Graphs of diameter two without small cycles

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ABSTRACT. It is known that triangle-free graphs of diameter 2 are just maximal triangle-free graphs. Kantor ([5]) showed that if G is a triangle-free and 4-cycle free graph of diameter 2, then G is either a star or a Moore-graph of diameter 2; if G is a 4-cycle free graph of diameter 2 with at least one triangle, then G is either a star-like graph or a polarity graph (defined from a finite projective plane with polarities) of order $r^2 + r + 1$ for some positive integer r (or *P_r-graph* for short). We study, by purely graph theoretical means, the structure of P_r-graphs and construct P_r-graphs for small value of r. Further we characterize graphs of diameter 2 without 5-cycles and 6-cycles respectively. In general one can characterize C_k-free graphs of diameter 2 with $k > 6$ with a similar approach.

1 Introduction

We consider, throughout this paper, finite simple undirected graphs. Terms and notations not specified follow Bondy and Murty [3].

Let V be the vertex set, and E be the edge set of a graph G. Let n be the order of G. Let $N(v)$ be the set of neighbors of v in G. Call $N(v)|$ the *degree* of v in G and denote it by $d(v)$. A *cycle* of length k is denoted by C_k. A graph is *C_k-free* if it contains no cycles of length k. A graph is $\{C_k, C_l\}$-*free* if it is C_k-free and C_l-free. The *distance* between two vertices in the graph is the length of any shortest path between these two vertices. The *diameter* $d(G)$ of G is the maximum distance between any two vertices in G. A graph is *diameter critical* if removing any edge increases the diameter. A *star-like graph* is a graph obtained by taking a star and adding a set of (possibly empty) independent edges between its end-vertices.

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Graphs of diameter 2 normally have a large number of edges, except when the maximum degree of the graph is $n-1$. One way to restrict these graphs from having too many edges is to impose the condition of nonexistence of certain small cycles. It turns out that triangle-free graphs of diameter 2 are just maximal triangle-free graphs. Barefoot, et al. ([1]) proved that maximal triangle-free graphs are diameter critical, and for $n \geq 5$ and $2n-5 \leq m \leq \lfloor (n-1)^2/4 \rfloor + 1$ there is a maximal triangle-free graph of size m with diameter 2.

Suppose G is a C_4-free graph of diameter 2. If u and v are two non-adjacent vertices in G, then u and v must have a unique common neighbor. Denote this common neighbor of u and v by $n(u,v)$. Clearly a C_4-free graph of diameter 2 may contain triangles. For example, a star-like graph (not a star) is C_4-free and diameter-2 which has at least one triangle. Besides star-like graphs, there are other graphs which are C_4-free and diameter-2, and contains at least one triangle. As shown in [2], these graphs are all of order r^2+r+1 with r a positive integer. For simplicity, call these graphs *P_r-graphs.*

Let P be a finite projective plane, and let π be a polarity of P (a one-to-one mapping of points onto lines such that $p \in \pi(q)$ whenever $q \in \pi(p)$). Then the *polarity graph* $G(P,\pi)$ is the graph with vertex set the points of P and edge set $\{(p,q) \mid p \in \pi(q), p \neq q\}$. Kantor [5] (independently, Bondy et al. [2]) proved that a graph is a P_r-graph if and only if it is a *polarity graph*:

Theorem 1 *(Kantor, [5]) Stars and Moore-graphs of diameter 2 are the only $\{C_3, C_4\}$-free graphs of diameter 2. Star-like graphs and polarity graphs are the only C_4-free graphs of diameter 2 with at least one triangle.*

Theoretically this result gives one way to construct P_r-graphs from finite projective planes with polarities of order r. But there is no good characterization of finite projective planes with polarities, and even if we know that a finite projective plane with polarities exists, finding all its polarities does not appear to be easy. As a consequence, it is more practical to use Theorem 1 (1) to show the non-existence of P_r-graphs for certain r from the known non-existence results of finite projective planes of order r with polarities; (2) to construct finite projective planes with polarities of order r from a P_r-graph (if any).

As an example of (1), we quote the following result on finite projective planes:

Theorem 2 (Bruck and Ryser, [4]) *If $r \equiv 1$ or $2 \pmod 4$ and r is not a sum of two integral squares, then there is no finite projective plane of order r.*

From Theorem 1 and Theorem 2 we have

Corollary 1 *If $r \equiv 1$ or 2 (mod 4) then there is no P_r-graph unless r is the sum of two integral squares.*

This implies that P_r graphs do not always exist for all r. For example, there is no P_6-graph. It is likely that P_r graphs are rare.

As for (2), one can easily construct at least one finite projective plane of order r with one polarity from a P_r-graph in the way described in [2].

In light of the above arguments, we investigate, in section 2, the existence of P_r-graphs from a purely graphical point of view. We discuss some other properties that C_4-free graphs of diameter 2 have. For instance, we prove that almost all of these graphs are maximal and diameter critical.

In section 3, we consider graphs of diameter 2 without other small cycles. Starting from the observation that $\{C_3, C_5\}$-free graphs of diameter 2 are just complete bipartite graphs, we characterize C_5-free graphs of diameter 2. The method can be extended to the characterization of diameter-2 graphs with no cycles of length k for $k \geq 6$. To illustrate, we further characterize C_6-free graphs of diameter 2.

2 P_r-graphs

Suppose G is a P_r graph. If $r = 1$, then G is a triangle, which is a star-like graph; if $r = 2$ then it is not difficult to see that the graph shown in Figure 1 is the unique P_2-graph.

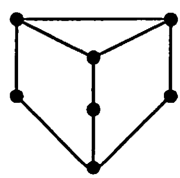


Figure 1: The unique P_2-graph.

Suppose G is a P_r-graph with $r \geq 2$. Let $T = \{1,2,3\}$ be a triangle in G. Let $M = M_1 \cup M_2 \cup M_3$ where $M_i = N(i) \setminus T$ for $i = 1,2,3$. Let $B = V \setminus (T \cup M)$. Here T stands for "top" or "triangle", M for "middle", B for "bottom". The following conclusions are drawn from the proof of Theorem 1:

(I) *Let $p = r - 1$. Then $|M_1| = |M_2| = |M_3| = p$, $|M| = 3p$ and $|B| = p^2$. Moreover each vertex in B has exactly one neighbor in M_i.*

(II) *Each vertex in M has exactly p neighbors in B;*

Let $M_1 = \{a_1, a_2, \cdots, a_p\}$ and $B_i = N(a_i) \cap B$ for $i = 1, 2, \cdots, p$.

(III) *The p sets B_i partition B in equal size; Moreover, every vertex in B_i has degree at most 1 in $G[B_i]$;*

Let $E(A, B)$ be the set of those edges with one end in A and the other in B. Sometimes we just list the elements of A or B or both.

(IV) *$E(M_j; B_k)$ is a matching for $j = 2, 3$ and $k = 1, 2, \cdots, p$;*

(V) *$E(B_i; B_j) \neq \emptyset$ if and only if $(a_i, a_j) \notin E$; if $E(B_i; B_j) \neq \emptyset$ for some $1 \leq i \neq j \leq p$, then $E(B_i; B_j)$ is a matching.*

Let $M_2 = \{b_1, b_2, \cdots, b_p\}$. Suppose $d_{1i} = n(a_1, b_i)$, and $c_i = n(d_{1i}, 3)$ for $i = 1, 2, \cdots, p$, then $b_i = n(d_{1i}, 2)$ and $M_3 = \{c_1, c_2, \cdots, c_p\}$. Let $d_{ij} = n(a_i, b_j)$, for $i = 2, 3, \cdots, p$ and $j = 1, 2, \cdots, p$. Then $B_i = \{d_{i1}, d_{i2}, \cdots, d_{ip}\}$ for $i = 1, 2, \cdots, p$.

(VI) *For any $1 \leq i \neq j, k \leq p$, d_{ik}, d_{jk} have no common neighbor in $M_3 \cup B$.*

(VII) *$(c_i, d_{ki}) \notin E$ for all $i = 1, 2, \cdots, p$ and $k = 2, 3, \cdots, p$.*

We make two observations before constructing P_r-graphs:

Lemma 1 *If, in a P_4-graph, the set M is always independent for every T, then no vertex in M has two neighbors u, v in B such that $(u, v) \in E$.*

Proof: (By contradiction) Suppose two neighbors d_{11} and d_{21} of b_1 are adjacent, then d_{11} cannot be adjacent to any vertex in B_1 since otherwise, if we take (d_{11}, b_1, d_{21}) as T, the corresponding M will not be independent, which violates our assumption. Now d_{11} can reach at most one vertex in M_2 (except b_1) through a vertex in B_i $(i = 3, 4)$. Thus one vertex in M_2 is at distance at least 3 from d_{11}, a contradiction. □

Lemma 2 *There is a triangle T in a P_5-graph G such that the corresponding M is not independent.*

Proof: Suppose, on the contrary, that for any given T, the corresponding M is independent. We keep the notation mentioned above except the adjacency between M and B.

Without loss of generality, suppose $(d_{11}, b_1), (d_{11}, c_1) \in E$ and $(b_1, d_{i1}), (c_1, d_{i2}) \in E$ for $i = 2, 3, \cdots, p$. We assume that $(d_{11}, d_{i3}) \in E$ for all $i =$

$2, \cdots, p$ in order to have $d(d_{11}, a_i) = 2$. We deduce, by symmetry, that $E(B_i; B_j)$ forms a matching for all $1 \leq i \neq j \leq p$.

Clearly d_{j3} ($j = 3, 4, \cdots, p$) is not adjacent to either d_{21} or d_{22}. We may assume that $(d_{23}, d_{p3}) \in E$. Consider $E(B_3; d_{23}, d_{43}, \cdots, d_{p3})$. As before, there is a vertex, say d_{i3} which is adjacent to d_{33} ($i \neq 1, 3$). We claim that $i \neq 2$; for otherwise, a 4-cycle occurs, namely $(d_{11}, d_{33}, d_{23}, d_{p3}, d_{11})$. Also $i \neq p$, otherwise we have 4-cycle $(d_{11}, d_{23}, d_{p3}, d_{33}, d_{11})$. Hence $2 < i < p$. This means we have two triangles (d_{11}, d_{23}, d_{p3}) and (d_{11}, d_{33}, d_{i3}) sharing a common vertex d_{11}. If we take one of these two triangles as T, then we end up with an M which is not independent, a contradiction. This completes the proof of Lemma 2. □

From the above preliminaries we can construct P_r-graphs for $r = 3, 4, 5$ respectively if such graphs exist.

Proposition 1 *There is a unique P_3-graph (see Figure 2).*

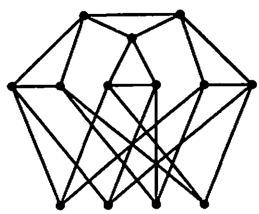


Figure 2: The unique P_3-graph.

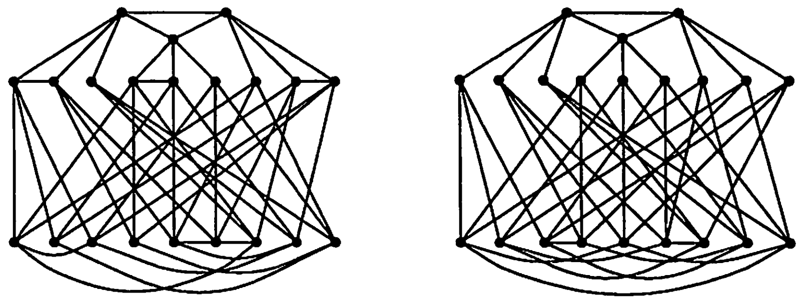
Proof: Suppose G is a P_3-graph with vertices labeled from 1 to 13. Let $T = \{1, 2, 3\}$ and $M_i = N(i) \cap M = \{2i+2, 2i+3\}$ for $i = 1, 2, 3$ and $N(i) \cap B = \{2i+2, 2i+3\}$ for $i = 4, 5$. Without loss of generality, we assume that $(10, 6), (10, 8), (8, 12) \in E$. By (IV) we have $(7, 11), (9, 11), (9, 13) \in E$. Now $(6, 12) \notin E$, since otherwise G has a 4-cycle (6,10,8,12,6). Hence $(6, 13), (7, 12) \in E$.

If there is no edge in $G[M]$, then to make vertex 10 have distance 2 from vertices 7 and 9, we have to have $(10, 11) \in E$; also vertex 10 has a neighbor in $\{12, 13\}$ so as to make 10 distance 2 from 5. By symmetry, we see that $G[B]$ is 2-regular, and hence has a 4-cycle, a contradiction. So $G[M]$ has at least one edge.

Suppose $(4, 5) \in E$. This implies that $E(10, 11; 12, 13) = \emptyset$, i.e., there is no edge with one end-vertex in $\{10, 11\}$ and the other in $\{12, 13\}$. If $(6, 7) \notin E$, then for a similar reason we have $(10, 11), (12, 13) \in E$. However if we take $\{1, 4, 5\}$ as T instead of $\{1, 2, 3\}$ we get a perfect matching in $G[M]$. So we can always assume that $(4, 5), (6, 7), (8, 9) \in E$. Now no more edges can be added to the graph. Furthermore the graph obtained (see

Figure 2) is C_4-free and is of diameter 2. Thus the P_3-graph exists and is unique. □

Proposition 2 *There are exactly two P_4-graphs (see Figure 3).*



(1) (2)

Figure 3: The two P_4-graphs.

Proof: Let $p = 3$. Consider two cases:

Case 1: M is not independent in G, say $(a_1, a_2) \in E$. Then $E(B_1; B_2) = \emptyset$.

If $(d_{1i}, d_{3i}) \notin E$ for all $i = 1, 2, 3$, we can assume, without loss of generality, that $(d_{11}, d_{32}) \in E$. Then $(d_{12}, d_{33}), (d_{13}, d_{31}) \in E$. Now $(d_{21}, d_{32}) \notin E$ (otherwise a 4-cycle $(b_1, d_{11}, d_{32}, d_{21}, b_1)$ occurs), and we have

$$(d_{21}, d_{33}), (d_{22}, d_{31}), (d_{23}, d_{32}) \in E.$$

To make $d(d_{11}, b_3) = 2$, we can add either (d_{11}, d_{13}) or (b_1, b_3) to G. But neither is possible, because of the paths: $(d_{11}, b_1, d_{31}, d_{13})$ and $(b_1, d_{21}, d_{33}, b_3)$. Thus we may assume that $(d_{11}, d_{31}) \in E$. Since $(d_{31}, c_1) \notin E$, assume, without loss of generality, that $(d_{31}, c_2) \in E$. Then $(c_3, d_{32}), (c_1, d_{33}) \in E$. By (VI) we have $(c_1, d_{22}), (c_2, d_{23}), (c_3, d_{21}) \in E$. Because of the path:$(d_{11}, d_{31}, c_2, d_{12})$ we see that $(d_{11}, d_{12}) \notin E$. To make d_{11} distance 2 from b_2 and b_3, we must have $(b_1, b_2), (d_{11}, d_{13}) \in E$. Since $(d_{11}, d_{31}) \in E$, by (V) $(d_{13}, d_{31}) \notin E$; also we have $(d_{13}, d_{33}) \notin E$ as there is a path $(d_{13}, d_{11}, c_1, d_{33})$. Thus we must have $(d_{13}, d_{32}) \in E$. By (V) we have $(d_{12}, d_{33}) \in E$.

Since $(d_{11}, d_{31}) \in E$, $(d_{21}, d_{31}) \notin E$ by (VI). G has a path $(d_{21}, b_1, b_2, d_{32})$ which implies that $(d_{21}, d_{22}) \notin E$. Thus $(d_{21}, d_{33}) \in E$. Now $(d_{22}, d_{33}) \notin E$, and $(d_{22}, d_{31}) \notin E$ as there is a path $(d_{22}, b_2, b_1, d_{31})$. Thus $(d_{22}, d_{32}) \in E$. Finally by (V), $(d_{23}, d_{31}) \in E$.

To make $d(d_{22}, b_3) = 2$, d_{22} must be adjacent to either d_{23} or d_{33}. Since $(d_{21}, d_{33}) \in E$, we know that $(d_{22}, d_{33}) \notin E$, thus $(d_{22}, d_{23}) \in E$.

From $(d_{12}, d_{13}), (d_{12}, d_{32}) \notin E$, we see that $(c_2, c_3) \in E$ in order to make $d(d_{12}, c_3) = 2$.

Now no more edges can be added to the graph G (see Figure 33.1) we have just obtained. As one can check, G is a P_4-graph.

Case 2: Suppose that for any T, the corresponding M is independent.

By Lemma 1 $(d_{ik}, d_{jk}) \notin E$ if $i \neq j$. Without loss of generality, suppose that $(d_{11}, d_{22}), (d_{11}, d_{33}) \in E$. Thus $(d_{12}, d_{23}), (d_{13}, d_{21}), (d_{12}, d_{31}), (d_{13}, d_{32}) \in E$. In turn we have (d_{23}, d_{31}), (d_{22}, d_{33}), $(d_{21}, d_{32}) \in E$. Consider the matching $E(M_3, B_3)$. Since G has a path $(c_1, d_{11}, d_{22}, d_{33})$, $(c_1, d_{33}) \notin E$. This gives $(c_1, d_{32}), (c_2, d_{33}), (c_3, d_{31}) \in E$.

Since $(c_1, d_{11}), (d_{11}, d_{22}) \in E$, by Lemma 1 we have $(c_1, d_{22}) \notin E$. Thus $(c_1, d_{23}), (c_2, d_{21}), (c_3, d_{22}) \in E$. By Lemma 1, each B_i is independent, so no more edges can be added to the graph G (see Figure 3.2) we have just constructed. It is easy to check that G is C_4-free with diameter 2 and thus a P_4-graph. The two P_4-graphs found are not isomorphic since the numbers of edges are different. In all we conclude that there are just two P_4-graphs. □

By a similar approach we prove that the P_5-graph is unique.

Proposition 3 *The graph shown in Figure 4 is the unique P_5-graph.*

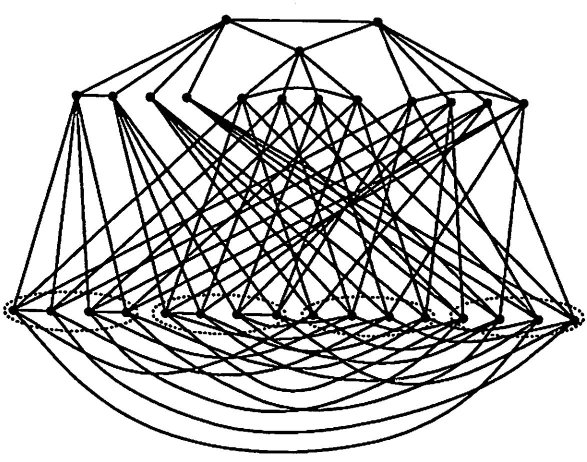


Figure 4: The unique P_5-graph.

By Corollary 1, there is no P_6-graph.

Corollary 2 *The unique finite projective plane on 7, 13 or 31 points has a unique polarity. Moreover, since the finite projective plane on 21 points is known to be unique, it follows from Proposition 2 that this finite projective plane has two different polarities.*

Remark 1 *It would be feasible to write a program, using a similar approach, to generate all the P_r-graphs for $r = 7, 8, 9$ or even $r = 11$. Since there is no finite projective plane of order 10 ([6]), there is no P_{10}-graph.*

It is clear that a maximal C_4-free graph has diameter at most 3. A diameter 3 graph without 4-cycles is not necessarily a maximal C_4-free graph. A 6-cycle would be a simple example. However, we have

Proposition 4 *Every C_4-free graph G of diameter 2 is maximal, except when G is a star-like graph with two or more end-vertices.*

Proof: Let G be a C_4-free graph of diameter 2. Without loss of generality, suppose G is not a star-like graph. Let u and v be two non-adjacent vertices in G. Since G is C_4-free with diameter 2, then u and v have a unique common neighbor, say $w = n(u, v)$. Since G is not a star-like graph, we deduce that $d(u), d(v) \geq 2$. Let $x \in N(u) \setminus \{w\}$, thus $x \neq v$ and $(x, v) \notin E$. Again, x and v have a common neighbor, say y. If $y \neq w$, then (u, w, v, y, x, u) is a 5-cycle. If $y = w$, then $d(u) > 2$. If x' is a vertex in $N(u) \setminus \{x, w\}$ then, as above, x', u, w, v and $n(x', v)$ form a 5-cycle. In sum u, w, v are in a 5-cycle. Now adding edge (u, v) to G would create a 4-cycle. Thus G is maximal. □

Proposition 5 *Every C_4-free graph G of diameter 2 is diameter critical, except when G is a star-like graph with at least one triangle.*

Proof: Again we assume that G is not a star-like graph. Let $e = (u, v)$ be an edge of G. If u and v have no common neighbor, then clearly the distance between u and v is more than 2 in $G \setminus e$. If u and v have a common neighbor w, then since G is not a star-like graph, we know that $d(u) > 2$. If x is a vertex in $N(u) \setminus \{v, w\}$, then $(x, v) \notin E$ as G is C_4-free. Also x and v have no common neighbor in $G \setminus e$, since otherwise, if y is a common neighbor of x and v in $G \setminus e$, then $y \neq v$ and (u, v, y, x, u) is a 4-cycle of G, a contradiction. Thus we see that the distance between x and v is at least 3. Since e can be any edge of G, we have proved that G is diameter critical. □

It is clear that a maximal C_4-free graph is not necessarily of diameter 2.

3 Further discussions

In this section we consider graphs of diameter 2 without other small cycles (not C_4). The following is an easy observation:

Theorem 3 *The only (C_3, C_5)-free graphs of diameter 2 are complete bipartite graphs.*

Clearly complete bipartite graphs and star-like graphs are C_5-free graphs of diameter 2. To characterize C_5-free graphs of diameter 2, we generalize the concept of star-like graph to what we call a W_k*-graph*: a graph obtained by taking a star $K_{1,n}$, say centered at o, and adding some edges among vertices in $V \setminus \{o\}$ such that no component of $G - o$ has a path on $k-1$ vertices. For example, a W_3-graph is just a star; a W_4-graph is just a star-like graph; and a W_5-graph is a graph with a vertex of degree $n-1$ such that every block which is not isomorphic to K_2 or K_4 consists a number of triangles with one edge in common.

By definition, a W_k-graph is also a W_{k+1}-graph. It is evident that W_k-graphs are C_k-free graphs which have a spanning star.

Theorem 4 *The only C_5-free graphs of diameter 2 are complete bipartite graphs and W_5-graphs.*

Proof: Suppose G is a C_5-free graph of diameter 2. If G is also triangle-free, then by Theorem 3, G is a complete bipartite graph. Now suppose G has a triangle, say (u, v, w) with $d(u) \geq d(v) \geq d(w)$. Let S be the set of common neighbors of u and v. We show that $N(v) = S \cup \{u\}$. In fact, if there is a vertex x in $N(v) \setminus (S \cup \{u\})$, then since $d(u) \geq d(v)$, we deduce that u has a neighbor, say y, which is not in $S \cup \{v, x\}$. As (x, v, w, u, y) is a path of G, we see that $(x, y) \notin E$. Now by the choice of x and y we have $(x, u), (y, v) \notin E$. Since G is of diameter 2, x and y must have a common neighbor z which is distinct from u and v. But this time we have a 5-cycle: (x, v, u, y, z, x), a contradiction.

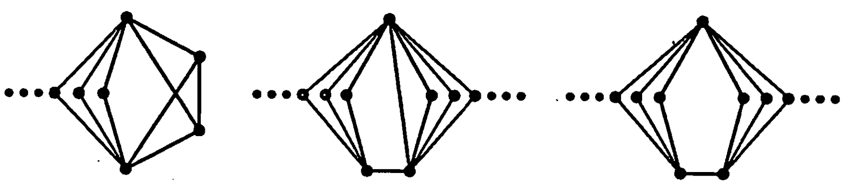
Next we show that every vertex in S is of degree 2 if $|S| \neq 2$. It is true if $|S| = 1$ because in this case we have $2 = d(v) \geq d(w) \geq 2$. Next assume that $|S| > 2$, say $x_1, x_2, x_3 \in S$. If two vertices x_1, x_2 in S are adjacent, then G has a 5-cycle (v, x_1, x_2, u, x_3, v), which is a contradiction. If x_1 is adjacent to a vertex y in $V \setminus (S \cup \{u, v\})$, then $(y, u), (y, v), (y, x_2) \notin E$ since we have paths $(y, x_1, v, x_2, u), (y, x_1, u, x_2, v), (y, x_1, u, v, x_2)$. Thus we must have $d(y, x_2) = 2$, and any common neighbor z of y and x_2 must be in $V \setminus (S \cup \{u, v\})$. But then we get a 5-cycle, (y, x_1, v, x_2, z, y), which is a contradiction. This proves that $d(x) = 2$ for any $x \in S$ if $|S| \neq 2$. Similarly when $|S| = 2$, say $S = \{w, x\}$, we can prove that either $d(x) = d(w) = 2$ or $d(x) = d(w) = 3$ with $(x, w) \in E$.

From the above we have $d(u) = n-1$ in order to make v distance 2 from other vertices except u and vertices in S (if any). In fact we have proved that if a block B of G is not isomorphic to K_2 or K_4, then B is the graph composed of a set of triangles with one edge in common. Hence G is a W_5-graph. This completes the proof of Theorem 4. □

Corollary 3 *The only* (C_5, C_6)*-free graphs of diameter 2 are* $K_{2,m}$ *and* W_5*-graphs.*

Proof: Suppose G is a (C_5, C_6)-free graph of diameter 2. Clearly every complete bipartite graph with no C_6 must have one part with cardinality no more than 2. Thus if G is a complete bipartite graph, then G is either a star which is a W_5-graph, or $K_{2,m}$ with $m > 1$. Notice that no W_5-graph has cycles of length greater than 4. By Theorem 4 we see that Corollary 3 holds. □

Theorem 5 *The only* C_6*-free graphs of diameter 2 are (i)* $K_{2,m}$*, (ii)* W_6*-graphs, and (iii) the three families of graphs shown in Figure 5.*



(a) (b) (c)

Figure 5: Three families of C_6-free graphs of diameter 2.

Proof: Suppose G is a C_6-free graph of diameter 2. If G is also C_5-free, then by Corollary 3, G is in class (i) or class (ii). Next suppose G has a 5-cycle: $(u_1, u_2, \cdots, u_5, u_1)$. For any $v \in A \subseteq V$, define $d_A(v) = |N(v) \cap A|$. Let $S = \{u_1, u_2, \cdots, u_5\}$ with $d_S(u_1) = \max\{d_S(u) \mid u \in S\}$.

We say a vertex u_i in S is *duplicated* if there is a vertex x which is adjacent to both u_{i+1} and u_{i-1}. Here, and in what follows, the subtraction and addition in the subscripts are taken modulo 5.

We make two observations based on the definition.

Claim 1: *No two consecutive vertices on a 5-cycle can be duplicated simultaneously.*

In fact if u_i and u_{i+1} are duplicated by x_1 and x_2 respectively, then we have a 6-cycle $(x_1, u_{i-1}, u_i, x_2, u_{i+2}, u_{i+1}, x_1)$, which is a contradiction.

Claim 2: *If u_i is duplicated by x, then $d(u_i) = d(x) = 2$.*

Since x duplicates u_i, by definition, $(x, u_{i+1}), (x, u_{i-1}) \in E$. Now G has paths $(u_i, u_{i+1}, x, u_{i-1}, u_{i+3}, u_{i+2})$, $(u_i, u_{i-1}, x, u_{i+1}, u_{i+2}, u_{i+3})$. Since G is C_6-free, we see that u_i is not adjacent to u_{i+2} and u_{i+3}. We prove that u_i is also not adjacent to any vertex in $V \setminus S$. In fact if y is a vertex in $V \setminus S$ such that $(u_i, y) \in E$, then, by Claim 1, y cannot duplicate either u_{i-1} or u_{i+1}. Therefore y is not adjacent to u_{i+2} nor u_{i+3}. Hence u_i is the only neighbor of y in S. Since $(u_i, u_{i+2}) \notin E$ and G is of diameter 2, y and u_{i+2} must have a common neighbor, say z, outside S. But this creates a 6-cycle $(y, u_i, u_{i-1}, u_{i+3}, u_{i+2}, z, y)$, which is a contradiction. This proves that $d(u_i) = 2$. By the same argument, using a new S in which x replaces u_i we see that x is also of degree 2.

Now we divide our proof into two cases:

Case 1: $d_S(u_1) = 4$.

Let $S' = N(u_1) \cap N(u_3)$ and $S'' = N(u_1) \cap N(u_4)$. By Claim 2, we see that $S' \cup S''$ is an independent set of G if $|S' \cup S''| > 2$. Now we show that u_3 (and u_4) has no neighbor outside $S \cup S' \cup S''$. If, to the contrary, $y \in V \setminus (S \cup S' \cup S'')$ is a neighbor of u_3, then because of the paths $(y, u_3, u_4, u_5, u_1, x), (y, u_3, u_2, u_1, u_4, z)$ and $(y, u_3, u_2, u_1, u_5, u_4)$ (where x is any vertex in S' and z is any vertex in S''), y is not adjacent to any vertex in $S' \cup S'' \cup \{u_4\}$. Since $y \notin S'$, we have $(u_1, y) \notin E$. Let w be the common neighbor of u_1 and y, then w is not in $S \cup S' \cup S''$. This again creates a 6-cycle $(y, w, u_1, u_5, u_4, u_3, y)$, which is a contradiction. Finally we have $d(u_1) = n - 1$ in order to have $d(G) = 2$. In this case G is C_6-free if and only if each component of $G - u_1$ has no path on five vertices; that is, G is a W_6-graph.

Case 2: $d_S(u_1) < 4$.

We make two more observations:

Claim 3: *Suppose x is a vertex in $V \setminus S$ such that $(x, u_i) \in E$. Then x duplicates exactly one of u_{i-1} and u_{i+1}.*

Clearly x is not adjacent to u_{i-1} or u_{i+1} since G has no 6-cycle. For the same reason, x cannot be adjacent to both u_{i+2} and u_{i+3} simultaneously. Now we show that x is adjacent to one of u_{i+2} and u_{i+3}. In fact, since $d_S(u_i) < 4$, without loss of generality suppose $(u_i, u_{i+2}) \notin E$. We know that x and u_{i+2} cannot have a common neighbor outside S, therefore x is adjacent to either u_{i+3} or u_{i+2} in order to make $d(x, u_{i+2} \le 2$. In all, x is adjacent to exactly one of u_{i+2} and u_{i+3}. This proves that x duplicates exactly one of u_{i-1} and u_{i+1}.

Claim 4: *If $d_S(u_i) = 3$, then u_i cannot be duplicated.*

Without loss of generality, suppose $(u_1, u_4) \in E$. If, to the contrary, u_1 is duplicated by a vertex $x \in V \setminus S$, then $(x, u_2), (x, u_5) \in E$. Thus G has a 6-cycle $(x, u_2, u_3, u_4, u_1, u_5, x)$, a contradiction.

Now consider the following three subcases:

(a) S has two chords, say $(u_1, u_3), (u_2, u_4) \in E$.

By Claim 4, u_1 through u_4 cannot be duplicated. Thus by Claim 3, $d(u_2) = d(u_3) = 3$. Also by Claim 3, $N(u_1) \setminus S = N(u_4) \setminus S$ (i.e. every vertex in the set duplicates u_5). Let $S' = N(u_1) \cap N(u_4)$. Then by Claim 2 and 3, all vertices in S' are of degree 2. Thus G belongs to class (iii-a).

(b) S has only one chord, say (u_1, u_3).

By Claim 4, u_1 and u_3 cannot be duplicated. By Claim 1, one of u_4 and u_5, say u_4 cannot be duplicated. Thus, by Claim 3, all neighbors of u_3 outside S duplicate u_2; all neighbors of u_4 outside S duplicate u_5; all neighbors of u_1 outside S duplicate either u_2 or u_5. Let $S' = N(u_1) \cap N(u_3)$ and $S'' = N(u_1) \cap N(u_4)$. Then, by Claim 2 and 3, the vertices in $S' \cup S''$ are all of degree 2. Therefore G belongs to class (iii-b).

(c) S has no chord.

By Claim 1, we can assume without loss of generality, that u_1, u_3 and u_4 cannot be duplicated. By Claim 3, $N(u_3) \setminus S, N(u_4) \setminus S \subseteq N(u_1) \setminus S$. By Claim 2 and 3, all vertices in $N(u_1) \cap (N(u_3) \cup N(u_4))$ are of degree 2. Thus G belongs to class (iii-c).

From the discussion above we see that G has to be in one of the three families shown in Figure 5.

Combining Corollary 3 with the two cases above, we see that Theorem 5 holds. □

Using this method it is, in theory, possible to characterize all the graphs of diameter 2 without cycles of length k for $k \geq 7$; but it would become more and more tedious with each increase in k.

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