# Intersections Of Triple Systems: Small Orders*

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ABSTRACT. The set of all possible intersection sizes between two simple triple systems $TS(v,\lambda_1)$ and $TS(v,\lambda_2)$ is denoted by $Int(v,\lambda_1,\lambda_2)$. In this paper, for $6 \leq v \leq 14$, and for all feasible $\lambda$'s, $Int(v,\lambda_1,\lambda_2)$ is determined.

## 1 Introduction

A *triple system* of *order* $v$ and *index* $\lambda$, $TS(v,\lambda)$, is a pair $(X,\mathcal{B})$, where $X$ is a $v$-set and $\mathcal{B}$ is a collection of 3-subsets of $X$ called *triples* in which every 2-subset of $X$ appears in precisely $\lambda$ triples. A triple system is *simple* if $\mathcal{B}$ contains no repeated triples.

For $(X,\mathcal{B}_1)$, a simple $TS(v,\lambda_1)$ and $(X,\mathcal{B}_2)$, a simple $TS(v,\lambda_2)$, the set of all possible sizes of intersections of $\mathcal{B}_1$ and $\mathcal{B}_2$ is denoted by $Int(v,\lambda_1,\lambda_2)$.

Kramer and Mesner[3] have indicated the importance of intersection problems. Lindner and Rosa[5] haved obtained a complete determination

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of $Int(v,1,1)$. (A comprehensive reference on this is [6].) But the determination of $Int(v,\lambda_1,\lambda_2)$ for all positive integers $v$, $\lambda_1$, and $\lambda_2$ remains unsolved.

It seems that in many combinatorial problems, small cases frequently behave rather peculiarly. Thus, the intent of this paper is to obtain the intersection numbers for triple systems of small orders ($6 \le v \le 14$). The techniques utilized here is a combination of the application of random permutations and the trade-off method[2]. To determine $Int(14,6,6)$ and $Int(12,4,2)$, we have used the method of combining $t$-designs [1], which is essentially the contents of Lemmas 3 and 4.

## 2 Some auxiliary lemmas and some constructions

It is well known that a necessary and sufficient condition for the existence of a simple $TS(v,\lambda)$ is that $\lambda_v|\lambda$ and $\lambda \le v-2$ in which $\lambda_v = \gcd(v-2,6)$.

**Lemma 1.** *Let $v$, $\lambda_1$, and $\lambda_2$ be three positive integers such that $\lambda_1 \equiv \lambda_2 \equiv 0 \pmod{\lambda_v}$. If $r \in Int(v,\lambda_1,\lambda_2)$, then $\lambda_2 v(v-1)/6 - r \in Int(v,v-2-\lambda_1,\lambda_2)$.*

**Proof:** Let $(X,\mathcal{B}_1)$ and $(X,\mathcal{B}_2)$ be two simple triple systems of order $v$ and indices $\lambda_1$ and $\lambda_2$, respectively, which intersect in exactly $r$ triples. Then $(X,P_3(X)\setminus\mathcal{B}_1)$ is a simple $TS(v,v-2-\lambda_1)$ and clearly $|P_3(X)\setminus\mathcal{B}_2\cap\mathcal{B}_1| = \lambda_2 v(v-1)/6 - r$. □

By replacing $\lambda_1$ with $v-2-\lambda_1$ in Lemma 1, we obtain the following result.

**Corollary 1.** *$Int(v,v-2-\lambda_1,\lambda_2) = \{\lambda_2 v(v-1)/6 - r \mid r \in Int(v,\lambda_1,\lambda_2)\}$.* □

In view of the above corollary we need only consider the case $\lambda_1,\lambda_2 \le (v-2)/2$. In some cases even we can essentially reduce the problem to the case $\lambda_1 = \lambda_2 = \lambda_v$. The main idea behind this is in the following definition and Lemma 2.

Let $v$, $\lambda_1$, and $\lambda_2$ be three positive integers such that $\lambda_1 \le \lambda_2 < v-2$, and $\lambda_1 \equiv \lambda_2 \equiv 0 \pmod{\lambda_v}$. Let $j$ be a nonnegative integer. We denote by $Int^j(v,\lambda_1,\lambda_2)$ the set of all integers $k$ such that there exist two simple triple systems of order $v$ and indices $\lambda_1$ and $\lambda_2$, e.g. $(X,\mathcal{B}_1)$) and $(X,\mathcal{B}_2)$, intersecting in exactly $k$ triples and $j$ mutually disjoint simple $TS(v,\lambda_v)$, say $(X,\Gamma_1),\dots,(X,\Gamma_j)$ such that

$$\mathcal{B}_1 \cap \Gamma_i = \mathcal{B}_2 \cap \Gamma_i = \emptyset, \qquad for\ i = 1,\dots,j.$$

**Lemma 2.** *Let $l$, $m$, $n$, $j$ be four positive integers such that $m+n-l \le j$. If $r \in Int^j(v,\lambda_1,\lambda_2)$, then*

$$l\lambda_v v(v-1)/6 + r \in Int(v,\lambda_1+m\lambda_v,\lambda_2+n\lambda_v).$$

**Proof:** Let $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ be two simple $TS(v, \lambda_1)$ and $TS(v, \lambda_2)$, respectively, intersecting in exactly $k$ triples and let $(X, \Gamma_1), \ldots, (X, \Gamma_j)$ be $j$ mutually disjoint simple $TS(v, \lambda_v)$ such that

$$\mathcal{B}_1 \cap \Gamma_i = \mathcal{B}_2 \cap \Gamma_i = \emptyset, \quad for\ i = 1, \ldots, j,$$

and let

$$\mathcal{C}_1 = \mathcal{B}_1 \cup (\bigcup_{i=1}^{m} \Gamma_i),$$
$$\mathcal{C}_2 = \mathcal{B}_2 \cup (\bigcup_{i=m-l+1}^{m+n-l} \Gamma_i).$$

Then clearly $(X, \mathcal{C}_1)$ and $(X, \mathcal{C}_2)$ are two simple $TS(v, \lambda_1 + m\lambda_v)$ and $TS(v, \lambda_2 + n\lambda_v)$, respectively, and

$$|\mathcal{C}_1 \cap \mathcal{C}_2| = |\mathcal{B}_1 \cap \mathcal{B}_2| + |\bigcup_{i=m-l+1}^{m} \Gamma_i| = r + l\lambda_v v(v-1)/6.$$

□

Utilizing above lemmas and tables one can determine $Int(v, \lambda_1, \lambda_2)$ for $v \leq 13$ with some gaps in the case $v = 12$. In the following lemma we fill in these gaps. To prove this lemma we need the following definition. If $G$ is a multigraph which is $l$-factorable, and $\mathcal{F} = \{F_1, \ldots, F_d\}$ and $\mathcal{G} = \{G_1, \ldots, G_d\}$ are two $l$-factorizations of $G$ such that $F_i$'s and $G_i$'s have no repeated edges, then we say $\mathcal{F}$ and $\mathcal{G}$ intersect in exactly $k$ edges if $\sum_{i=1}^{d} |F_i \cap G_i| = k$. Let $J(G, l)$ be the set of all $k$'s such that there exist a pair of $l$-factorizations of $G$ having exactly $k$ edges in common.

**Lemma 3.** *Let $l \in \{1, 2\}$. If $r, s \in Int(7, l, l)$, and $t, k \in J(K_6, l)$, then $r + s + t + m \in Int(12, 2l, 2l)$.*

**Proof:** Let $X = \{1, \cdots, 12\}$, and define

$$X_0 = \{1, \cdots, 7\}, \quad Y_1 = \{8, \cdots, 12\},$$

$$X_1 = \{1, \cdots, 6\}, \quad Y_2 = \{7, \cdots, 12\},$$

$$X_2 = \{1, \cdots, 5\}, \quad Y_3 = \{6, \cdots, 12\}.$$

Let $(X_0, \mathcal{B}_{10})$ , $(Y_3, \mathcal{B}_{13})$ , $(X_0, \mathcal{B}_{20})$ , *and* $(Y_3, \mathcal{B}_{23})$ be four simple triple systems of order 7 and index $l$ such that $|\mathcal{B}_{10} \cap \mathcal{B}_{20}| = r$, and $|\mathcal{B}_{13} \cap \mathcal{B}_{23}| = s$. For $i \in \{1, 2\}$ let $\mathcal{F}_i = \{F_j^i | 1 \leq j \leq 5\}$ and $\mathcal{G}_i = \{G_j^i | 1 \leq j \leq 5\}$ be two $l$-factorizations of $lK_6$ on $X_1$ and $Y_2$, respectively such that $\mathcal{F}_1$ and $\mathcal{F}_2$ intersect in exactly $t$ edges, while $\mathcal{G}_1$ and $\mathcal{G}_2$ intersect in exactly $m$ edges, and define

$$\mathcal{B}_1 = \mathcal{B}_{10} \cup \mathcal{B}_{13} \cup (\cup_{i=1}^{5} (i+7) * F_i^1) \cup (\cup_{i=1}^{5} i * G_i^1),$$
$$\mathcal{B}_2 = \mathcal{B}_{20} \cup \mathcal{B}_{23} \cup (\cup_{i=1}^{5} (i+7) * F_i^2) \cup (\cup_{i=1}^{5} i * G_i^2).$$

Here, $i * F_i^j = \{A \cup \{i\} | A \in F_i^j\}$. Then it is an easy exercise to check that $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ are simple triple systems of index $2l$, and clearly $|\mathcal{B}_1 \cap \mathcal{B}_2| = r + s + t + m$. □

**Corollary 2.** $\{27, 30, 33\} \in Int(12, 2, 2)$.

**Proof:** It is an easy exercise to construct two 1-factorizations of $K_6$ intersecting in exactly 3,6,9, or 15 edges, and by Table 2 we have $\{0, 1, 3, 7\} \subset Int(7, 1, 1)$. Now the assertion is a straightforward consequence of Lemma 3. □

**Corollary 3.** $\{31, 34, 38, 71, 74, 77, 78, 80\} \in Int(12, 4, 4)$.

**Proof:** It is an easy exercise to construct two 2-factorizations of $2K_6$ intersecting in exactly 0,3,6,26,27, or 30 edges, and by Table 2 we have $\{2, 5, 8, 14\} \subset Int(7, 2, 2)$. Now the assertion is a straightforward consequence of Lemma 3. □

**Lemma 4.** *If $r \in Int(6, 2, 2)$, $s \in Int(10, 4, 4)$, $0 \le t \le 4$, and $0 \le \ell \le 2$, then*

$$b = r + s + 10t + 36\ell \in Int(14, 6, 6).$$

**Proof:** Let $X = \{1, \cdots, 14\}$, and denote

$$X_0 = \{1, \cdots, 6\}, \quad Y_1 = \{7, \cdots, 14\},$$

$$X_1 = \{1, \cdots, 5\}, \quad Y_2 = \{6, \cdots, 14\},$$

$$X_2 = \{1, \cdots, 4\}, \quad Y_3 = \{5, \cdots, 14\}.$$

Let $(X_0, \mathcal{B}_{10})$ and $(X_0, \mathcal{B}_{20})$ be two simple TS(6,2) intersecting in exactly $r$ triples, let $(Y_3, \mathcal{B}_{13})$ and $(Y_3, \mathcal{B}_{23})$ be two TS(10,4) intersecting in exactly $s$ triples, and let

$$\begin{aligned} F_1 &= \{\{i,j\} | 1 \le i < j \le 5, \ \ and \ \ i - j \equiv 1 \ or \ -1 \ (mod\ 5)\}, \\ F_2 &= P_2(X_1) \setminus F_1, \\ G_1 &= \{\{i,j\} | 6 \le i < j \le 14, \ \ and \ \ i - j \equiv 1, 3, 5 \ or \ 7 \ (mod\ 9)\}, \\ G_2 &= P_2(Y_2) \setminus G_1. \end{aligned}$$

Now for $1 \le j \le 8$, and $1 \le i \le 4$, define

$$H_j = \begin{cases} F_1, & 5 - t \le j \le 8 - t, \\ F_2, & otherwise, \end{cases} \qquad K_i = \begin{cases} G_1, & 3 - \ell \le i \le 4 - \ell, \\ G_2, & otherwise, \end{cases}$$

and

$$\begin{aligned} \mathcal{B}_1 &= \mathcal{B}_{10} \cup \mathcal{B}_{13} \cup (\cup_{i=7}^{14} i * H_{i-6}) \cup (\cup_{i=1}^{4} i * K_i) \\ \mathcal{B}_2 &= \mathcal{B}_{20} \cup \mathcal{B}_{23} \cup (\cup_{i=7}^{10} i * F_1) \cup (\cup_{i=11}^{14} i * F_2) \cup (\cup_{i=1}^{2} i * G_1) \cup (\cup_{i=3}^{4} i * G_2), \end{aligned}$$

then, it is an easy exercise to show that $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ are two simple TS(14,6) intersecting in exactly b triples. □

**Theorem.** *Let* $6 \le v \le 14$, $\lambda_1, \lambda_2 \equiv 0 \; mod(\lambda_v)$, $\lambda_1 \le \lambda_2 \le (v-2)/2$, *and* $m = \lambda_1 v(v-1)/6$. *Then*

*i) For* $v = 6$, $\lambda_1 = \lambda_2 = 2$, $v = 10$, $\lambda_1 = \lambda_2 = 4$, *and* $v = 14$, $\lambda_1 = \lambda_2 = 6$, *we have*

$$Int(v, \lambda_1, \lambda_2) = \{0, \dots, m\} \setminus A(v, \lambda_1, \lambda_2),$$

*where* $A(v, \lambda_1, \lambda_2) = \{m - i | i = 1, 2, 3, 5\} \cup \{i | i = 1, 2, 3, 5\}$.

*ii) For* $9 \le v \le 14$,

$$Int(v, \lambda_1, \lambda_2) = \begin{cases} \{0, \dots, m\} \setminus A(v, \lambda_1, \lambda_2) & \lambda_1 = \lambda_2, \\ \{0, \dots, m\} & \textit{otherwise}, \end{cases}$$

*where* $A(v, \lambda_1, \lambda_2) = \{m - i | i = 1, 2, 3, 5\}$, *except for* $A(9, 1, 1) = \{5, 7, \dots, 11\}$.

*iii)* $Int(7, 1, 1) = \{0, 1, 3, 7\}; Int(7, 1, 2) = \{1, 4, 7\}; Int(7, 2, 2) = \{2, 5, 8, 14\}$.

**Proof:** If $6 \le v \le 9$, the result is given in Tables 1,2, and 3. For $v = 10$, Table 4 together with Lemma 2 give rise to the result. For $v = 11$ or 13, Tables 5 and 7 prove the assertion. For $v = 12$, Table 6 together with Lemmas 2 and 3 establishes the statement, and finally for $v = 14$ the assertion is an immediate consequence of Lemma 4. □

## 3 Tables of intersection numbers ($6 \le v \le 13$)

In Tables 1-6, except for $Int(12, 4, 2)$, $Int(v, \lambda_1, \lambda_2)$ is given explicitly for $6 \le v \le 13$, and for all feasible $\lambda$'s.

## References


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**Table 1.** $v = 6$
Design used in Table 1.

$$BD(1) = \{\{1,2,3\},\{1,2,4\},\{1,3,5\},\{1,4,6\},\{1,5,6\},\{2,3,6\},\{2,4,5\},\{2,5,6\},\{3,4,5\},\{3,4,6\}\}.$$

| First Design | | Second Design | | $Int(6,\lambda_1,\lambda_2)$ |
|---|---|---|---|---|
| Name | Index | Rule of Construction | Index | |
| BD(1) | 2 | (3 5 4 6)BD(1) | 2 | 0 |
| BD(1) | 2 | (1 2)BD(1) | 2 | 4 |
| BD(1) | 2 | (3 5)(4 6)BD(1) | 2 | 6 |
| BD(1) | 2 | BD(1) | 2 | 10 |

**Table 2.** $v = 7$
Designs used in Table 2.

$$BD(1) = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\},$$
$$BD(2) = BD(1) + (3\ 4)(5\ 6\ 7)BD(1).$$

| First Design | | Second Design | | $Int(7,\lambda_1,\lambda_2)$ |
|---|---|---|---|---|
| Name | Index | Rule of Construction | Index | |
| BD(1) | 1 | ( 3 4 )( 5 6 7 )BD(1) | 1 | 0 |
| BD(1) | 1 | ( 5 6 7 )BD(1) | 1 | 1 |
| BD(1) | 1 | ( 6 7 )BD(1) | 1 | 3 |
| BD(1) | 1 | BD(1) | 1 | 7 |
| BD(2) | 2 | ( 5 7 6 )BD(1) | 1 | 1 |
| BD(2) | 2 | ( 6 7 )BD(1) | 1 | 4 |
| BD(2) | 2 | BD(1) | 1 | 7 |
| BD(2) | 2 | ( 3 4 5 6 )BD(2) | 2 | 2 |
| BD(2) | 2 | ( 5 6 7 )BD(2) | 2 | 5 |
| BD(2) | 2 | ( 6 7 )BD(2) | 2 | 8 |
| BD(2) | 2 | BD(2) | 2 | 14 |

**Table 3.** $v = 9$

Designs used in Table 3.

$BD(1) = \{\{1,2,3\},\{1,4,7\},\{1,5,9\},\{1,6,8\},\{2,4,9\},\{2,5,8\},$
$\{2,6,7\},\{3,4,8\},\{3,5,7\},\{3,6,9\},\{4,5,6\},\{7,8,9\}\}$

$BD(2) = BD(1) + (3\ 4)(6\ 8\ 9)BD(1)$

$BD(3) = (BD(2) \setminus \{\{1,2,4\},\{1,3,7\},\{3,4,8\},\{2,7,8\}\})$
$\cup\{\{1,2,7\},\{1,3,4\},\{2,4,8\},\{3,7,8\}\}$

$BD(4) = BD(2) + (3\ 5\ 6)(4\ 8\ 9)BD(1)$

$BD(5) = BD(3) + (3\ 4)(6\ 9\ 7\ 8)BD(1)$

$BD(6) = (3\ 6\ 9\ 4\ 7\ 8)BD(3) + (3\ 7\ 8\ 4)(5\ 6)BD(1)$

| First Design | | Second Design | | $Int(9, \lambda_1, \lambda_2)$ |
|---|---|---|---|---|
| Name | Index | Rule of Construction | Index | |
| BD(1) | 1 | ( 3 4 )( 6 8 9 ) BD(1) | 1 | 0 |
| BD(1) | 1 | ( 6 7 8 9 ) BD(1) | 1 | 1 |
| BD(1) | 1 | ( 6 7 )( 8 9 ) BD(1) | 1 | 2 |
| BD(1) | 1 | ( 7 8 9 ) BD(1) | 1 | 3 |
| BD(1) | 1 | ( 5 6 7 9 ) BD(1) | 1 | 4 |
| BD(1) | 1 | ( 8 9 ) BD(1) | 1 | 6 |
| BD(1) | 1 | BD(1) | 1 | 12 |
| BD(3) | 2 | ( 3 4 )( 6 9 7 8 ) BD(1) | 1 | 0 |
| BD(3) | 2 | ( 5 7 9 6 8 ) BD(1) | 1 | 1 |
| BD(3) | 2 | ( 6 9 8 7 ) BD(1) | 1 | 2 |
| BD(3) | 2 | ( 6 8 7 9 ) BD(1) | 1 | 3 |
| BD(3) | 2 | ( 7 9 8 ) BD(1) | 1 | 4 |
| BD(3) | 2 | ( 7 9 ) BD(1) | 1 | 5 |
| BD(3) | 2 | ( 7 8 9 ) BD(1) | 1 | 6 |
| BD(3) | 2 | ( 7 8 ) BD(1) | 1 | 7 |
| BD(3) | 2 | ( 6 8 9 ) BD(1) | 1 | 8 |
| BD(3) | 2 | ( 8 9 ) BD(1) | 1 | 9 |
| BD(3) | 2 | ( 5 8 ) BD(1) | 1 | 10 |
| BD(3) | 2 | BD(1) | 1 | 11 |
| BD(2) | 2 | BD(1) | 1 | 12 |

Table 3 (Cont.)

| First Design | | Second Design | | $Int(9, \lambda_1, \lambda_2)$ |
|---|---|---|---|---|
| Name | Index | Rule of Construction | Index | |
| BD(2) | 2 | ( 3 5 6 )( 4 8 9 ) BD(2) | 2 | 0 |
| BD(2) | 2 | ( 3 5 9 6 )( 4 8 7 ) BD(2) | 2 | 1 |
| BD(2) | 2 | ( 4 5 8 6 7 9 ) BD(2) | 2 | 2 |
| BD(2) | 2 | ( 4 5 6 7 8 9 ) BD(2) | 2 | 3 |
| BD(2) | 2 | ( 5 7 8 6 9 ) BD(2) | 2 | 4 |
| BD(2) | 2 | ( 6 8 7 9 ) BD(2) | 2 | 5 |
| BD(2) | 2 | ( 6 8 )( 7 9 ) BD(2) | 2 | 6 |
| BD(2) | 2 | ( 6 7 9 8 ) BD(2) | 2 | 7 |
| BD(2) | 2 | ( 6 7 8 9 ) BD(2) | 2 | 8 |
| BD(2) | 2 | ( 7 8 9 ) BD(2) | 2 | 9 |
| BD(2) | 2 | ( 6 7 )( 8 9 ) BD(2) | 2 | 10 |
| BD(2) | 2 | ( 5 6 8 9 7 ) BD(2) | 2 | 11 |
| BD(2) | 2 | ( 7 9 ) BD(2) | 2 | 12 |
| BD(2) | 2 | ( 5 6 8 9 ) BD(2) | 2 | 13 |
| BD(2) | 2 | ( 7 8 ) BD(2) | 2 | 14 |
| BD(2) | 2 | ( 4 7 )( 5 8 )( 6 9 ) BD(2) | 2 | 15 |
| BD(2) | 2 | ( 8 9 ) BD(2) | 2 | 16 |
| BD(2) | 2 | ( 2 3 )( 4 7 )( 6 9 ) BD(2) | 2 | 17 |
| BD(2) | 2 | ( 5 8 ) BD(2) | 2 | 18 |
| BD(3) | 2 | ( 5 8 ) BD(2) | 2 | 20 |
| BD(2) | 2 | BD(2) | 2 | 24 |
| BD(5) | 3 | ( 3 5 4 )( 6 7 9 8 )BD(1) | 1 | 0 |
| BD(5) | 3 | ( 3 5 7 6 4 ) BD(1) | 1 | 1 |
| BD(5) | 3 | ( 5 7 6 8 ) BD(1) | 1 | 2 |
| BD(5) | 3 | ( 6 9 8 7 ) BD(1) | 1 | 3 |
| BD(5) | 3 | ( 5 6 7 8 9 ) BD(1) | 1 | 4 |
| BD(5) | 3 | ( 7 9 8 ) BD(1) | 1 | 5 |
| BD(5) | 3 | ( 6 7 ) BD(1) | 1 | 6 |
| BD(5) | 3 | ( 7 8 ) BD(1) | 1 | 7 |
| BD(5) | 3 | ( 7 8 9 ) BD(1) | 1 | 8 |
| BD(5) | 3 | ( 6 8 9 ) BD(1) | 1 | 9 |
| BD(5) | 3 | ( 8 9 ) BD(1) | 1 | 10 |
| BD(5) | 3 | BD(1) | 1 | 11 |
| BD(5) | 3 | ( 3 4 )( 6 9 7 8 ) BD(1) | 1 | 12 |

Table 3 (Cont.)

| First Design | | Second Design | | $Int(9, \lambda_1, \lambda_2)$ |
|---|---|---|---|---|
| Name | Index | Rule of Construction | Index | |
| BD(4) | 3 | ( 2 6 4 5 8 9 ) BD(3) | 2 | 0 |
| BD(4) | 3 | ( 3 6 5 )( 4 9 8 ) BD(3) | 2 | 1 |
| BD(4) | 3 | ( 3 6 8 5 )( 4 7 9 ) BD(3) | 2 | 2 |
| BD(4) | 3 | ( 4 8 )( 5 7 6 ) BD(3) | 2 | 3 |
| BD(4) | 3 | ( 4 5 )( 6 9 )( 7 8 ) BD(3) | 2 | 4 |
| BD(4) | 3 | ( 6 9 7 8 ) BD(3) | 2 | 5 |
| BD(4) | 3 | ( 6 8 )( 7 9 ) BD(3) | 2 | 6 |
| BD(4) | 3 | ( 6 9 )( 7 8 ) BD(3) | 2 | 7 |
| BD(4) | 3 | ( 7 8 9 ) BD(3) | 2 | 8 |
| BD(4) | 3 | ( 6 7 8 9 ) BD(3) | 2 | 9 |
| BD(4) | 3 | ( 6 7 9 ) BD(3) | 2 | 10 |
| BD(4) | 3 | ( 6 7 8 ) BD(3) | 2 | 11 |
| BD(4) | 3 | ( 7 8 ) BD(3) | 2 | 12 |
| BD(4) | 3 | ( 7 9 ) BD(3) | 2 | 13 |
| BD(4) | 3 | ( 6 7 )( 8 9 ) BD(3) | 2 | 14 |
| BD(4) | 3 | ( 5 7 9 8 6 ) BD(3) | 2 | 15 |
| BD(4) | 3 | ( 8 9 ) BD(3) | 2 | 16 |
| BD(4) | 3 | ( 6 7 ) BD(3) | 2 | 17 |
| BD(4) | 3 | ( 4 7 ) BD(3) | 2 | 18 |
| BD(4) | 3 | ( 4 7 )( 6 9 ) BD(3) | 2 | 19 |
| BD(4) | 3 | ( 2 3 )( 4 5 8 )( 7 9 )BD(3) | 2 | 20 |
| BD(4) | 3 | BD(3) | 2 | 21 |
| BD(4) | 3 | ( 2 6 4 9 3 8 7 5 ) BD(3) | 2 | 22 |
| BD(4) | 3 | ( 2 4 8 )( 3 6 5 9 ) BD(3) | 2 | 23 |
| BD(4) | 3 | BD(2) | 2 | 24 |

Table 3 (Cont.)

| First Design | | Second Design | | $Int(9,\lambda_1,\lambda_2)$ |
|---|---|---|---|---|
| Name | Index | Rule of Construction | Index | |
| BD(5) | 3 | ( 1 7 )( 2 6 )( 3 4 )( 5 9 )BD(5) | 3 | 0 |
| BD(6) | 3 | BD(4) | 3 | 1 |
| BD(5) | 3 | ( 2 5 6 3 4 8 9 7 ) BD(5) | 3 | 2 |
| BD(5) | 3 | ( 1 4 )( 2 5 6 7 9 )( 3 8 ) BD(5) | 3 | 3 |
| BD(5) | 3 | ( 2 3 6 8 4 7 9 5 ) BD(5) | 3 | 4 |
| BD(5) | 3 | ( 2 3 7 6 5 4 9 ) BD(5) | 3 | 5 |
| BD(5) | 3 | ( 3 4 8 6 )( 5 9 7 ) BD(5) | 3 | 6 |
| BD(5) | 3 | ( 3 5 9 7 4 8 6 ) BD(5) | 3 | 7 |
| BD(5) | 3 | ( 3 4 8 9 7 5 6 ) BD(5) | 3 | 8 |
| BD(5) | 3 | ( 4 5 7 9 )( 6 8 ) BD(5) | 3 | 9 |
| BD(5) | 3 | ( 4 5 6 )( 7 8 9 ) BD(5) | 3 | 10 |
| BD(5) | 3 | ( 4 5 7 8 9 6 ) BD(5) | 3 | 11 |
| BD(5) | 3 | ( 5 7 6 8 ) BD(5) | 3 | 12 |
| BD(5) | 3 | ( 4 5 7 6 )( 8 9 ) BD(5) | 3 | 13 |
| BD(5) | 3 | ( 5 6 8 7 9 ) BD(5) | 3 | 14 |
| BD(5) | 3 | ( 5 6 8 7 ) BD(5) | 3 | 15 |
| BD(5) | 3 | ( 5 6 )( 7 8 9 ) BD(5) | 3 | 16 |
| BD(5) | 3 | ( 6 8 7 9 ) BD(5) | 3 | 17 |
| BD(5) | 3 | ( 7 8 9 ) BD(5) | 3 | 18 |
| BD(5) | 3 | ( 6 7 8 9 ) BD(5) | 3 | 19 |
| BD(5) | 3 | ( 6 8 )( 7 9 ) BD(5) | 3 | 20 |
| BD(5) | 3 | ( 5 6 8 ) BD(5) | 3 | 21 |
| BD(5) | 3 | ( 7 8 ) BD(5) | 3 | 22 |
| BD(5) | 3 | ( 6 8 9 ) BD(5) | 3 | 23 |
| BD(5) | 3 | ( 6 7 ) BD(5) | 3 | 24 |
| BD(5) | 3 | ( 3 4 )( 6 8 9 ) BD(5) | 3 | 25 |
| BD(5) | 3 | ( 6 8 ) BD(5) | 3 | 26 |
| BD(5) | 3 | ( 3 4 )( 6 8 7 9 ) BD(5) | 3 | 27 |
| BD(5) | 3 | ( 8 9 ) BD(5) | 3 | 28 |
| BD(5) | 3 | ( 2 3 )( 4 7 )( 5 9 ) BD(5) | 3 | 29 |
| BD(5) | 3 | ( 3 7 )( 5 8 ) BD(5) | 3 | 30 |
| BD(5) | 3 | ( 1 2 )( 4 7 )( 5 9 )( 6 8 )BD(5) | 3 | 32 |
| BD(5) | 3 | BD(5) | 3 | 36 |

**Table 4.** $v = 10$

Designs used in Table 4.

$$\begin{aligned}BD(1) = \{&\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,6\},\{1,5,7\},\{1,6,7\},\\ &\{1,8,9\},\{1,8,10\},\{1,9,10\},\{2,3,4\},\{2,3,8\},\{2,5,9\},\\ &\{2,6,9\},\{2,6,10\},\{2,7,8\},\{2,7,10\},\{3,5,9\},\{3,5,10\},\\ &\{3,6,10\},\{3,7,8\},\{3,7,9\},\{4,5,8\},\{4,5,10\},\{4,6,8\},\\ &\{4,6,9\},\{4,7,9\},\{4,7,10\},\{5,6,7\},\{5,6,8\},\{8,9,10\}\}\end{aligned}$$

$$BD(2) = BD(1) + (4\ 7\ 6\ 10)(5\ 8)BD(1)$$

$$BD(3) = (BD(2) \setminus \{\{1,2,4\},\{1,3,6\},\{2,3,7\},\{4,6,7\}\}) \cup \{\{1,2,3\},\{1,4,6\},\{2,4,7\},\{3,6,7\}\}$$

| First Design | | Second Design | | $Int(10,\lambda_1,\lambda_2)$ |
|---|---|---|---|---|
| Name | Index | Rule of Construction | Index | |
| BD(1) | 2 | ( 4 7 6 10 )( 5 8 ) BD(1) | 2 | 0 |
| BD(1) | 2 | ( 4 7 )( 5 6 10 8 9 ) BD(1) | 2 | 1 |
| BD(1) | 2 | ( 4 5 9 10 )( 6 8 ) BD(1) | 2 | 2 |
| BD(1) | 2 | ( 4 5 6 7 9 8 10 ) BD(1) | 2 | 3 |
| BD(1) | 2 | ( 4 5 6 7 8 10 9 ) BD(1) | 2 | 4 |
| BD(1) | 2 | ( 4 5 )( 6 8 )( 9 10 ) BD(1) | 2 | 5 |
| BD(1) | 2 | ( 5 8 6 10 9 ) BD(1) | 2 | 6 |
| BD(1) | 2 | ( 5 6 )( 7 9 8 10 ) BD(1) | 2 | 7 |
| BD(1) | 2 | ( 6 7 10 )( 8 9 ) BD(1) | 2 | 8 |
| BD(1) | 2 | ( 7 9 8 10 ) BD(1) | 2 | 9 |
| BD(1) | 2 | ( 7 9 )( 8 10 ) BD(1) | 2 | 10 |
| BD(1) | 2 | ( 7 8 9 10 ) BD(1) | 2 | 11 |
| BD(1) | 2 | ( 6 7 8 ) BD(1) | 2 | 12 |
| BD(1) | 2 | ( 7 8 9 ) BD(1) | 2 | 13 |
| BD(1) | 2 | ( 7 8 )( 9 10 ) BD(1) | 2 | 14 |
| BD(1) | 2 | ( 4 5 )( 6 7 )( 8 9 10 ) BD(1) | 2 | 15 |
| BD(1) | 2 | ( 7 9 ) BD(1) | 2 | 16 |
| BD(1) | 2 | ( 5 6 7 ) BD(1) | 2 | 17 |
| BD(1) | 2 | ( 7 8 ) BD(1) | 2 | 18 |
| BD(1) | 2 | ( 8 9 10 ) BD(1) | 2 | 19 |
| BD(1) | 2 | ( 6 7 ) BD(1) | 2 | 20 |
| BD(1) | 2 | ( 5 8 6 9 7 10 ) BD(1) | 2 | 21 |
| BD(1) | 2 | ( 8 9 ) BD(1) | 2 | 22 |
| BD(1) | 2 | ( 5 6 7 )( 8 9 10 ) BD(1) | 2 | 23 |
| BD(1) | 2 | ( 9 10 ) BD(1) | 2 | 24 |
| BD(1) | 2 | ( 1 8 )( 3 4 )( 5 7 )( 9 10 ) BD(1) | 2 | 26 |
| BD(1) | 2 | BD(1) | 2 | 30 |

Table 4 (Cont.)

| First Design | | Second Design | | $Int(10, \lambda_1, \lambda_2)$ |
|---|---|---|---|---|
| Name | Index | Rule of Construction | Index | |
| BD(2) | 4 | ( 3 7 4 9 )( 5 6 10 8 ) BD(1) | 2 | 0 |
| BD(2) | 4 | ( 1 5 6 9 2 )( 3 8 4 )( 7 10 ) BD(1) | 2 | 1 |
| BD(2) | 4 | ( 1 5 6 9 2 )( 3 8 4 )( 7 10 ) BD(1) | 2 | 2 |
| BD(2) | 4 | ( 3 7 5 )( 4 9 6 8 ) BD(1) | 2 | 3 |
| BD(2) | 4 | ( 3 4 6 10 8 5 9 ) BD(1) | 2 | 4 |
| BD(2) | 4 | ( 4 9 8 10 7 5 6 ) BD(1) | 2 | 5 |
| BD(2) | 4 | ( 4 6 8 7 5 9 ) BD(1) | 2 | 6 |
| BD(2) | 4 | ( 4 5 9 )( 6 8 ) BD(1) | 2 | 7 |
| BD(2) | 4 | ( 4 5 6 9 7 8 10 ) BD(1) | 2 | 8 |
| BD(2) | 4 | ( 4 5 6 9 8 10 ) BD(1) | 2 | 9 |
| BD(2) | 4 | ( 5 6 )( 7 10 8 9 ) BD(1) | 2 | 10 |
| BD(2) | 4 | ( 5 6 )( 7 9 )( 8 10 ) BD(1) | 2 | 11 |
| BD(2) | 4 | ( 7 9 8 10 ) BD(1) | 2 | 12 |
| BD(2) | 4 | ( 7 10 8 9 ) BD(1) | 2 | 13 |
| BD(2) | 4 | ( 7 9 )( 8 10 ) BD(1) | 2 | 14 |
| BD(2) | 4 | ( 7 9 10 8 ) BD(1) | 2 | 15 |
| BD(2) | 4 | ( 7 9 10 ) BD(1) | 2 | 16 |
| BD(2) | 4 | ( 7 8 9 10 ) BD(1) | 2 | 17 |
| BD(2) | 4 | ( 7 8 10 9 ) BD(1) | 2 | 18 |
| BD(2) | 4 | ( 6 7 8 9 ) BD(1) | 2 | 19 |
| BD(2) | 4 | ( 7 8 9 ) BD(1) | 2 | 20 |
| BD(2) | 4 | ( 8 9 10 ) BD(1) | 2 | 21 |
| BD(2) | 4 | ( 8 10 9 ) BD(1) | 2 | 22 |
| BD(2) | 4 | ( 8 10 ) BD(1) | 2 | 23 |
| BD(2) | 4 | ( 8 9 ) BD(1) | 2 | 24 |
| BD(2) | 4 | ( 6 7 )( 8 10 ) BD(1) | 2 | 25 |
| BD(2) | 4 | ( 9 10 ) BD(1) | 2 | 26 |
| BD(2) | 4 | ( 4 7 9 6 5 ) BD(1) | 2 | 27 |
| BD(2) | 4 | ( 5 8 )( 6 10 )( 7 9 ) BD(1) | 2 | 28 |
| BD(3) | 4 | ( 2 3 6 8 5 7 )( 4 10 ) BD(1) | 2 | 29 |
| BD(2) | 4 | BD(1) | 2 | 30 |

**Table 4 (Cont.)**

| First Design | | Second Design | | $Int(10, \lambda_1, \lambda_2)$ |
|---|---|---|---|---|
| Name | Index | Rule of Construction | Index | |
| BD(2) | 4 | ( 1 5 )( 2 6 )( 3 9 )( 4 8 )( 7 10 ) BD(2) | 4 | 6 |
| BD(2) | 4 | ( 1 5 )( 2 4 )( 3 7 )( 6 8 )( 9 10 ) BD(2) | 4 | 8 |
| BD(2) | 4 | ( 1 5 )( 2 6 )( 3 4 )( 7 10 )( 8 9 ) BD(2) | 4 | 10 |
| BD(2) | 4 | ( 1 4 8 5 )( 2 6 )( 3 9 )( 7 10 ) BD(2) | 4 | 12 |
| BD(2) | 4 | ( 1 2 3 8 7 6 9 5 )( 4 10 ) BD(2) | 4 | 13 |
| BD(2) | 4 | ( 2 6 )( 3 4 )( 7 10 )( 8 9 ) BD(2) | 4 | 14 |
| BD(2) | 4 | ( 2 3 4 6 8 5 9 ) BD(2) | 4 | 15 |
| BD(2) | 4 | ( 3 7 4 9 )( 5 6 10 8 ) BD(2) | 4 | 16 |
| BD(2) | 4 | ( 2 4 6 8 7 3 5 10 9 ) BD(2) | 4 | 17 |
| BD(2) | 4 | ( 4 5 6 7 9 8 10 ) BD(2) | 4 | 18 |
| BD(2) | 4 | ( 4 5 6 7 9 )( 8 10 ) BD(2) | 4 | 19 |
| BD(2) | 4 | ( 4 6 7 10 8 9 ) BD(2) | 4 | 20 |
| BD(2) | 4 | ( 4 5 6 )( 7 9 8 10 ) BD(2) | 4 | 21 |
| BD(2) | 4 | ( 5 6 )( 7 9 )( 8 10 ) BD(2) | 4 | 22 |
| BD(2) | 4 | ( 5 6 7 9 )( 8 10 ) BD(2) | 4 | 23 |
| BD(2) | 4 | ( 5 6 7 10 )( 8 9 ) BD(2) | 4 | 24 |
| BD(2) | 4 | ( 7 9 8 10 ) BD(2) | 4 | 25 |
| BD(2) | 4 | ( 5 6 )( 7 10 )( 8 9 ) BD(2) | 4 | 26 |
| BD(2) | 4 | ( 6 7 10 8 9 ) BD(2) | 4 | 27 |
| BD(2) | 4 | ( 7 10 )( 8 9 ) BD(2) | 4 | 28 |
| BD(2) | 4 | ( 6 7 9 8 10 ) BD(2) | 4 | 29 |
| BD(2) | 4 | ( 7 9 )( 8 10 ) BD(2) | 4 | 30 |
| BD(2) | 4 | ( 7 8 10 9 ) BD(2) | 4 | 31 |
| BD(2) | 4 | ( 7 9 10 ) BD(2) | 4 | 32 |
| BD(2) | 4 | ( 6 7 )( 8 9 10 ) BD(2) | 4 | 33 |
| BD(2) | 4 | ( 7 8 9 10 ) BD(2) | 4 | 34 |
| BD(2) | 4 | ( 6 7 10 8 ) BD(2) | 4 | 35 |
| BD(2) | 4 | ( 7 8 9 ) BD(2) | 4 | 36 |
| BD(2) | 4 | ( 8 9 10 ) BD(2) | 4 | 37 |
| BD(2) | 4 | ( 7 10 ) BD(2) | 4 | 38 |
| BD(2) | 4 | ( 5 6 10 ) BD(2) | 4 | 39 |
| BD(2) | 4 | ( 7 8 )( 9 10 ) BD(2) | 4 | 40 |
| BD(2) | 4 | ( 4 5 7 )( 6 9 )( 8 10 ) BD(2) | 4 | 41 |
| BD(2) | 4 | ( 8 9 ) BD(2) | 4 | 42 |
| BD(2) | 4 | ( 5 9 )( 6 10 )( 7 8 ) BD(2) | 4 | 43 |
| BD(2) | 4 | ( 9 10 ) BD(2) | 4 | 44 |
| BD(2) | 4 | ( 2 3 )( 4 8 )( 5 6 )( 9 10 ) BD(2) | 4 | 45 |
| BD(2) | 4 | ( 6 9 ) BD(2) | 4 | 46 |
| BD(2) | 4 | B( 2 6 )( 3 5 )( 4 7 )( 9 10 ) BD(2) | 4 | 47 |
| BD(2) | 4 | ( 8 10 ) BD(2) | 4 | 48 |
| BD(2) | 4 | ( 2 6 )( 3 5 )( 9 10 ) BD(2) | 4 | 49 |
| BD(2) | 4 | ( 6 10 ) BD(2) | 4 | 50 |
| BD(2) | 4 | ( 2 8 )( 3 10 )( 4 6 )( 7 9 ) BD(3) | 4 | 51 |
| BD(2) | 4 | ( 3 8 )( 4 9 ) BD(2) | 4 | 52 |
| BD(2) | 4 | ( 2 3 )( 4 7 )( 6 9 )( 8 10 ) BD(2) | 4 | 53 |
| BD(2) | 4 | BD(3) | 4 | 56 |
| BD(2) | 4 | BD(2) | 4 | 60 |

**Table 5.** $v = 11$

Designs used in Table 5.

$BD(1) = \{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{1,6,7\},$
$\{1,6,8\},\{1,6,9\},\{1,7,10\},\{1,7,11\},\{1,8,9\},\{1,8,10\},\{1,9,11\},$
$\{1,10,11\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{2,6,8\},\{2,6,9\},\{2,6,10\},$
$\{2,7,9\},\{2,7,10\},\{2,7,11\},\{2,8,9\},\{2,8,11\},\{2,10,11\},\{3,4,5\},$
$\{3,6,9\},\{3,6,10\},\{3,6,11\},\{3,7,8\},\{3,7,9\},\{3,7,11\},\{3,8,10\},$
$\{3,8,11\},\{3,9,10\},\{4,6,7\},\{4,6,10\},\{4,6,11\},\{4,7,8\},\{4,7,9\},$
$\{4,8,10\},\{4,8,11\},\{4,9,10\},\{4,9,11\},\{5,6,7\},\{5,6,8\},\{5,6,11\},$
$\{5,7,8\},\{5,7,10\},\{5,8,9\},\{5,9,10\},\{5,9,11\},\{5,10,11\}\}$

$BD(2) = (BD(1) \setminus \{\{1,2,3\},\{1,10,11\},\{2,7,10\},\{3,6,9\},\{3,7,11\},$
$\{5,6,7\},\{5,9,11\},\{3,7,11\}\}) \cup \{\{1,2,10\},\{1,3,11\},\{2,3,7\},$
$\{3,6,7\},\{7,10,11\},\{3,9,11\},\{5,6,9\},\{5,7,11\}\}$

| First Design | | Second Design | | $Int(11,\lambda_1,\lambda_2)$ |
|---|---|---|---|---|
| Name | Index | Rule of Construction | Index | |
| BD(2) | 3 | ( 1 6 )( 2 8 )( 5 9 )( 7 10 11 ) BD(1) | 3 | 1 |
| BD(1) | 3 | ( 1 11 8 )( 2 7 9 5 10 6 ) BD(1) | 3 | 2 |
| BD(1) | 3 | ( 1 11 9 2 7 6 )( 5 10 8 ) BD(1) | 3 | 3 |
| BD(2) | 3 | ( 1 8 )( 2 7 5 9 3 10 4 11 6 ) BD(1) | 3 | 4 |
| BD(1) | 3 | ( 1 3 8 9 11 4 5 10 7 6 ) BD(1) | 3 | 5 |
| BD(1) | 3 | ( 3 4 6 7 9 5 11 10 ) BD(1) | 3 | 6 |
| BD(1) | 3 | ( 4 6 8 5 11 9 7 ) BD(1) | 3 | 7 |
| BD(1) | 3 | ( 4 6 )( 5 7 11 )( 8 10 ) BD(1) | 3 | 8 |
| BD(1) | 3 | ( 4 6 )( 5 7 8 10 11 ) BD(1) | 3 | 9 |
| BD(1) | 3 | ( 4 6 )( 5 7 )( 8 9 10 ) BD(1) | 3 | 10 |
| BD(1) | 3 | ( 4 6 )( 5 7 )( 8 10 ) BD(1) | 3 | 11 |
| BD(1) | 3 | ( 4 6 )( 5 7 )( 9 10 11 ) BD(1) | 3 | 12 |
| BD(1) | 3 | ( 4 6 )( 5 7 )( 8 9 11 10 ) BD(1) | 3 | 13 |
| BD(1) | 3 | ( 4 6 )( 5 7 )( 10 11 ) BD(1) | 3 | 14 |
| BD(1) | 3 | ( 5 6 7 9 11 10 ) BD(1) | 3 | 15 |
| BD(1) | 3 | ( 5 6 )( 7 8 9 10 ) BD(1) | 3 | 16 |
| BD(1) | 3 | ( 5 6 )( 7 8 10 11 ) BD(1) | 3 | 17 |
| BD(1) | 3 | ( 5 6 )( 8 9 10 ) BD(1) | 3 | 18 |
| BD(1) | 3 | ( 5 6 )( 9 10 11 ) BD(1) | 3 | 19 |
| BD(1) | 3 | ( 5 6 )( 8 10 9 11 ) BD(1) | 3 | 20 |
| BD(1) | 3 | ( 5 6 )( 9 11 ) BD(1) | 3 | 21 |
| BD(1) | 3 | ( 5 6 )( 8 10 )( 9 11 ) BD(1) | 3 | 22 |
| BD(1) | 3 | ( 5 6 )( 9 10 ) BD(1) | 3 | 23 |
| BD(1) | 3 | ( 5 6 )( 8 11 )( 9 10 ) BD(1) | 3 | 24 |
| BD(1) | 3 | ( 5 6 )( 10 11 ) BD(1) | 3 | 25 |
| BD(1) | 3 | ( 5 7 10 ) BD(1) | 3 | 26 |
| BD(1) | 3 | ( 5 6 )( 7 10 ) BD(1) | 3 | 27 |
| BD(1) | 3 | ( 5 6 7 )( 8 10 )( 9 11 ) BD(1) | 3 | 28 |

Table 5 (Cont.)

| First Design | | Second Design | | $Int(11, \lambda_1, \lambda_2)$ |
|---|---|---|---|---|
| Name | Index | Rule of Construction | Index | |
| BD(1) | 3 | ( 4 6 )( 5 9 ) BD(1) | 3 | 29 |
| BD(1) | 3 | ( 6 7 9 8 11 10 ) BD(1) | 3 | 30 |
| BD(1) | 3 | ( 7 9 11 8 10 ) BD(1) | 3 | 31 |
| BD(1) | 3 | ( 7 8 11 9 ) BD(1) | 3 | 32 |
| BD(1) | 3 | ( 7 8 10 11 ) BD(1) | 3 | 33 |
| BD(1) | 3 | ( 8 9 10 11 ) BD(1) | 3 | 34 |
| BD(1) | 3 | ( 7 8 9 10 ) BD(1) | 3 | 35 |
| BD(1) | 3 | ( 8 9 10 ) BD(1) | 3 | 36 |
| BD(1) | 3 | ( 7 8 9 11 10 ) BD(1) | 3 | 37 |
| BD(1) | 3 | ( 9 10 11 ) BD(1) | 3 | 38 |
| BD(1) | 3 | ( 8 10 ) BD(1) | 3 | 39 |
| BD(1) | 3 | ( 7 8 )( 9 10 11 ) BD(1) | 3 | 40 |
| BD(1) | 3 | ( 9 11 ) BD(1) | 3 | 41 |
| BD(1) | 3 | ( 7 10 11 ) BD(1) | 3 | 42 |
| BD(1) | 3 | ( 9 10 ) BD(1) | 3 | 43 |
| BD(1) | 3 | ( 4 5 )( 6 7 )( 8 10 )( 9 11 ) BD(1) | 3 | 44 |
| BD(1) | 3 | ( 8 9 ) BD(1) | 3 | 45 |
| BD(1) | 3 | ( 6 7 )( 8 10 9 11 ) BD(1) | 3 | 46 |
| BD(1) | 3 | ( 10 11 ) BD(1) | 3 | 47 |
| BD(1) | 3 | ( 3 4 )( 6 7 )( 8 11 )( 9 10 ) BD(1) | 3 | 48 |
| BD(1) | 3 | ( 6 7 )( 8 10 )( 9 11 ) BD(1) | 3 | 49 |
| BD(1) | 3 | ( 3 4 ) BD(1) | 3 | 51 |
| BD(1) | 3 | BD(1) | 3 | 55 |

**Table 6.** $v = 12$
Designs used in Table 6.

$BD(1) = \{\ \{1,2,3\},\{1,2,4\},\{1,3,4\},\{1,5,7\},\{1,5,9\},\{1,6,8\},\{1,6,10\},$
$\{1,7,11\},\ \{1,8,12\},\{1,9,11\},\{1,10,12\},\{2,3,4\},\{2,5,8\},$
$\{2,5,10\},\{2,6,7\},\{2,6,9\},\{2,7,12\},\{2,8,11\},\{2,9,12\},$
$\{2,10,11\},\{3,5,11\},\{3,5,12\},\{3,6,11\},\{3,6,12\},\{3,7,9\},$
$\{3,7,10\},\{3,8,9\},\{3,8,10\},\{4,5,11\},\{4,5,12\},\{4,6,11\},$
$\{4,6,12\},\{4,7,9\},\{4,7,10\},\{4,8,9\},\{4,8,10\},\{5,6,7\},$
$\{5,6,8\},\{5,9,10\},\{6,9,10\},\{7,8,11\},\{7,8,12\},\{9,11,12\},$
$\{0,11,12\}\}$

$BD(2) = \{\ \{1,2,5\},\{1,2,6\},\{1,3,10\},\{1,3,11\},\{1,4,9\},\{1,4,12\},\{1,5,6\},$
$\{1,7,10\},\{1,7,12\},\{1,8,9\},\{1,8,11\},\{2,3,9\},\{2,3,12\},$
$\{2,4,10\},\{2,4,11\},\{2,5,6\},\{2,7,9\},\{2,7,11\},\{2,8,10\},$
$\{2,8,12\},\{3,4,11\},\{3,4,12\},\{3,5,7\},\{3,5,8\},\{3,6,7\},$
$\{3,6,8\},\{3,9,10\},\{4,5,7\},\{4,5,8\},\{4,6,7\},\{4,6,8\},\{4,9,10\},$
$\{5,9,11\},\{5,9,12\},\{5,10,11\},\{5,10,12\},\{6,9,11\},\{6,9,12\},$
$\{6,10,11\},\{6,10,12\},\{7,8,9\}\{7,8,10\},\{7,11,12\},\{8,11,12\}\}$

$BD(3) = (5\ 7\ 11\ 9\ 6\ 8\ 12\ 10)BD(2)$

$BD(4) = (3\ 9\ 12\ 6\ 4\ 10\ 11\ 5)(7\ 8)BD(1)$

$BD(5) = (3\ 11\ 9\ 7\ 6\ 4\ 12\ 10\ 8\ 5)BD(1)$

$BD(6) = (BD(1) \setminus \{\{1,2,3\},\{1,5,7\},\{1,10,12\},\{2,5,10\},\{2,7,12\},$
$\{3,5,12\},\{3,7,10\}\}) \cup \{\{1,2,5\},\{1,3,10\},\{1,7,12\},\{2,3,12\},$
$\{2,7,10\},\{3,5,7\},\{5,10,12\}\}$

$BD(7) = (BD(1) \setminus \{\{1,2,3\},\{1,5,7\},\{2,5,10\},\{2,7,12\},\{3,5,12\},$
$\{3,7,10\}\}) \cup \{\{1,2,7\},\{1,3,5\},\{2,3,10\},\{2,5,12\},\{3,7,12\},$
$\{5,7,10\}\}$

$BD(8) = (BD(1) \setminus \{\{2,5,10\},\{2,7,12\},\{3,5,12\},\{3,7,10\}\})$
$\cup\{\{2,5,12\},\{2,7,10\},\{3,5,10\},\{3,7,12\}\}$

$BD(A) = \sum_{j\in A} BD(j) \quad for\ A \subset \{1,\dots,5\}$

It is an easy exercise to check that $\{BD(i)|1 \le i \le 5\}$ is a large set of disjoint triple systems of $TS(12,2)$'s.
Hence, for every $A \subset \{1,\dots,5\}$, $BD(A)$ is a simple $TS(12,2|A|)$. For any $j \in A$, both designs ( First and Second) in the table are disjoint from $BD(j)$.

## Table 6

| First Design | Second Design | $Int(12, \lambda_1, \lambda_2)$ | $A$ |
|---|---|---|---|
| BD(2) | BD(1) | 0 | {3,4,5} |
| BD(5) | BD(6) | 1 | {3,4} |
| BD(5) | ( 2 3 6 9 8 5 11 12 7 10 4 )BD(6) | 2 | {4} |
| BD(4) | ( 7 8 )BD(6) | 3 | {3,5} |
| BD(3) | ( 3 5 )( 4 6 )( 7 11 8 12 )BD(1) | 4 | {1,4} |
| BD(5) | ( 5 7 11 9 )( 6 8 12 10 )BD(7) | 5 | {2,4} |
| BD(2) | BD(6) | 6 | {3,4} |
| BD(5) | ( 5 12 )( 6 11 )( 7 10 )( 8 9 )BD(6) | 7 | {2,3} |
| BD(4) | ( 5 9 11 7 6 10 12 8 )BD(6) | 8 | {2,3} |
| BD(2) | ( 3 7 )( 4 8 )( 9 11 10 12 )BD(6) | 9 | {1,4} |
| BD(4) | ( 3 7 10 5 4 8 9 6 )BD(7) | 10 | {1,2} |
| BD(2) | ( 10 11 )BD(6) | 11 | {5} |
| BD(5) | ( 7 12 8 9 10 11 )BD(6) | 12 | {3} |
| BD(1) | ( 6 7 9 8 )( 11 12 )BD(6) | 13 | {5} |
| BD(1) | ( 6 8 )( 7 10 9 )( 11 12 )BD(6) | 14 | {5} |
| BD(2) | ( 7 12 )( 8 9 10 11 )BD(6) | 15 | {3} |
| BD(2) | ( 7 12 8 10 11 )BD(6) | 16 | {3} |
| BD(1) | ( 6 8 )( 7 11 )( 9 12 )BD(6) | 17 | {3} |
| BD(3) | ( 6 9 11 10 12 )BD(6) | 18 | {5} |
| BD(2) | ( 7 12 )( 8 10 11 )BD(6) | 19 | {3} |
| BD(2) | ( 7 12 8 9 10 11 )BD(6) | 20 | {3} |
| BD(1) | ( 10 12 11 )BD(6) | 21 | {5} |
| BD(1) | ( 10 11 )BD(6) | 22 | {5} |
| BD(2) | ( 3 5 )( 4 6 )( 7 11 9 10 )( 8 12 )BD(6) | 23 | {3} |
| BD(2) | ( 1 3 6 10 8 4 5 9 7 2 )( 11 12 )BD(6) | 24 | {5} |
| BD(2) | ( 1 3 6 10 8 4 5 9 7 2 )BD(6) | 25 | {5} |
| BD(3) | ( 1 3 8 4 7 2 )( 9 12 )( 10 11 )BD(6) | 26 | {4} |
| BD(2) | ( 3 5 )( 4 6 )( 7 11 )( 8 12 )BD(7) | 28 | {4} |
| BD(2) | ( 3 5 )( 4 6 )( 7 11 )( 8 12 )BD(6) | 29 | {4} |
| BD(1) | ( 7 8 )( 11 12 )BD(7) | 31 | {5} |
| BD(1) | ( 7 8 )BD(6) | 32 | {3,5} |
| BD(3) | ( 3 7 10 5 4 8 9 6 )BD(7) | 34 | {5} |
| BD(3) | ( 3 7 )( 4 8 )( 9 11 10 12 )BD(6) | 35 | {1,4} |
| BD(1) | ( 11 12 )BD(1) | 36 | {4,5} |
| BD(1) | ( 5 12 )( 6 11 )( 7 10 )( 8 9 )BD(6) | 37 | {2,3} |

Table 6 (Cont.)

| Firt Design | Second Design | $Int(12, \lambda_1, \lambda_2)$ | $A$ |
|---|---|---|---|
| BD(1) | BD(7) | 38 | {5} |
| BD(2) | ( 3 5 )( 4 6 )( 7 11 8 12 )BD(1) | 40 | {1,4} |
| BD(1) | BD(1) | 44 | {2,3} |
| BD(1,3) | ( 10 11 )BD(6) | 27 | {5} |
| BD(2,5) | ( 7 12 )( 8 9 10 11 )BD(6) | 30 | {3} |
| BD(1,2) | ( 10 11 )BD(6) | 33 | {5} |
| BD(1,3) | ( 5 7 11 9 )( 6 8 12 10 )BD(7) | 39 | {2} |
| BD(1,4) | BD(8) | 41 | {2} |
| BD(1,3) | BD(8) | 42 | {2} |
| BD(1,5) | ( 5 7 11 9 )( 6 8 12 10 )BD(7) | 43 | {2} |
| BD(4,5) | BD(8) | 2 | {2} |
| BD(3,4) | ( 10 11 )BD(6) | 11 | {5} |
| BD(1,4) | ( 7 12 8 9 10 11 )BD(6) | 12 | {3} |
| BD(1,3) | ( 1 3 6 10 8 4 5 9 7 2 )( 11 12 )BD(6) | 13 | {5} |
| BD(1,4) | ( 1 3 6 10 8 4 5 9 7 2 )( 11 12 )BD(6) | 14 | {5} |
| BD(1,4) | ( 6 9 11 10 12 )BD(6) | 15 | {5} |
| BD(2,3) | ( 10 11 )BD(6) | 16 | {5} |
| BD(2,4) | ( 10 11 )BD(6) | 17 | {5} |
| BD(1,2) | ( 6 9 11 10 12 )BD(6) | 18 | {5} |
| BD(2,4) | ( 6 9 11 10 12 )BD(6) | 19 | {5} |
| BD(1,2) | ( 9 12 )( 10 11 )BD(1) | 20 | {5} |
| BD(1,4) | ( 6 8 )( 7 10 9 )( 11 12 )BD(6) | 21 | {5} |
| BD(1,3) | ( 6 7 9 8 )( 11 12 )BD(6) | 22 | {5} |
| BD(1,3) | ( 6 8 )( 7 10 9 )( 11 12 )BD(6) | 23 | {5} |
| BD(3,4) | ( 9 12 )( 10 11 )BD(1) | 24 | {5} |
| BD(1,3) | ( 6 9 11 10 12 )BD(6) | 25 | {5} |
| BD(3,4) | ( 6 9 11 10 12 )BD(6) | 26 | {5} |
| BD(1,4) | ( 10 11 )BD(6) | 28 | {5} |
| BD(2,3) | ( 6 9 11 10 12 )BD(6) | 29 | {5} |

## Table 7. $v = 13$

Designs used in Table 7.

$BD(1) = \{\{1,2,12\},\{1,3,4\},\{1,5,6\},\{1,7,9\},\{1,8,13\},\{1,10,11\},$
$\{2,3,13\},\{2,4,9\},\{2,5,10\},\{2,6,11\},\{2,7,8\},\{3,5,11\},$
$\{3,6,7\},\{3,8,12\},\{3,9,10\},\{4,5,8\},\{4,6,12\},\{4,7,11\},$
$\{4,10,13\},\{5,7,13\},\{5,9,12\},\{6,8,10\},\{6,9,13\},\{7,10,12\},$
$\{8,9,11\},\{11,12,13\}\}$

$BD(j) = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)^{j-1} BD(1)$ *for* $2 \le j \le 11$,

$BD(12) = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\},\{1,10,11\},$
$\{1,12,13\},\{2,4,6\},\{2,5,7\},\{2,8,10\},\{2,9,12\},\{2,11,13\},$
$\{3,4,8\},\{3,5,12\},\{3,6,10\},\{3,7,11\},\{3,9,13\},\{4,7,9\},$
$\{4,10,13\},\{4,11,12\},\{5,6,13\},\{5,8,11\},\{5,9,10\},\{6,8,12\},$
$\{6,9,11\},\{7,8,13\},\{7,10,12\}\}$

$BD(A) = \sum_{j \in A} BD(j)$ *for* $A \subset \{1,\dots,11\}$

It is well known that $\{BD(i) | 1 \le i \le 11\}$ is a large set of disjoint triple systems $TS(13,1)$'s[4]. Hence, for every $A \subset \{1,\dots,11\}$, $BD(A)$ is a simple $TS(13,|A|)$. For any $j \in A$, both designs (First and Second) in the table are disjoint from $BD(j)$.

| First Design | Second Design | $Int(13,\lambda_1,\lambda_2)$ | $A$ |
|---|---|---|---|
| BD(1) | BD(2) | 0 | {3,4,5,6,7,8,9,10,11} |
| BD(2) | ( 3 8 7 11 12 )( 4 10 9 5 6 13 )BD(1) | 1 | {4,5,6,7,8,9,11} |
| BD(10) | ( 3 8 7 11 12 )( 4 10 9 5 6 13 )BD(1) | 2 | {4,5,6,7,8,9,11} |
| BD(10) | ( 3 4 10 5 13 6 11 7 )( 8 9 12 )BD(1) | 3 | {1,2,3,5,6,7,11} |
| BD(4) | ( 3 4 7 9 13 )( 5 11 )( 6 10 )BD(1) | 4 | {2,5,6,7,8,9,10} |
| BD(9) | ( 3 4 10 5 13 6 11 7 )( 8 9 12 )BD(1) | 5 | {1,2,3,5,6,7,11} |
| BD(11) | ( 2 4 12 11 10 9 6 13 8 3 5 )BD(1) | 6 | {3,4,5,6,7,9,10} |
| BD(9) | ( 3 9 7 5 11 12 6 )( 4 13 10 8 )BD(1) | 7 | {1,3,5,7,8,10,11} |
| BD(4) | ( 3 4 10 5 13 6 11 7 )( 8 9 12 )BD(1) | 8 | {1,2,3,5,6,7,11} |
| BD(11) | ( 3 4 7 9 13 )( 5 11 )( 6 10 )BD(1) | 9 | {2,5,6,7,8,9,10} |
| BD(8) | ( 3 4 10 5 13 6 11 7 )( 8 9 12 )BD(1) | 10 | {1,2,3,5,6,7,11} |
| BD(1) | ( 2 4 12 11 10 9 6 13 8 3 5 )BD(1) | 11 | {3,4,5,6,7,9,10} |
| BD(4) | ( 3 9 7 5 11 12 6 )( 4 13 10 8 )BD(1) | 12 | {1,3,5,7,8,10,11} |
| BD(5) | ( 3 9 6 5 13 12 7 11 10 )( 4 8 )BD(1) | 13 | {2,3,4,7,9,10,11} |
| BD(9) | ( 4 12 5 6 11 13 9 8 7 )BD(1) | 14 | {2,3,4,8,10,11} |
| BD(4,5) | (4 10 )( 5 11 13 7 8 )( 9 12)BD(1) | 14 | {1,2,8,9,10,11} |
| BD(3) | (4 7 8 5 )(6 10 13 9 11)BD(1) | 15 | {1,2,5,7,10,11} |
| BD(7) | ( 3 13 11 6 10 5 12 9 )( 4 8 7 )BD(1) | 16 | {2,4,5,6,8,9,11} |
| BD(6) | ( 1 12 11 7 10 6 2 9 3 )( 5 8 13 )BD(1) | 17 | {1,2,3,4,5,9,11} |
| BD(7) | ( 3 10 12 8 5 4 13 6 11 9 )BD(12) | 18 | {1,2,4,5,6,8,9} |
| BD(7) | ( 3 6 11 13 10 4 8 12 5 7 9 )BD(1) | 19 | {1,2,3,6,8,11} |
| BD(4,6) | (4 10 )( 5 11 13 7 8 )( 9 12)BD(1) | 19 | {1,2,8,9,10,11} |
| BD(11) | ( 1 12 2 5 7 8 13 10 9 4 11 6 3 )BD(1) | 20 | {1,2,4,5,6,8,9} |
| BD(3,11) | ( 1 12 2 5 7 8 13 10 9 4 11 6 3 )BD(1) | 21 | {1,2,4,5,6,8,9} |
| BD(3) | ( 3 8 7 11 12 )( 4 10 9 5 6 13 )BD(1) | 22 | {4,5,6,7,8,9,11} |
| BD(1,3) | ( 3 8 7 11 12 )( 4 10 9 5 6 13 )BD(1) | 23 | {4,5,6,7,8,9,11} |
| BD(3,10) | ( 3 8 7 11 12 )( 4 10 9 5 6 13 )BD(1) | 24 | {4,5,6,7,8,9,11} |
| BD(5) | $(BD(1,5) \setminus \{\{1,2,12\},\{2,3,13\},\{4,12,13\},\{1,3,4\}\}) \cup \{\{1,2,3\},\{1,4,12\},\{2,12,13\},\{3,4,13\}\}$ | 25 | {4,7,8,9,10,11} |
| BD(1) | BD(1) | 26 | {2,3,4,5,6,7,8,9,10,11} |

Table 7 (Cont.)

| First Design | Second Design | $Int(13, \lambda_1, \lambda_2)$ |
|---|---|---|
| BD(1,2,3,4,5) | BD(6,7,8,9,10) | 0 |
| BD(5,8,9,10,11) | (BD(2,3,4,6,7) $\backslash$ {{1, 2, 3}, {1, 4, 12}, {2, 12, 13}, {3, 4, 13}}) ∪{{1, 2, 12}, {1, 3, 4}, {2, 3, 13}, {4, 12, 13}} | 1 |
| BD(1,2,3,4,10) | ( 1 12 2 5 7 8 13 10 9 4 11 6 3 )BD(1)+BD(5,6,8,9) | 3 |
| BD(1,2,3,4,7) | ( 1 12 2 5 7 8 13 10 9 4 11 6 3 )BD(1) + BD(5,6,8,9) | 4 |
| BD(1,2,4,7,10) | ( 1 12 2 5 7 8 13 10 9 4 11 6 3 )BD(1) + BD(5,6,8,9) | 5 |
| BD(1,2,3,7,10) | ( 1 12 2 5 7 8 13 10 9 4 11 6 3 )BD(1) + BD(5,6,8,9) | 6 |
| BD(3,6,9,10,11) | ( 4 9 )( 5 11 )( 6 10 )( 7 12 )BD(1) + BD(2,4,5,8) | 7 |
| BD(5,8,9,10,11) | ( 1 12 11 7 10 6 2 9 3 )( 5 8 13 )BD(1) + BD(1,2,3,4) | 8 |
| BD(7,8,9,10,11) | ( 1 12 11 7 10 6 2 9 3 )( 5 8 13 )BD(1) + BD(1,2,3,4) | 9 |
| BD(4,5,7,9,10) | ( 6 13 )BD(1) + BD(2,3,6,8) | 10 |
| BD(1,2,6,10,11) | ( 3 8 4 9 13 6 7 5 12 11 )BD(1) + BD(3,7,8,9) | 11 |
| BD(1,8,9,10,11) | ( 4 11 7 13 )( 5 12 6 8 10 9 )BD(1) + BD(2,3,6,7) | 12 |
| BD(1,5,6,10,11) | ( 3 8 4 9 13 6 7 5 12 11 )BD(1) + BD(3,7,8,9) | 13 |
| BD(1,4,8,9,11) | ( 4 11 7 13 )( 5 12 6 8 10 9 )BD(1) + BD(2,3,6,7) | 14 |
| BD(4,8,9,10,11) | ( 4 11 7 13 )( 5 12 6 8 10 9 )BD(1) + BD(2,3,6,7) | 15 |
| BD(1,4,8,9,10) | ( 4 11 7 13 )( 5 12 6 8 10 9 )BD(1) + BD(2,3,6,7) | 16 |
| BD(4,5,8,9,11) | ( 4 11 7 13 )( 5 12 6 8 10 9 )BD(1) + BD(2,3,6,7) | 17 |
| BD(5,6,7,9,11) | ( 1 12 11 7 10 6 2 9 3 )( 5 8 13 )BD(1) + BD(1,2,3,4) | 18 |
| BD(1,3,9,10,11) | ( 4 9 )( 5 11 )( 6 10 )( 7 12 )BD(1) + BD(2,4,5,8) | 19 |
| BD(1,7,9,10,11) | ( 4 9 )( 5 11 )( 6 10 )( 7 12 )BD(1) + BD(2,4,5,8) | 20 |
| BD(1,2,3,4,11) | ( 1 12 2 5 7 8 13 10 9 4 11 6 3 )BD(1) + BD(5,6,8,9) | 21 |
| BD(1,2,4,10,11) | ( 1 12 2 5 7 8 13 10 9 4 11 6 3 )BD(1) + BD(5,6,8,9) | 22 |
| BD(1,2,4,7,11) | ( 1 12 2 5 7 8 13 10 9 4 11 6 3 )BD(1) + BD(5,6,8,9) | 23 |
| BD(1,2,3,7,11) | ( 1 12 2 5 7 8 13 10 9 4 11 6 3 )BD(1) + BD(5,6,8,9) | 24 |
| BD(1,2,7,10,11) | ( 1 12 2 5 7 8 13 10 9 4 11 6 3 )BD(1) + BD(5,6,8,9) | 25 |
| BD(1,2,3,4,5) | BD(5,6,7,8,9) | 26 |