All Parity Realizable Trees

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Abstract

Given a graph $G = (V, E)$ and a vertex subset $D \subseteq V$, a subset $S \subseteq V$ is said to realize "parity assignment" D if for each vertex $v \in V$ with closed neighborhood $N[v]$ we have that $| N[v] \cap S |$ is odd if and only if $v \in D$. Graph G is called all parity realizable if every parity assignment D is realizable. This paper presents some examples and provides a constructive characterization of all parity realizable trees.

1 Introduction

For a graph $G = (V, E)$ the open neighborhood of vertex $v \in V(G)$ is the set of vertices adjacent to v, $N(v) = \{w \in V(G) : vw \in E(G)\}$, and the closed neighborhood is $N[v] = N(v) \cup \{v\}$.

Theorem 1 (Sutner[8]) *For every graph G there exists a subset $S \subseteq V(G)$ such that $| N[v] \cap S |$ is odd for every $v \in V$.*

We first describe Sutner's theorem as a result in domination theory. The standard domination problem for a graph $G = (V, E)$ is to find a (minimum cardinality) set $S \subseteq V(G)$ such that each vertex is either in S or adjacent to at least one vertex in S, that is, each $v \in V(G)$ has $| N[v] \cap S | \geq 1$. (See, for example, [3] which is an entire issue of *DiscreteMathematics* devoted to domination theory.) More generally, as in Jacobson and Peters [4] S is a k-dominating set if $| N[v] \cap S | \geq k$ for all $v \in V$. Even more generally, for $V(G) = \{v_1, v_2, ..., v_n\}$ to each v_i assign a set R_i of nonnegative integers, and we can ask if there exists a set $S \subseteq V(G)$ such that $| N[v_i] \cap S | \in R_i$ for $1 \leq i \leq n$. For standard domination every $R_i = \{1, 2, ..., n\}$ and clearly there always exists a dominating set, for example V itself; for k-domination every $R_i = \{k, k + 1, ..., n\}$ and there exists a k-dominating set if and only if the minimum degree is at least $k - 1$; for Sutner's odd parity problem

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every $R_i = \{1, 3, 5, ..\}$, and the not so obvious result is that a suitable S can always be found.

Letting N be the closed neighborhood matrix (the binary n-by-n matrix with $N_{ij} = 1$ if either $i = j$ or $v_iv_j \in E(G)$ and $N_{ij} = 0$ otherwise), and letting $\mathbf{1}$ be the all 1's n-tuple, the equation $N \cdot X = \mathbf{1} \bmod 2$ can be solved in polynomial time. Each solution vector X is the characteristic function of a set S for Sutner's odd parity problem. Recently Dawes [2] showed how to find in linear time a minimum cardinality set S for Sutner's problem when the graph is a tree. (While Sutner [8] observed that the problem of determining the minimum cardinality of an all-odd parity solution is NP-Hard for arbitrary graphs.) In [1] we presented a linear time algorithm for a minimum cardinality solution for the more general class of series-parallel graphs and for an arbitrary parity assignment. For an arbitrary parity assignment each R_i is allowed to be either {1,3,5,7,...} or {0,2,4,6,...}.

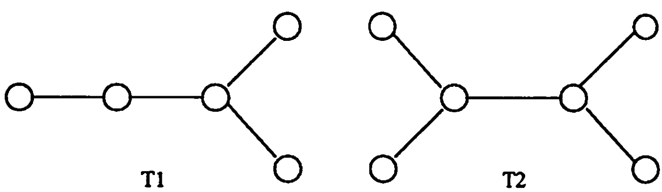


Figure 1: Tree T1 is APR, but T2 is not.

In [1], a graph G was defined to be all parity realizable (APR) if for any subset $D \subseteq V(G)$ with $R_i = \{1, 3, 5, ...\}$ for $v_i \in D$ and $R_i = \{0, 2, 4, 6, ...\}$ for $v_i \notin D$ there exists a subset $S \subseteq V(G)$ such that $| N[v_i] \cap S | \in R_i$ for $1 \le i \le n$. Such a set S will be called a D-parity set, and specifically an all-even parity set is a $\emptyset$-parity set and all-odd parity set is a $V(G)$-parity set. Letting $N1$ and $N2$ be the closed neighborhood matrices for trees $T1$ and $T2$, respectively, of Figure 1, one can verify that $N1$ has rank five over Z_2 so $N1 \cdot X = B$ always has a unique binary solution vector X for any binary n-tuple B, and $T1$ is APR. That this is not the case for $T2$ is easily deduced from the following.

Theorem 2 [1] *A graph G is APR if and only if the only all-even parity set is $S = \emptyset$ (that is, the only solution with every $R_i = \{0, 2, 4, , 6, ...\}$ is the empty set $S = \emptyset$ so that $| N[v_i] \cap S | = 0$ for $1 \le i \le n$).*

Corollary 1 *If every vertex in G has odd degree then G is non-APR.*

Proof. It suffices to note that $S = V(G)$ would a non-empty all-even parity set.

By Corollary 1 tree $T2$ is not APR. The next theorem generalizes the result for tree $T1$.

Theorem 3 *If tree T has exactly one vertex of even degree, then T is APR.*

Proof. Assume tree T is a smallest possible counterexample with the degree $deg(v)$ even and $deg(u)$ odd for all $u \in V(T)$ with $u \neq v$. Let S be a non-empty all-even parity set. Such a set S with $| N[w] \cap S |$ even for every $w \in V(T)$ exists by Theorem 2.

First, suppose $v \notin S$. Let v' be a vertex in S closest to v with $v', w, ..., v$ being the $v' - v$ path in T (possibly having $w = v$). Then the subtree formed by the component $T - v'w$ containing v' would be a smaller counterexample. Thus $v \in S$. Consider T as being rooted at v. Because $v \in S$, $deg(v)$ is even, and $| N[v] \cap S |$ is even, it follows that some child of v, say v_1, is not in S. Because $| N[v_1] \cap S |$ is even, some child of v_1 is in S, say v_2 is such a vertex. The component $T - v_1v_2$ containing v_2 would be a smaller counterexample, completing the proof.

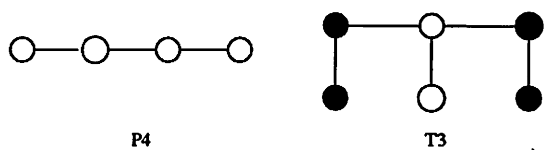


Figure 2: Path P4 is APR; tree T3 is not.

Proposition 1 [1,8] *Path P_n is APR if and only if $n \neq 3k + 2$, $k \geq 0$.*

With reference to Figure 2, note that path P_4 and tree T_3 have exactly two vertices of even degree, but P_4 is APR and T_3 is not. (The darkened vertices form a non-empty all-even parity set of T_3.)

2 Some Examples

In this section, we consider some special classes of trees, namely, paths, spiders, caterpillars, and complete k-ary trees, and obtain characterizations of APR trees (or equivalently of non-APR trees) in each class.

2.1 Paths

A path P_n on n vertices may be denoted by the sequence $(v_1, v_2, ..., v_n)$ of vertices along the path. For paths $P_i = (u_1, u_2, ..., u_i)$ and $P_j = (v_1, v_2, ..., v_j)$ by $P_i.x.P_j$ we denote the path $(u_1, ..., u_i, x, v_1, ..., v_j)$.

Proposition 2 *The set P^* of all non-APR paths is defined as follows.*
1) $P_2 \in P^*$.
2) If P_i and P_j, not necessarily distinct, are in P^* then so is $P_i.x.P_j$.

Proof. First, let S be a non-empty all-even parity set for P_i. If an end point, say u_1, satisfies $u_1 \notin S$, then let h be the smallest value such that $u_h \in S$. Then, we have $| N[u_{h-1}] \cap S | = 1$, a contradiction. Thus, $\{u_1, u_i\} \subseteq S$. Clearly P_2 is in P^*. The statement 2) follows, from Theorem 2, by noting that if S_i and S_j are non-empty all-even parity sets of P_i and P_j, respectively, then $S_i \cup S_j$ contains $\{u_i, v_1\}$ and is a non-empty all-even parity set of $P_i.x.P_j$.

To see that no other path is in P^*, let P_n be a path other than P_2 with S being its non-empty all-even parity set. Since $n > 2$, $S \neq V(P_n)$. Let $v \notin S$ be an internal vertex of P_n. Then $P_n - v$ consists of two paths, say P_i and P_j, each of which is non-APR with $S \cap V(P_i)$ and $S \cap V(P_j)$ being respective non-empty all-even parity sets, thus $P_n = P_i.v.P_j$, completing the proof.

By noting that $P_n \in P^*$ if and only if $n = 3k + 2$, $k \geq 0$, it follows that a path P_n is APR if and only if $n \neq 2 \bmod 3$. We observe that two-thirds of all paths are APR.

2.2 Spiders

A spider $T_{sp} = (v, P_{n1}, P_{n2}, \ldots, P_{nk})$, $k > 2$, is a tree obtained from paths $P_{n1}, P_{n2}, \ldots, P_{nk}$, where $P_{ni} = (u_{i1}, u_{i2}, \ldots, u_{i(ni)})$, by adding vertex v and the edges (v, u_{i1}) for $i = 1, 2, \ldots, k$. The APR spiders may be characterized in terms of the sequence $(n1, n2, \ldots, nk)$ of path lengths.

Proposition 3 *Let $T_{sp} = (v, P_{n1}, P_{n2}, \ldots, P_{nk})$, $k > 2$, be a spider, and let $t_i = | \{j : n_j = i \bmod 3\} |$, $0 \leq i \leq 2$. Then T_{sp} is non-APR if and only if either 1) $t_2 \geq 2$, or 2) $t_2 = 0$ and t_1 is odd.*

Proof. Let T_{sp} be non-APR with S being a non-empty all-even parity set. If v is not in S, then $S_i = S \cap V(P_{ni})$, $1 \leq i \leq k$, is an all-even parity set of P_{ni}. Further, by Proposition 1, S_i can be non-empty if and only if $ni = 2 \bmod 3$. Moreover, the parity of v being even requires that the number of non-empty S_i's must necessarily be even, hence $t_2 \geq 2$.

If v is in S, then consider the vertex u_{i1} along the path P_{ni}. If $u_{i1} \in S$ and $ni > 1$, then the parity of u_{i1} is even requires that u_{i2} not be in S. Now $(T_{sp} - u_{i2})$ contains a subpath P of P_{ni} which is non-APR, hence $ni - 2 = 2 \bmod 3$ or $ni = 1 \bmod 3$. Since the number of such paths must necessarily be odd so that parity of v is even, we have that t_1 is odd. If u_{i1} is not in S, then $ni > 1$ (since the parity of u_{i1} is even) and $(T_{sp} - u_{i1})$ contains a subpath P of P_{ni} which is non-APR. Hence $ni - 1 = 2 \bmod 3$ or $ni = 0 \bmod 3$. This implies that $t_2 = 0$.

Conversely, if $t_2 \geq 2$, and if P_{n1} and P_{n2} are paths, $n1$ and $n2$ each equals 2 mod 3, with S_1 and S_2 being non-empty all-even parity sets of P_{n1} and P_{n2}, respectively, then $S_1 \cup S_2$ is an all-even parity set of T_{sp}. If $t_2 = 0$ and t_1 is odd, then let S_i denote the non-empty all-even parity set of $(v, u_{i1}, u_{i2}, \ldots, u_{i(ni)})$ if $ni = 1 \bmod 3$, and of $(u_{i2}.u_{i3}, \ldots, u_{i(ni)})$ if $ni = 0 \bmod 3$. Then it is easy to see that the union of these S_i's is a non-empty all-even parity set of T_{sp}. The conclusion then follows from Theorem 2.

Equivalently, a spider is APR if and only if $t_2 = 1$, or $t_2 = 0$ and t_1 is even. Thus, for large k, most spiders are not APR.

2.3 Caterpillars

A caterpillar T_c is a tree in which the internal vertices induce a path. The path is called the spine of the caterpillar (as in [5,6,7]). Let $(u_1, u_2, \ldots, u_k)$ be the spine of T_c and let $(n_1, n_2, \ldots, n_k)$ denote the sequence where n_i denotes the number of end vertices adjacent to vertex u_i of the spine. We observe that the only caterpillars with all vertices of odd degrees are those of the forms illustrated by T_1 and $T_{2,j}, j \geq 0$, in Figure 3.

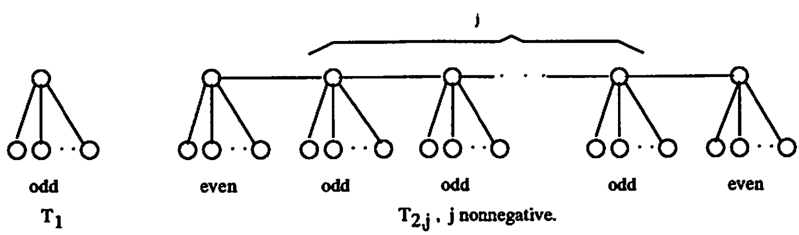


Figure 3: Caterpillars with all vertices of odd degree.

For caterpillars T_{c1} and T_{c2} with spines $(u_1, u_2, \ldots, u_{k1})$ and $(v_1, v_2, \ldots, v_{k2})$ by $T_{c1}.x.T_{c2}$ we denote a caterpillar with spine

$(u_1, u_2, \ldots, u_{k1}, x, v_1, v_2, \ldots, v_{k2})$ in which the number of end vertices adjacent to vertex x can be arbitrary and the number of end vertices adjacent to each u_i and each v_k is the same as in T_{c1} and T_{c2}, respectively.

Proposition 4 *The set T_c^* of all non-APR caterpillars is defined by:*
1) each tree of type T_1 or $T_{2,j}, j \geq 0$, is in T_c^*, and
2) if T_{c1} and T_{c2} are in T_c^*, then so is $T_{c1}.x.T_{c2}$.

Proof. Since every caterpillar T_c of type T_1 or $T_{2,j}$, $j \geq 0$, has all vertices of odd degree, by Corollary 1, it follows that T_c is non-APR. Next, let T_{c1} and T_{c2} be non-APR with non-empty all-even parity sets being S_1 and S_2, respectively. Note that if T_{ci} is of type T_1 then the center vertex is in S_i, and in general the end points of the spine of T_{ci} can be seen to be in S_i. Thus, $S_1 \cup S_2$ is a non-empty all-even parity set of $T_{c1}.x.T_{c2}$. The conclusion then follows from Theorem 2.

To see that no other caterpillar is a member of T_c^*, consider T_c in T_c^* which is not of type T_1 or $T_{2,j}$, $j \geq 0$. Let S be a non-empty all-even parity set of T_c, then clearly $S \neq V(T_c)$. Let the spine of T_c be $(u_1, u_2, \ldots, u_k)$. We observe that an end vertex v of T_c is in S if and only if the vertex u_i, $1 \leq i \leq k$, on the spine adjacent to v is in S. Further, as noted, u_1 and u_k both must be in S. Because $S \neq V_{T_c}$, for some $i, 1 < i < k$, u_i is not in S. Then $T_c - u_i$ consists of two caterpillars T_{c1} and T_{c2}, and possibly some isolated vertices. Further $S_j = S \cap V(T_{cj})$ is a non-empty all-even parity set of T_{cj}, $j = 1, 2$. Hence, by Theorem 2, each T_{cj} is non-APR. Thus, $T_c = T_{c1}.u_i.T_{c2}$, completing the proof.

With a T_c we can associate a parity sequence $(p_1, p_2, \ldots, p_k)$ where p_i is even or odd depending on whether the number of end vertices adjacent to vertex u_i on its spine is even or odd. There are 2^k parity sequences of length k. The number A_k of these parity sequences which correspond to non-APR caterpillars satisfies the following recurrences: $A_1 = A_2 = 1$, and $A_k = 2A_{k-2} + 2A_{k-3} + \ldots + 2A_1$. It is straightforward to show that $A_k = (2^k \pm 1)/3$.

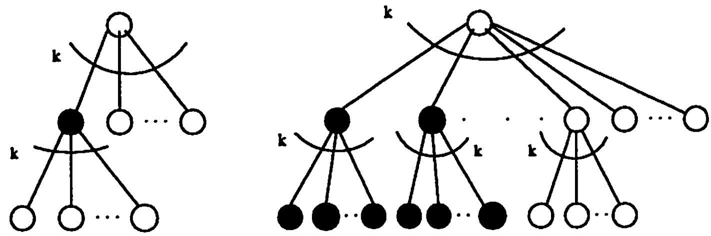
2.4 K-ary Trees

A k-ary tree, $k \geq 2$, is a rooted tree in which the root vertex has k children and every other vertex is either a leaf vertex or has k children.

Proposition 5 *Every k-ary tree with k even is APR.*

Proof. The proposition follows from Theorem 3 by noting that if k is even then a k-ary tree has exactly one vertex of even degree, namely, the root.

The level $L(v)$ of a vertex v in a rooted tree is defined as follows. If v is the root vertex then $L(v) = 1$, otherwise $L(v) = L(parent(v)) + 1$. The depth of a rooted tree is the maximum level of its leaf vertices. Note that the depth of a k-ary tree is at least 2. A k-ary tree of depth d is called complete if 1) every leaf vertex exists at level $d-1$ or d, and 2) if $d > 2$, then for each vertex, except possibly one, at level $d-2$ it is the case that all its children are leaf vertices, or each of its children has k children.



(a) Exactly one vertex of even degree.

(b) The number of leaf vertices at level 3 is at least k.

Figure 4: Complete k-ary tree with odd k and depth 3.

Proposition 6 *A complete k-ary tree T with $k \geq 2$ is APR if and only if it contains exactly one vertex of even degree.*

Proof. If k is even then the conclusion follows from Proposition 5. Assume k is odd ($k \geq 3$). The only complete k-ary tree with exactly one vertex of even degree is as shown in Figure 4a, which is APR by Theorem 3.

If $d = 2$, then every vertex in T has odd degree, hence T is not APR. If $d = 3$ and the number of leaf vertices at depth 3 is $> k$, then T is not APR as can be seen in Figure 4b where the darkened vertices form an all-even parity set.

If $d \geq 4$, then observe that for a complete k-ary tree T there exists two vertices x and y both at level $d-1$ or $d-2$ with $parent(x) = parent(y)$, and each of x and y has k children which are leaves of T. It is easy to see that x, y and all their children comprise an all-even parity set of T, completing the proof.

3 APR Trees

We conclude with a constructive characterization of APR trees. Note that a graph G is APR if and only if for each subset $D \subseteq V(G)$ there is a unique

D-parity set S. Equivalently, no two of the 2^n distinct subsets of $V(G)$ are D-parity sets for the same D. By Theorem 2, this is equivalent to having the empty set $\emptyset$ as the unique all-even parity set. For each $Y \subseteq V(G)$ let $ODS(Y) = \{v \in V(G) :| N[v] \cap Y \mid \textit{is odd}\}$, so that Y is an $ODS(Y)$-parity set.

The proof of Theorem 2 in [1] can be easily strengthened to prove the next theorem.

Theorem 4 *If $S \subseteq V(G)$ is a D-parity set then the number of D-parity sets is the same as the number of all-even parity sets.*

Let (T_i, x_i) denote a tree T_i with a designated vertex $x_i \in V(T_i)$. If T_i is APR we let X_i denote the unique vertex set with $ODS(X_i) = \{x_i\}$, so that $\mid N[v] \cap X_i \mid$ is odd if and only if $v = x_i$.

Definition 1 *Assume T_1 and T_2 are APR and for (T_1, x_1) we have $x_1 \notin X_1$. (Note that for (T_2, x_2), x_2 might or might not be in X_2.) The tree $T^* = T_1 \cup T_2 + x_1x_2$ formed from the disjoint union of T_1 and T_2 by adding edge x_1x_2 is said to be obtained from (T_1, x_1) and (T_2, x_2) by a TYPE 1 operation.*

Definition 2 *Given (T_1, x_1), $(T_2, x_2), \ldots, (T_{2k}, x_{2k})$ where T_i is APR and $x_i \in X_i$ for $1 \leq i \leq 2k$, the tree $T^{**} = T_1 \cup T_2 \cup \ldots \cup T_{2k} + v + x_1v + x_2v + \ldots + x_{2k}v$ formed from the disjoint union of T_i's by adding vertex v and edges vx_i, for $i = 1, 2, \ldots, 2k$, is said to be obtained from (T_1, x_1), $(T_2, x_2), \ldots, (T_{2k}, x_{2k})$ by a TYPE 2 operation.*

Proposition 7 *If T_1 and T_2 are APR and T^* is the tree obtained from (T_1, x_1) and (T_2, x_2) by a TYPE 1 operation, then T^* is also APR.*

Proof. Assume $S \subseteq V(T^*)$ is a non-empty all-even parity set. Let $S_i = S \cap V(T_i)$ for $i = 1, 2$. If $S \cap \{x_1, x_2\} = \emptyset$ we note that at least one S_i is non-empty, and if $S_i \neq \emptyset$ then S_i is a non-empty all-even parity set for T_i, a contradiction because T_i is APR. If $\mid S \cap \{x_1, x_2\} \mid = 1$ say $S \cap \{x_1, x_2\} = \{x_i\}$, then S_i is a non-empty all-even parity set for the APR tree T_i, a contradiction. Finally, assume $\mid S \cap \{x_1, x_2\} \mid = 2$, then $x_1 \in S_1$ and $x_1 \notin X_1$ and $ODS(S_1) = \{x_1\} = ODS(X_1)$, again a contradiction because T_1 is APR.

Proposition 8 *If $T_1, T_2, \ldots, T_{2k}$ are APR (with $k \geq 1$) and T^{**} is the tree obtained from $T_1, T_2, \ldots, T_{2k}$ by a TYPE 2 operation, then T^{**} is also APR.*

Proof. Assume $S \subseteq V(T^{**})$ is a non-empty all-even parity set. Let $S_i =$

$S \cap V(T_i)$. If $v \notin S$, then some $S_i \neq \emptyset$, and we would have a non-empty all-even parity set S_i for APR tree T_i, a contradiction. If $v \in S$ then, because $|N[v] \cap S|$ is even, at least one $x_i \notin S_i$. Now S_i and X_i are distinct subsets of $V(T_i)$ with $ODS(S_i) = \{x_i\} = ODS(X_i)$, a contradiction because T_i is APR.

Theorem 5 *Tree T is APR if and only if T is K_1 or T can be obtained from a set of APR trees by a TYPE 1 or TYPE 2 operation.*

Proof. By two previous propositions every tree obtained by a TYPE 1 or TYPE 2 operation is APR.

We now show that every APR tree on $p > 1$ vertices can be obtained by a TYPE 1 or TYPE 2 operation. Note that star $K_{1,m}$ is APR if and only if m is even, and $K_{1,2k}$ is obtainable from $2k$ copies of K_1 by a TYPE 2 operation. Assume T is an APR tree of diameter at least three, and let x be a vertex of degree $deg(x) = d + 1$ which is adjacent to d end vertices.

Case 1: d is odd.

Let $d = 2k + 1$, with $N(x) = \{w, v_1, v_2, \ldots, v_{2k+1}\}$ and $deg(v_i) = 1$ for $1 \leq i \leq 2k+1$. Because T is APR there is a unique vertex set $Y \subset V(T)$ such that $ODS(Y) = \{w, v_1, v_2, \ldots, v_{2k+1}\}$. Figure 5 illustrates the two cases corresponding to whether or not x is in Y which are considered next.

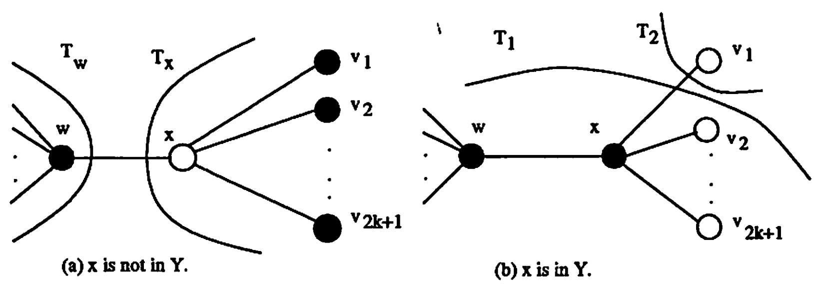


Figure 5: Tree T with deg(x) even and set Y with ODS(Y) = N(x).

First, suppose $x \notin Y$. Because $x \notin Y$ and $v_i \in ODS(Y)$, $v_i \in Y$ for $1 \leq i \leq 2k + 1$. Further, $x \notin ODS(Y)$ and $x \notin Y$ implies that $w \in Y$. Let T_w and T_x be the components of $T - wx$ containing w and x, respectively. Let $Y1 = Y \cap V(T_w)$, then in T_w we have $w \in Y1$ and $ODS(Y1) = \{w\}$. If T_w is not APR then, applying Theorem 2, let $Y2 \subseteq V(T_w)$ with $Y1 \neq Y2$ and $ODS(Y1) = ODS(Y2) = \{w\}$, a contradiction; and if $w \notin Y2$ then $Y2 \cup \{x, v_1, v_2, \ldots, v_{2k+1}\}$ would be a non-empty all-even parity set of T, again a contradiction. Thus, T_w is APR and T can be obtained from T_w

and $2k+1$ copies of K_1 by a TYPE 2 operation.

Secondly, suppose $x \in Y$. Let $T_1 = T - v_1$, and assume T_1 is not APR. Let S be a non-empty all-even parity set of T_1. If $x \notin S$ then S would also be a non-empty all-even parity set of T, contradicting the fact that T is APR. Thus $x \in S$ and if $k \geq 1$ then $\{v_2, v_3, \ldots, v_{2k+1}\} \subset S$. Also, w is an element of S. But now we have that in T, $Y1 = (S - x) \cup \{v_1\}$ has $ODS(Y1) = \{w, v_1, v_2, \ldots, v_{2k+1}\}$. Thus, we have in T, $Y \neq Y1$ and $ODS(Y) = ODS(Y1)$, a contradiction. Therefore T_1 is APR.

Also because $x \in Y$ and $ODS(Y) = \{w, v_1, v_2, \ldots v_{2k+1}\}$, each $v_i \notin Y$ for $1 \leq i \leq 2k+1$ and $w \in Y$. Now for $X_1 = Y - \{x\}$ we have $ODS(X_1) = \{x\}$ in T_1 with $x \notin X_1$. Thus, T can be obtained from T_1 and T_2 with $V(T_2) = \{v_1\}$, by a TYPE 1 operation.

Case 2: d is even.

Let $d = 2k$ with $N(x) = \{w, v_1, v_2, \ldots, v_{2k}\}$ with $deg(v_i) = 1$ for $1 \leq i \leq 2k$. Again let Y be the unique subset of $V(T)$ with $ODS(Y) = \{w, v_1, v_2, \ldots, v_{2k}\}$. Let T_w and T_x be the components of $T - wx$, and let $Y1 = Y \cap V(T_w)$. Note that T_x is APR. We consider two cases corresponding to whether or not x is in Y, as illusrated in Figure 6.

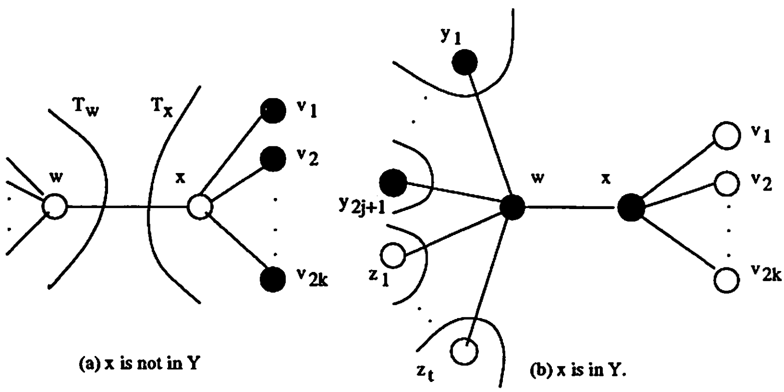


Figure 6: Tree T with deg(x) odd and set Y with ODS(Y) = N(x).

First, suppose $x \notin Y$. Then $v_i \in Y$ for $1 \leq i \leq 2k$ and $w \notin Y$. Now $w \notin Y1$ and in T_w we have $ODS(Y1) = \{w\}$. If T_w is not APR then there exits $Y2 \subseteq V(T_w)$, $Y2 \neq Y1$ with $ODS(Y1) = ODS(Y2) = \{w\}$. If $w \notin Y2$ then in T we have $ODS(Y2) = ODS(Y1)$, a contradiction; and if $w \in Y2$ then $Y2 \cup \{x, v_2, v_2, \ldots, v_{2k}\}$ would be a non-empty all-even parity set for T, again a contradiction. Thus T_w is APR, and T can be obtained from T_w and T_x by a TYPE 1 operation.

Second, suppose $x \in Y$. Then each $v_i \notin Y$ and $w \in Y$. Further, $| N(w) \cap Y1 |$ is odd, so in T we have $N(w) = \{x, y_1, y_2, \ldots, y_{2j+1}, z_1, z_2, \ldots, z_t\}$ with $y_i \in Y$ for $1 \leq i \leq 2j+1$ and $z_i \notin Y$ for $1 \leq i \leq t$. Let T_{y_i} (respectively T_{z_i}) be the component of $T_w - w$ containing y_i (respectively z_i). To see that T_{y_i} is APR, assume to the contrary. Let $Y^* = Y \cap V(T_{y_i})$, then $y_i \in Y^*$ and $ODS(Y^*) = \{y_i\}$. Since T_{y_i} is assumed to be not APR there is $Y^{**} \subset V(T_{y_i})$, $Y^* \neq Y^{**}$, with $ODS(Y^*) = ODS(Y^{**}) = \{y_i\}$. If y_i is also in Y^{**} then $ODS(Y^*) = ODS(Y^{**})$ in T, a contradiction; if y_i is not in Y^{**} then $Y^{**} \cup (Y - Y^*) \cup \{v_1, v_2, \ldots, v_{2k}\}$ is a non-empty all-even parity set of T, a contradiction. Thus, T_{y_i} is APR and likewise one can show that each T_{z_i} is also APR with $z_i \notin Z_i$, $Z_i \subset V(T_{z_i})$ and $ODS(Z_i = \{z_i\}$. Let $T_0 = T - (V(T_{z_1}) \cup V(T_{z_2}) \cup \ldots \cup V(T_{z_t}))$, and for $1 \leq i \leq t$, let T_i be the subtree of T with $V(T_i) = V(T_{i-1}) \cup V(T_{z_i})$. Then T_0 can be obtained from APR trees $T_x, T_{y_1}, \ldots, T_{y_{2j+1}}$ by a TYPE 2 operation, and each T_i with $i \geq 1$ can be obtained from APR trees T_{z_i} and T_{i-1} by a TYPE 1 operation, completing the proof.

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