Note, however, that for each UEC that does not admit such a path, one can change to another cycle extension and get a UEC that admits a covering path. For example, if two Q_i's meet x and a'' then no such path will cover the uncovered edges in UEC C3, but we can change the cycle extension to get the UEC C2, and we see that now there is a path through K_1, joining the Q_i's, and covering the uncovered edges.

We now see that if two Q_i's meet K_1 at x and y, and do not meet K_2 at b' and b'', that these two paths may be extended to cycles covering the uncovered edges either directly or by means of a change in the cycle extensions at K_2. If the two Q_i's do meet K_2 at b and b'' then we take the Q_i meeting x and the third path meeting K_1 and observe that these two paths meet consecutive vertices on the bounding cycles of both K_1 and K_2 and thus we can extend the paths to cycles covering the uncovered edges.

Next we have the case where we don't have paths meeting both x and y, and by symmetry we don't have paths meeting both z and w. In this case it is easy to see that we can find two paths that meet consecutive vertices on the bounding cycles of both K_1 and K_2, and thus the covering cycle can be found.

If $G' = T$, then G must be one of the two planar 3-connected graphs with five vertices (Figure 12). The theorem is easily verified in these two cases.

We need now to consider what happens if K_1 and K_2 have nonempty intersection, for this might cause our choice of paths through K_1 and K_2 to create a cycle that intersects itself (and thus it would not be a cycle). By the choice of b, we have that $\{a', a''\} \cap \{b', b''\}$ is empty. Cycle modifications are simply choices of whether the modified path passes through a', or a'' (or b' or b'') and thus if a cycle modification caused a cycle to intersect itself, it would have intersected itself before the two vertices were split. We do, however, have to treat several cases in the construction of the new cycle. There are nine ways that K_1 and K_2 can intersect. These are shown in Figures 3–11. In Figure 3 the new circuit is indicated by dotted lines. Every covering by the original circuits admits an extension that covers e_1 and e_2 in K_1 (See Figure). After the circuits have been extended for K_1 we may then choose an extension in K_2 (based on the circuits created by the extension for K_1).

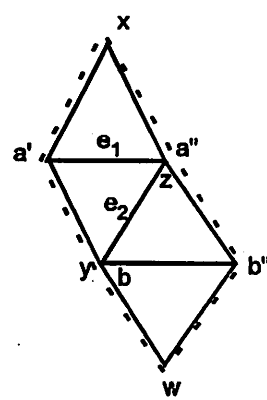


Figure 3

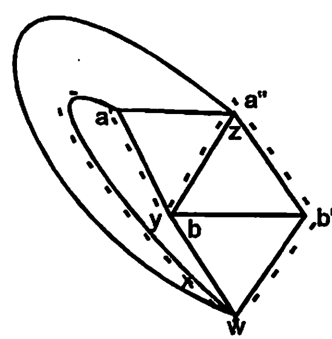
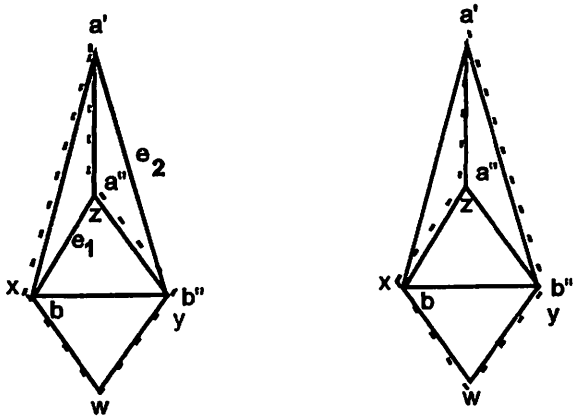


Figure 4

In Figure 4 we choose the new circuit as shown, and as in the argument for Figure 3, the extensions exist to cover the other edges. In Figure 5 we choose the circuit indicated in Figure 5A. If there is no extension that covers e_1 and e_2 then we choose the circuit indicated in Figure 5B and the necessary extension will exist. We can then find an extension for K_2.

In Figures 6 and 7, the vertex x must be one of the Q_i's. For K_1 the path x and any other path will serve for the two connecting paths, thus there will be a pair of connecting paths one of which is x, and the paths through K_1 and K_2 will not cause the new circuit to intersect itself. In figures 8, 9 and 10, x and y must be two of the paths. If we choose these for the connecting paths then again the new cycle will not intersect itself.



5A 5B

Figure 5

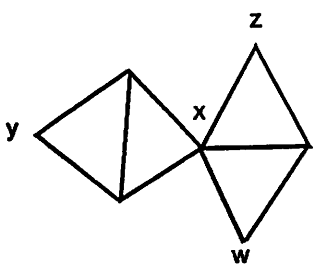


Figure 6

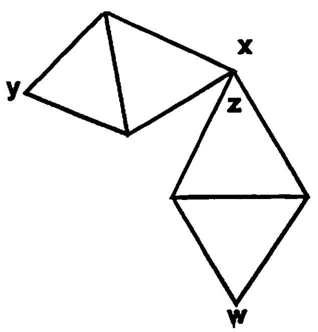


Figure 7

The configuration in Figure 11 presents a problem. There are no choices for the paths that will work. In this case we observe that the graph L consisting of the cycle C made up of edges e_1, e_2 and e_3 together with the vertices and edges enclosed by C will be a planar 3-connected graph. By Lemma 1 we can choose a removable edge in L independent of e_3. We use that as our second removable edge. If this choice also creates a configuration as in Figure 11, we repeat this process. The new graph L will be a proper subgraph of the graph L in the first application of this argument, thus the process eventually ends.

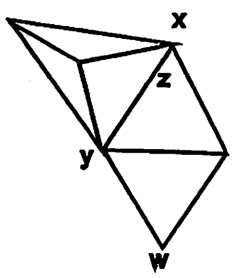


Figure 8

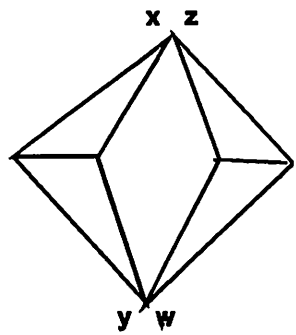


Figure 9

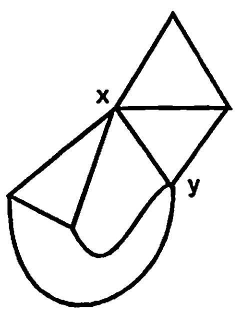


Figure 10

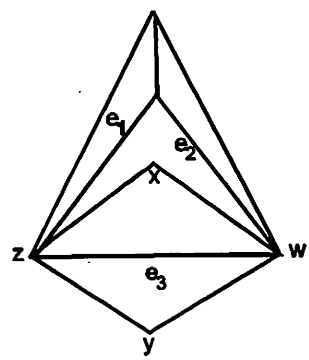


Figure 11

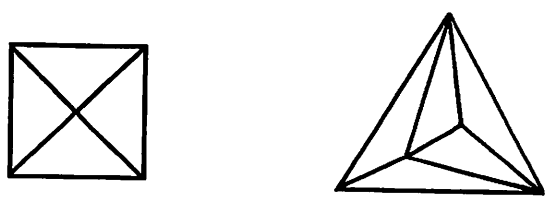


Figure 12

Similar but much simpler arguments hold if there are splittings of type II or III involved. If a splitting of type II is involved we take three paths that meet the vertices of the triangle that is produced. This is accomplished by joining the vertex z to each vertex of the triangle. If a splitting of type III is involved we choose three paths from $a'a''$ in the following way. Let $a'a'''$ be an edge meeting e. We join the new vertex z to the three vertices a', a'' and a'''. Then we take our three paths from z to w. One of these paths will pass through a'''. We delete the edge from z to a''' and substitute $a'a'''$. Now we have three paths from $a'a'''$ to the other configuration with one of the paths containing a''. The argument now is similar to the above argument.

In each case the addition of one cycle gives us a covering of the edges and we have used at most $[(n+1)/2]$ cycles. □

We note that the $(2k+1)$-wheel requires $k+1$ cycles to cover the $(2k+1)$-valent vertex, thus for even n the bound is sharp. Using the $2k$-wheels one sees that the bound is within one of being sharp for odd n. The second graph in Figure 12 shows that the bound is sharp for $n = 5$.

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