Balanced Part Ternary Designs: Some New Results

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ABSTRACT. A balanced part ternary design (BPTD) is a balanced ternary design (BTD) with a specified number of blocks, say b_2, each having repeated elements. We prove some necessary conditions on b_2 and show the existence of some particular BPTDs. We also give constructions of infinite families of BPTDs with $b_1 = 0$, including families of ternary t-designs with $t \geq 3$.

1 Introduction

A *balanced ternary design*, BTD$(V, B; \rho_1, \rho_2, R; K, \Lambda)$, is an arrangement of V elements into B multisets, or *blocks*, each of cardinality K ($K \leq \Lambda$), satisfying

1. Each element appears $R = \rho_1 + 2\rho_2$ times altogether, with multiplicity one in exactly ρ_1 blocks and with multiplicity two in exactly ρ_2 blocks.
2. Every pair of distinct elements appears Λ times; i.e., if m_{vb} is the multiplicity of the vth element in the bth block, then for every pair of distinct elements v and w, we have $\sum_{b=1}^{B} m_{vb} m_{wb} = \Lambda$.

A *balanced part ternary design* [7], BPTD$(V; b_1, b_2, B; \rho_1, \rho_2, R; K, \Lambda)$, is a BTD that also satisfies

3. There exist exactly $b_2 = B - b_1$ blocks each containing at least one element of multiplicity two.

Hence, every BTD is a BPTD for some choice of b_1 and b_2.

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For example,

$$\begin{array}{ccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
3 & 3 & 4 & 3 & 3 & 2 & 2 & 3 & 4 & 3 & 3 & 4 & 4 & 4 & 4 \\
4 & 3 & 4 & 4 & 4 & 2 & 2 & 3 & 4 & 3 & 3 & 4 & 4 & 4 & 4
\end{array}$$

and

$$\begin{array}{ccccccccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
2 & 2 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
4 & 2 & 3 & 2 & 2 & 3 & 2 & 3 & 4 & 3 & 3 & 4 & 4 & 4 & 4 \\
4 & 3 & 4 & 3 & 4 & 4 & 2 & 3 & 4 & 3 & 3 & 4 & 4 & 4 & 4
\end{array}$$

are BPTDs with parameters (4; 1,14,15; 3,6,15; 4,11) and (4; 0,15,15; 3,6,15; 4,11), respectively. On the other hand, there cannot exist a BPTD(4; 2,13,15; 3,6,15; 4,11) (see Proposition 2.8).

Earlier [7] we studied various constructions and existence conditions for the parameters of a BPTD, and in an upcoming article [8] we present a table of BPTD parameters, listing the known and possible values of b_2 (and hence b_1) for all non-trivial possible values of $(V, B; \rho_1, \rho_2, R; K, \Lambda)$ with $R \leq 15$ from [2]. The main thrust of this work is to decide the existence of BPTDs with small parameters from the table, so the results in Section 2 are not necessarily interlinked or very general. There remain several open questions regarding such BPTDs, as listed in the table [8].

Section 3 contains some constructions of BPTDs. Note that any such construction serves double-duty, since one may view its result as either a BPTD or a BTD. The first of these, based on one-factorizations of complete graphs, is similar to a construction of Khodkar [6]. This construction is interesting as it provides two designs as part of a single family which were originally constructed with different techniques by Saha [10] and Murty and Das [9]. Billington [1, p. 54] referred to Nigam's doubly balanced n-ary designs, which by definition satisfy

$$\sum_{\substack{b=1 \\ i,j,k \text{ distinct}}}^{B} m_{ib} m_{jb} m_{kb} = \text{ constant, say } \Lambda_3,$$

as n-ary 3-designs. Another construction in Section 3 is interesting as it constructs a BPTD(8;0,56,56; 21,7,49; 5,18) as the first member of an infinite family; note that this BPTD is not a multiple of a smaller design and is a ternary 3-design with $\Lambda_3 = 7$.

2 Existence conditions

By simple counting arguments, one can prove the following results:

Proposition 2.1. *Let s be a nonnegative integer. If a BPTD$(V; b_1, b_2, B; \rho_1, 1, R; K, 4+s)$ exists, and if $B \leq 2p - s$, then $b_2 = V$.*

Proposition 2.2. *For a BPTD$(V; b_1, b_2, B; \rho_1, \rho_2, R; K, 4)$ to exist, it is necessary that*

$$B \geq 1 + (\rho_1 + \rho_2 - 1)\left\lceil \frac{V\rho_2}{b_2} \right\rceil .$$

Proposition 2.3. *For a BPTD$(V; b_1, b_2, B; \rho_1, \rho_2, R; K, 5)$ to exist, it is necessary that*

$$B \geq 1 + (\rho_1 + \rho_2 - 1)t - \frac{t(t-1)}{2},$$

where $t = \lceil V\rho_2 / b_2 \rceil$.

These results are proven by first assuming that one block contains 2 (in Proposition 2.1) or t (in Propositions 2.2 and 2.3) repeated elements.

Proposition 2.4. *Let $K = 4$. If there exists a block with two repeated elements, then*

$$V\rho_2 - B + 2\rho_1 - \Lambda + 4 \leq b_2.$$

Proof: Let $\beta > 0$ stand for the number of blocks with 2 repeat elements. We have a lower bound on $B - \beta$, since there must be (at least) $2\rho_1 - \Lambda + 4$ blocks containing (at least) one single element. Combining this with $\beta = V\rho_2 - b_2$ gives the desired inequality. □

We recall Lemma 2.8 from [2]:

Proposition 2.5. *There exists a symmetric BTD with parameters $(4l+3, 4l+3; 1, 2l+1, 4l+3; 4l+3, 4l+2)$ if and only if there exists a symmetric BTD with parameters $(4l+3, 4l+3; 2l+1, 1, 2l+3; 2l+3, l+2)$. Given one design, one can construct the other by interchanging 1's and 2's in the incidence matrix.*

The above proposition and the one that follows have a useful corollary.

Proposition 2.6. *If $\rho_1 = 1$, $V = B$, and $K \geq 3$ is odd, then $b_2 = B$. Furthermore, each block contains both single and double elements.*

Corollary 2.7. *If either design of Proposition 2.5 exists, then $b_2 = B$ for both designs.*

Proposition 2.8. *If $\rho_1 = 1$ and Λ is odd, then $V = K$ is even and $b_2 = B - 1$.*

Proposition 2.9. *If Λ and K are both odd, and if $V + B - 1 = \rho_1 V$, then $K = V$ and $b_2 = B - 1$.*

Proof: In this proof, "set" should be interpreted as possibly a multiset.

Consider all partitions of the set of $V\rho_1$ single elements into B nonempty sets. We can construct one such partition from the design (assuming it exists) by striking out its repeated elements.

In any such partition, whether or not it comes from a design, when two sets of size $m+1$ and $n+1$ (n and $m > 0$) are replaced by sets of size $m+n+1$ and 1, the total number of pairs of not-necessarily-distinct elements occurring in the partition increases. Thus a partition consisting of $B-1$ singleton sets and one set of size V has the largest number of such pairs, namely $\binom{V}{2}$. Such a partition exists by hypothesis.

On the other hand, Λ is odd, forcing every pair of elements to occur singly at least once in the design. Consequently, $\binom{V}{2}$ is the smallest possible number of pairs of distinct elements in the partition corresponding to the given design.

(That is, for $\binom{V}{2}$ pairs to occur singly in the design, there must be one complete block and $B-1$ blocks with exactly one single element.) □

Proposition 2.9 has as a corollary an earlier result of the authors [7, Thm. 6.1]. Entries 50 and 186 of the parameter list [8] show that there exist designs to which Proposition 2.9 applies but its corollary does not.

If one modifies four blocks of a BPTD with parameters $(V; b_1, b_2, B; \rho_1, \rho_2, R; 6, \Lambda)$ as follows

$$\begin{array}{cccc} 1&1&1&1\\ 1&1&2&2\\ 2&2&3&3\\ 2&2&4&4\\ 3&4&5&5\\ 3&4&6&6 \end{array} \quad\longrightarrow\quad \begin{array}{cccc} 1&1&1&1\\ 1&1&2&2\\ 2&2&3&4\\ 2&2&3&4\\ 3&3&5&5\\ 4&4&6&6 \end{array}$$

one obtains a BPTD with the parameters $(V; b_1-2, b_2+2, B; \rho_1, \rho_2, R; 6, \Lambda)$. Beginning with the BPTD(6; 0,10,10; 0,5,10; 6,8)

$$\begin{array}{cccccccccc} 1&1&1&1&4&4&4&4&2&2\\ 1&1&1&1&4&4&4&4&2&2\\ 2&2&5&5&6&6&5&5&6&6\\ 2&2&5&5&6&6&5&5&6&6\\ 3&4&3&6&1&3&2&3&3&5\\ 3&4&3&6&1&3&2&3&3&5 \end{array}$$

and adjoining complete blocks, we obtain the next result by repeated applications of this modification.

Proposition 2.10. *For b_1 any nonnegative integer, for b_2 any even integer between 10 and 20 (inclusive), and for $B = b_1 + b_2$, there exists a BPTD$(6; b_1, b_2, B; B-10, 5, B; 6, B-2)$.*

While preparing the parameter-list in [8], the authors proved the nonexistence of certain BPTDs using straightforward arguments pertaining to specific parameters. For completeness, we collect those results in the next proposition and briefly indicate some of their proofs.

Proposition 2.11. *There do not exist BPTDs with the following parameters:*

$$(7;0,9,9;3,3,9;7,8) \tag{2.11.a}$$

$$(6;3,9,12;4,2,8;4,4) \tag{2.11.b}$$

$$(5;b_1,b_2,10;4,2,8;4,5) \quad (b_2 < 9) \tag{2.11.c}$$

$$(18;b_1,b_2,18;5,2,9;9,4) \quad (b_2 < 18) \tag{2.11.d}$$

$$(12;3,9,12;3,3,9;9,6) \tag{2.11.e}$$

$$(18;10,8,18;9,3,15;15,12) \tag{2.11.f}$$

Proof of (2.11.a): Assume the design exists. There must be either 0, 2, 4, or 6 blocks with two repeat elements. One can rule out each of these possibilities. □

Proof of (2.11.b): It is impossible that the three blocks each containing two repeat elements are disjoint. Keeping that in mind, the first three rows of the $V \times B$ incidence matrix must be completed as shown:

$$\begin{array}{cccccccccccc}
2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 2
\end{array}$$

The third block containing two repeated elements must correspond to one of the last two columns, which forces

$$\begin{array}{cccccccccccc}
2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 2 \\
0 & 0 & & & & & 0 & 0 & 0 & 0 & 2 & 0/1
\end{array}$$

and this cannot be completed. □

Proof of (2.11.c): If $b_2 < 9$, then, without loss of generality, three rows of the incidence matrix are as follows:

$$\begin{array}{cccccccccc}
2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 \\
2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 2 \\
0 & 0 & 0 & 0/1 & 1/0 & & & & 2 &
\end{array}$$

The third row can't be completed. □

Proof of (2.11.d): If $b_2 < 18$, then there must be at least one block with three repeat elements, but this is inconsistent with the other parameters. □

Proof of (2.11.e): If there were nine blocks with a repeated element, since K is odd, there would be exactly one single element in each of the nine. It follows that at least one pair would occur an even number of times. □

Proof of (2.11.f): If b_2 were 8, then the total number of pairs that occur in the design would be odd. □

Note added in proof: 2.11.d, 2.11.e, 2.11.f are special cases of the fact that the only possible value of b_2 in a symmetric BTD is B.

3 Constructions of BPTDs with $b_1 = o$

Our first result is a general construction (with non-trivial hypotheses).

Theorem 3.1. *If all $\binom{V}{K}$ K-subsets of a V-set $\{1,\ldots,V\}$ can be partitioned into V classes $C_1,\ldots,C_V$ each of equal size and such that each is a 1-design on $\{1,\ldots,V\}\setminus\{i\}$, then a BPTD exists with parameters $V; b_1 = 0, b_2 = \binom{V}{K} = B; \rho_1 = \binom{V-1}{K-1}, \rho_2 = \frac{1}{V}\binom{V}{K}$; blocksize $K+2$, and $\Lambda = \frac{4}{V-1}\binom{V-1}{K-1} + \binom{V-2}{K-2}$.*

Proof: Adjoin $\{i,i\}$ to each set in C_i, □

We now prove the existence of the required partition for certain cases.

When $K = 2$ and V is odd, the partition can be obtained from the one-factorization on $V+1$ elements, a concept which we briefly review.

Let K_n denote the complete graph with vertices $\{1,\ldots,n\}$ and $\binom{n}{2}$ distinct edges. A collection of n disjoint edges – that is, having no vertex in common – from the graph K_{2n} is said to be a one-factor of K_{2n}, and a collection of $2n-1$ disjoint one-factors of K_{2n} – that is, having no edge in common – is called a one-factorization of K_{2n}. As a consequence, every edge of K_{2n} appears exactly once in any one-factorization, just as every vertex of K_{2n} appears exactly once in any one-factor.

It can be proven that there exist one-factorizations of any K_{2n}. For example, K_8 has the following one-factorization, in which its 28 edges are partitioned into 7 one-factors, each containing 4 disjoint edges.

F_1:	(1,8)	(7,2)	(6,3)	(5,4)
F_2:	(2,8)	(1,3)	(7,4)	(6,5)
F_3:	(3,8)	(2,4)	(1,5)	(7,6)
F_4:	(4,8)	(3,5)	(2,6)	(1,7)
F_5:	(5,8)	(4,6)	(3,7)	(2,1)
F_6:	(6,8)	(5,7)	(4,1)	(3,2)
F_7:	(7,8)	(6,1)	(5,2)	(4,3)

See Stanton and Golden [11] for the construction of one-factorizations and applications to design theory.

From the above one-factorization of K_8 we form the following BPTD:

B_1:	$\{7,2,1,1\}$	$\{6,3,1,1\}$	$\{5,4,1,1\}$
B_2:	$\{1,3,2,2\}$	$\{7,4,2,2\}$	$\{6,5,2,2\}$
B_3:	$\{2,4,3,3\}$	$\{1,5,3,3\}$	$\{7,6,3,3\}$
B_4:	$\{3,5,4,4\}$	$\{2,6,4,4\}$	$\{1,7,4,4\}$
B_5:	$\{4,6,5,5\}$	$\{3,7,5,5\}$	$\{2,1,5,5\}$
B_6:	$\{5,7,6,6\}$	$\{4,1,6,6\}$	$\{3,2,6,6\}$
B_7:	$\{6,1,7,7\}$	$\{5,2,7,7\}$	$\{4,3,7,7\}$

In general, we get the following family:

Corollary 3.2. *For every positive integer n, there exists a BPTD with parameters* $(2n+1;0,(2n+1)n,(2n+1)n;2n,n,4n;4,5)$.

Earlier, Khodkar [6] has constructed more general families using one-factorizations of complete graphs with loops.

If $K \geq 3$, the partition needed in Theorem 3.1 would be guaranteed by the existence the Steiner System $S(K, K+1, V)$, a collection of $K+1$-subsets of a set of size V such that each K-subset occurs exactly once. For example, when $K = 3$, it is well known that $S(3,4,V)$ exists when $V \equiv 2$ or 4 (mod 6). To obtain the desired partition $\{C_1, \ldots, C_V\}$, consider all blocks which contain an element i. The sets obtained from these blocks by deleting the element i will form the required partition class C_i. In case $K = 3$, the parameters of the resulting BPTD are

$$\left(V;0\binom{V}{3},\binom{V}{3};\binom{V-1}{2},\frac{(V-1)(V-2)}{6},\frac{5(V-1)(V-2)}{6};5,3(V-2)\right).$$

All of these designs are 3-designs with $\Lambda_3 = 7$. For example, the first design of this family is a BPTD(8; 0,56,56; 21,7,49; 5,18) with $\Lambda_3 = 7$ which, as mentioned in the introduction, is not a multiple of a smaller design.

It is well known that the blocks in each partition class obtained from $S(3,4,V)$ are actually a BIBD$(V-1,3,1)$, called the *Derived Triple System*, which suggests the following construction:

Proposition 3.3. *From a BIBD(v,b,r,k,λ), one can construct a BPTD with $V = v+1; b_1 = 0, b_2 = (v+1)b = B$; $\rho_1 = rv, \rho_2 = b$, (and therefore $R = rv + 2b$); $K = k+2$, and $\Lambda = (v-1)\lambda + 4r$.*

Proof: For each $j \in \{0,1,\ldots,v\}$, let B_j be the BIBD on $\{0,1,\ldots,j-2,j+1,\ldots,v\}$ isomorphic to the given BIBD. To each block of B_j add $\{j,j\}$; the desired BPTD is the collection of all resulting blocks. □

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Appendix A

These examples, appearing for what we believe to be the first time, settle certain existence questions relevant to our earlier work [8].
BPTD(4; 1,14,15; 3,6,15; 4,11):

1	1	1	1	1	1	1	1	1	2	2	2	2	3	3
2	2	2	1	1	1	1	1	1	2	2	2	2	3	3
3	3	3	2	2	3	3	4	4	3	3	4	4	4	4
4	3	3	2	2	3	4	4	4	3	4	4	4	4	4

BPTD(4; 0,15,15; 3,6,15; 4,11);

1	1	1	1	1	1	1	1	1	2	2	2	2	2	3
2	3	2	1	1	1	1	1	1	3	3	2	2	2	3
3	4	2	2	2	3	3	4	4	4	3	3	3	4	4
3	4	4	2	2	3	3	4	4	4	4	4	3	4	4

BPTD(6; 16,5,21; 12,1,14; 4,8):

1	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
1	2	2	2	2	3	3	3	3	3	3	4	4	3	3	3	3	3	3	4	4
2	3	4	5	6	4	4	4	4	5	5	5	5	4	4	4	4	5	5	5	5
2	3	4	5	6	5	5	6	6	6	6	6	6	5	5	6	6	6	6	6	6

See [5] for a BPTD(9; 0,12,12; 4,4,12; 9,11) relevant to #80 of [8].

Appendix B

Disregarding redundancies, here are the designs from the BPTD table [8] to which results in either this or our earlier paper [7] applies.

The result:	applies to the designs [8]:
Lemma 3.7 [7]:	50,169.
Theorem 5.2 [7]:	6,13,20,32,45,67,93,119,158.
Theorem 5.3 [7]:	1,3,5,12,18,30,43,58,89,112,155.
Theorem 6.1 [7]:	14,38,83,101,138,146,176,187.
Examples [7]:	1,13,18,20,24,30,32,45,67.
Proposition 2.1:	2,9,17,27,42,55,87,111,151.
Proposition 2.2:	15,26,40,47,62,63,81,85,91,125,149.
Proposition 2.3:	39.
Proposition 2.4:	10,11,57,65,66,105.
Proposition 2.6:	7,8,23,53,103,104,189,190.
Corollary 2.7:	4,16,41,86,150.
Proposition 2.8:	52,102,140,179,188.
Proposition 2.9:	50,186.
Proposition 2.10:	52,82,100,134,172.
Proposition 2.11:	10,11,19,21,34,46,70,94,122,161,164.
Corollary 3.2:	65.
Examples:	105,179.