Asymmetric Digraphs with Prescribed m-Median and m-Periphery

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ABSTRACT. Let $n \geq 2$ be an arbitrary integer. We show that for any two asymmetric digraphs D and F with $m\text{-rad}F \geq \max\{4, n+1\}$, there exists an asymmetric digraph H such that $mM(H) \cong D$, $mP(H) \cong F$, and $md(D, F) = n$. Furthermore, if K is a nonempty asymmetric digraph isomorphic to an induced subdigraph of both D and F, then there exists a strong asymmetric digraph H such that $mM(H) \cong D$, $mP(H) \cong F$, and $mM(H) \cap mP(H) \cong K$ if $m\text{-rad}_{H_0}F \geq 4$, where H_0 is a digraph obtained from D and F by identifying vertices similar to those in K.

1 Introduction

Let D be a strong digraph. For vertices u and v of D, the directed distance from u to v $d(u,v)$ is the length of a shortest directed u-v path in D. The maximum distance between u and v is defined as $md(u,v) = \max\{d(u,v), d(v,u)\}$. The m-eccentricity of a vertex v in D, $me(v)$, is $\max\{md(v,w) \mid w \in V(D)\}$. The m-radius of D, $m\text{-rad}D$, is defined as $\min\{me(v) \mid v \in V(D)\}$. The m-diameter of D, $m\text{-diam}D$, is $\max\{me(v) \mid v \in V(D)\}$. The m-center $mC(D)$ of D is the subdigraph induced by those vertices v with $me(v) = m\text{-rad}D$. The m-periphery $mP(D)$ of D is the subdigraph induced by those vertices v with $me(v) = m\text{-diam}D$. The m-status of a vertex v is $ms(v) = \sum_{w \in V(D)} md(v,w)$. The m-median of D, denoted by $mM(D)$, is the subdigraph of D induced by those vertices having minimum m-status.

The maximum distance between two subdigraphs D_1 and D_2 of D, denoted $md(D_1, D_2)$, is defined by $md(D_1, D_2) = \min\{md(u,v) \mid u \in V(D_1),$

$v \in V(D_2)\}$. Observe that if D_1 and D_2 are disjoint in D then $md(D_1, D_2) \geq 2$. Otherwise, $md(D_1, D_2) = 0$.

In [1] m-center, m-periphery, and m-median of digraphs have been investigated. Chartrand and Tian [2] studied the relative location of m-center and m-median of digraphs. In [3], similar results have been done for center and median of digraphs defined under directed distance. In this paper, we study the relative location of m-median and m-periphery of an asymmetric digraph. For other graph theory terminology, we follow [4].

2 Noninteresting Medians and Peripheries

We consider the case where the m-median and the m-periphery are nonintersecting. The following theorem will provide a necessary condition of the nonintersecting median and periphery in an asymmetric digraph.

Theorem 1. *Let H be a strong asymmetric digraph. Then,*

$$m\text{ - rad}mP(H) \geq md(mM(H), mP(H)) + 1.$$

Proof: Since $mP(H)$ contains at least two vertices, m-rad$mP(H) \geq 2$. Therefore, the theorem holds when $md(mM(H), mP(H)) = 0$. Assume that $md(mM(H), mP(H)) > 0$, i.e. $mP(H) \cap mM(H) = \emptyset$. Let v be a vertex of $mP(H)$ such that $me_{mP(H)}(v) = m$-rad$mP(H)$. Let w be a vertex of $mP(H)$ such that $md(v, w) = m$-diam$H$. Then

$$m\text{ - rad}mP(H) = me(v) \geq md_{mP(H)}(v, w) \geq md_H(v, w) = m\text{ - diam}H.$$

Consider an arbitrary vertex $x \in V(mM(H))$. Since $mP(H) \cap mM(H) = \emptyset$, it follows that $x \notin V(mP(H))$. Thus, $md(x, v) \leq m$-diam$H - 1$. Therefore, m-rad$mP(H) \geq m$-diam$H \geq md(x, v) + 1 \geq md(mM(H), mP(H)) + 1$.

In order to embed an arbitrary asymmetric digraph D as the m-median of some asymmetric digraph H, we first show that it is possible to embed D into an asymmetric digraph having m-diameter 2.

Lemma 2. *For every asymmetric digraph D, there exists a digraph H containing D as an induced subdigraph such that $me(v) = 2$ for all $v \in V(H)$.*

Proof: Assume that $V(D) = \{v_1, v_2, \dots, v_p\}$. Without loss of generality, we assume that $p \geq 3$. Let

$$V(H) = V(D) \cup \{x_1, x_2, \dots, x_{p-1}\} \cup \{y_1, y_2, \dots, y_{p-1}\}$$

and

$$\begin{aligned} E(H) = E(D) &\cup \{v_i y_i, y_i x_i, x_i v_i \mid 1 \leq i \leq p-1\} \cup \{y_i v_j, v_j x_i, \mid 1 \leq i < j \leq p\} \\ &\cup \{v_1 y_t, x_t, v_1 \mid 2 \leq t \leq p-1\} \cup \{x_{i+1} y_i \mid 1 \leq i \leq p-2\} \\ &\cup \{x_1 y_{p-1}, x_{p-1} y_1\} \cup \{x_i x_j, y_j y_i \mid 1 \leq i < j \leq p-1\}. \end{aligned}$$

Figure 1 illustrates the construction of H when $p = 4$.

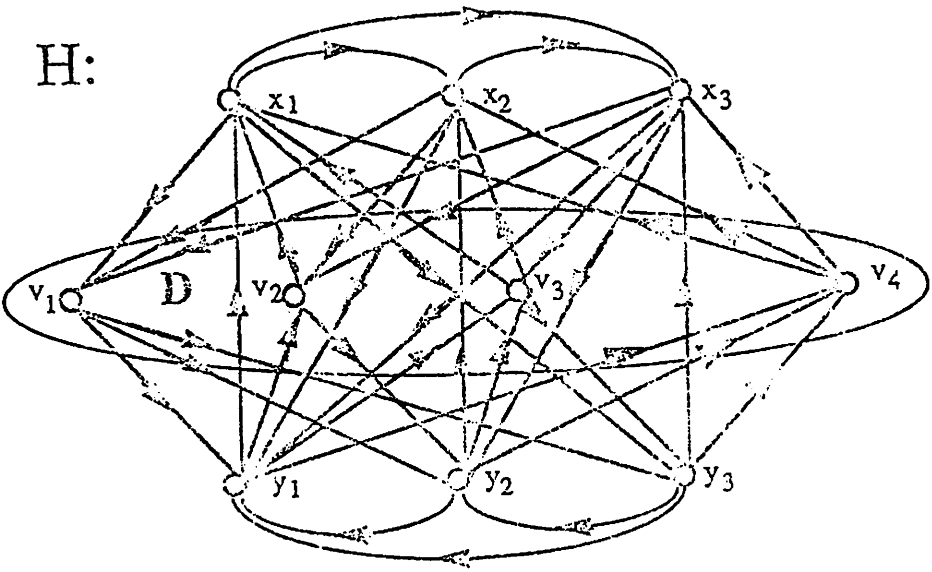

Figure 1

Clearly H is an asymmetric digraph containing D as an induced subdigraph. Therefore, $md(u, v) \geq 2$ for $u, v \in V(H)$. To prove the lemma, it suffices to show that $md(u, v) \leq 2$ for $u, v \in V(H)$. Let $1 \leq i < j \leq p$ be integers. By construction, there exists a 4-cycle v_i, y_i, v_j, x_i, v_i. Thus, $md(v_i, v_j) \leq 2$. Note that H contains 3-cycles x_i, x_j, v_j, x_i and y_i, v_j, y_j, y_i. Therefore, $md(x_i, x_j) = md(y_i, y_j) = md(x_i, v_j) = md(y_i, v_j) = 2$. Next, for $1 \leq i \leq p-1$, there exists a 3-cycle v_1, y_i, x_i, v_1. Thus, $md(v_1, x_i) = md(v_i, y_i) = 2$. There also exists 3-cycles x_1, y_{p-1}, v_p, x_1 and x_i, y_{i-1}, v_p, x_i for $2 \leq i \leq p-1$. Therefore, $md(v_p, x_i) = md(v_p, x_i) = 2, 1 \leq i \leq p-1$. Observe that H contains 4-cycles $x_{p-1}, v_1, y_j, x_1, x_{p-1}$ and $y_{p-1}, y_1, v_i, x_1, y_{p-1}$, $2 \leq i \leq p-1$. Thus, $md(x_{p-1}, v_i) = md(y_{p-1}, v_i) = 2$ for $2 \leq i \leq p-2$. Finally, the existence of cycle x_i, v_1, y_j, v_p, x_i implies that $md(x_i, y_j) = 2$ for $1 \leq i, j \leq p-1$.

Theorem 3. *Let D and F be asymmetric digraphs. Let $n \geq 2$ be an integer. If $m\text{-}radF \geq \max\{4, n+1\}$, then there exists an asymmetric digraph H such that $mM(H) = D$, $mP(H) = F$, and $md(mM(H), mP(H)) = n$.*

Proof:

Case 1. Assume that $n \geq 3$.

First, we define H_0 by applying Lemma 1 on D. Thus, $md(v, u) = 2$, for $v, u \in V(H_0)$. Next, we define H_1 by

$$V(H_1) = V(H_0) \cup V(F) \cup \{w_1, \ldots, w_{n-1}\}$$

and

$$E(H_1) = E(H_0) \cup E(P) \cup \{w_{i-1}w_i \mid 2 \le i \le n-1\}$$
$$\cup \{w_{n-1}u, vw_1, uv, \mid u \in V(H_0), v \in V(F)\}.$$

We define H_2 by

$$V(H_2) = V(H_1) \cup \{x_0, y_0, x_1, y_1\}$$

and

$$E(H_2) = E(H_1) \cup \{x_0y_0, x_1y_1, x_0y_1, x_1y_0, w_{n-2}x_0, x_0w_{n-1}, w_{n-2}x_1, x_1, w_{n-1}\}$$
$$\cup \{ux_0, y_0u, ux_1, y_1u \mid u \in V(D)\} \cup \{y_0v, y_1v \mid v \in V(F)\}$$

(See Figure 2).

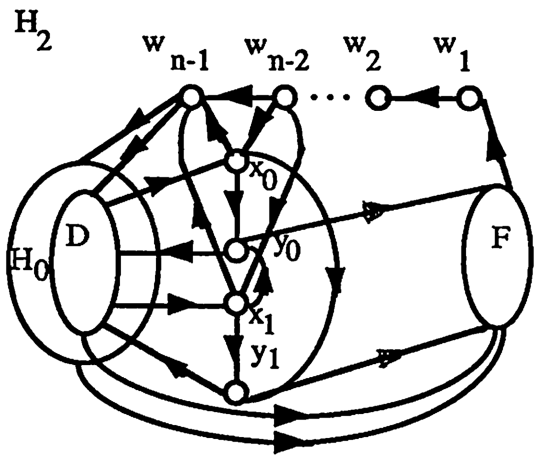

Figure 2

Clearly, $md_{H_2}(u,v) = n$ for $u \in V(H_0)$, $v \in V(F)$. Thus, $md_{H_2}(D,F) = n$. To prove $mP(H_2) = F$, we now calculate the eccentricities of vertices in H_2. We first show that $me(x_0) = me(y_0) = n$. By the construction, there exists an $n+1$-cycle $x_0, y_0, v, w_1 \dots w_{n-2}, x_0$ for every $v \in V(F)$. It follows that $md(x_0, z) \le n$ and $md(y_0, z) \le n$, where $z \in V(F) \cup \{w_i \mid 1 \le i \le n-2\}$. Furthermore, $md(x_1, w_{n-2}) \le n$ and $md(y_0, v) \le n$. Consider a vertex $u \in V(D)$. Since H_2 contains 3-cycles u, x_0, y_0, u and w_{n-1}, u, x_0, w_{n-1}, it follows that $md(x_0, u) = md(y_0, u) = md(x_1, w_{n-1}) = 2 < n$. By the construction in the proof of Lemma 2, for every vertex $u \in V(H_0) - V(D)$, there exist vertices $v, w \in V(D)$ such that $uv, wu \in E(H_0)$. Therefore, H_2 contains cycles u, v, x_0, w_{n-1}, u and u, v, x_0, y_0, w, u.

Thus, $md(y_0, u) = 3$ and $md(x_0, u) = 2$ for $u \in V(H_0) - V(D)$. By the construction, $md(x_0, y_1) = 2$, and $md(x_0, x_1) = 3$. Therefore, $me(x_0) = me(y_0) = n$. Since x_1 and y_1 are similar to x_0 and y_0 respectively in H_2, we have $me(x_1) = me(y_1) = n$.

Let u be an arbitrary vertex of H_0 and v be any vertex of F. Then, $me(u) = md(u, v) = n$, and $me(w_1) = md(w_1, v) = n$. It follows that $me(w_i) = md(w_i, w_{i-1}) = n$, for $2 \leq i \leq n-1$. Let x be an arbitrary vertex in F. Since m-rad$F \geq n+1$, there exists a vertex $y \in V(F)$ such that $md_F(x, y) = me(x) \geq n+1$. Let u be an arbitrary vertex of D. Then, $x, w_1, w_2, \ldots, w_{n-1}, v, y$ is a shortest $x-y$ path of length $n+1$ in H_2. Therefore, $md_{H_2}(x, y) = n+1$. So, $me(x) \geq n+1$. On the other hand, $md_{H_2}(x, z) \leq n+1$, for all $z \in V(H_2)$. Thus, $me(x) = n+1$. Therefore, $mP(H_2) = F$.

Observe that the vertices of H_0 have the same m-status, say k, in H_2. Let $t = k - \min\{ms(v) \mid v \in V(H_2)\}$. We define H as follows:

$$V(H) = V(H_2) \cup \{x_i, y_i \mid 2 \leq i \leq t+2\}$$

and

$$\begin{aligned} E(H) = E(H_2) &\cup \{x_iy_i \mid 2 \leq i \leq t+2\} \cup \{x_iy_j \mid 0 \leq i, j \leq t+2, i \neq j\} \\ &\cup \{ux_i, y_iu \mid u \in V(D), 2 \leq i \leq t+2\} \\ &\cup \{y_iv \mid v \in V(F), 2 \leq i \leq t+2\} \\ &\cup \{x_iw_{n-1}, w_{n-2}x_i \mid 2 \leq i \leq t+2\} \end{aligned}$$

Observe that, $md_H(u, v) = md_{H_2}(u, v)$, for $u, v \in V(H_2)$. Therefore,

$$ms_H(u) = ms_{H_2}(u) + \sum_{2 \leq i \leq t+2} md(u, x_i) + md(u, y_i).$$

If $u \in V(D)$, then $md(u, x_i) = md(u, y_i) = 2$, $2 \leq i \leq t+2$. Therefore, $ms_H(u) = ms_{H_2}(u) + 4(t+1)$, for $u \in V(D)$. If $u \in V(H_2) - V(D)$, then $md(u, x_i) + md(u, y_i) \geq 5$, $2 \leq i \leq t+2$. Thus $ms_H(u) \geq ms_{H_2}(u) + 5(t+1)$. Since $t = k - \min\{ms(v) \mid v \in V(H_2)\}$, it follows that $ms_{H_2}(u) + t \geq \min\{ms(v) \mid v \in V(H_2)\} + t = k$. Thus,

$$ms_H(u) \geq ms_{H_2}(u) + 5(t+1) > k + 4(t+1) = ms_H(v),$$

for $u \in V(H_2) - V(D)$ and $v \in V(D)$. Since $x_0 \in V(H_2) - V(D)$, it follows that $ms_H(x_0) \geq k + 5(t+1)$. Since x_i and x_0 are similar vertices in H, it follows that $ms_H(x_i) = ms_H(x_0) > k + 4(t+1)$, $2 \leq i \leq t+2$. Similarly, $ms(y_i) = ms_H(y_0) > k + 4(t+1)$, $2 \leq i \leq t+2$. Therefore, $mM(H) = D$.

Case 2. Assume that $n = 2$.

First, we define H_0 as the same digraph in Case 1. Next, we define H_1 by

$$V(H_1) = V(H_0) \cup V(F) \cup \{w_1, w_2\}$$

and

$$E(H_1) = E(H_0) \cup E(F) \cup \{w_2w_1\} \\ \cup \{vw_1, w_1u, uw_2, w_2v \mid u \in V(H_0), v \in V(F)\}.$$

We then define H_2 by

$$V(H_2) = V(H_1) \cup \{x_0, y_0, x_1, y_1\}$$

and

$$E(H_2) = E(H_1) \dot\cup \{ux_0, ux_1, y_0u, y_1u \mid u \in V(D))\} \\ \cup \{w_1x_0, w_1x_1, y_0w_2, y_1w_2\} \\ \cup \{x_0y_0, x_1y_1, x_0y_1, x_1y_0\}$$

(See Figure 3).

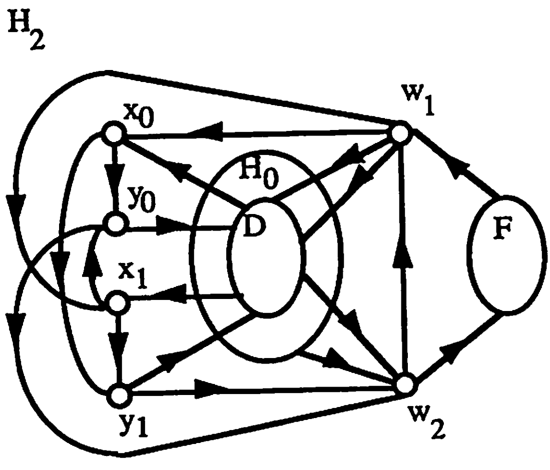

Figure 3

By a similar argument as in Case 1, we have the following:

1. $md(x_0, y_0) = md(x_0, u) = md(x_0, y_1) = 2$ for $u \in V(D)$,
2. $md(x_0, x_1) = md(y_0, y_1) = md(x_0, v) = md(y_0, v) = 3$ for $v \in V(H_0) - V(D)$.

Observe that H_2 contains an 4-cycle $w_1, x_0, y_0, w_2 w_1$ and an 5-cycle v, w_1, x_0, y_0, w_2, v for each $v \in V(F)$. Thus, $md(x_0, w_1) = md(y_0, w_2) = 3$, $md(x_0, w_2) = md(y_0, w_1) = 2$, and $md(x_0, v) = md(y_0, v) = 3$ for $v \in V(F)$. Therefore, in H_2, $me(x_0) = me(x_1) = me(y_0) = me(y_1) = 3$. Observe that H_2 contains an 3-cycle u, w_2, w_1, u, for each $u \in V(H_0)$, and an 4-cycle v, w_1, u, w_2, v, for each $v \in V(F)$. Thus, in H_2, $me(w_1) = me(w_2) = 3$, $me(u) = 2$ for all $u \in V(D)$, and $me(x) = 3$ for all $x \in V(H_0) - V(D)$. Let x be an arbitrary vertex of F. Since $m\text{-rad}F \geq \max\{n+1, 4\}$ and $n = 2$, it follows that $m\text{-rad}F \geq 4$. Thus, there exists a vertex y in F such that $md_F(x, y) \geq 4$. Let v be an arbitrary vertex in H_0. We see that x, w_1, v, w_2, y is a shortest $x - y$ path in H_2. Therefore, $md_F(x, y) = 4$ implying that $me(x) = 4$ in H_2. Thus, $mP(H_2) = F$.

Observe that $md_{H_2}(x_0, u) + md_{H_2}(y_0, u) = 4$, for $u \in V(D)$, and that $md(x_0, v) + md(y_0, v) \geq 5$ in for $v \in V(H_2) - V(D)$. With this fact, we then define H by a similar process utilized in Case 1. Let $t = k - \min\{ms(v) \mid v \in V(H_2)\}$ where $k = ms(u)$ for $u \in V(H_0)$. We then define H by

$$V(H) = V(H_2) \cup \{x_i, y_i \mid 2 \leq i \leq t+2\}$$

and

$$\begin{aligned} E(H) = E(H_2) &\cup \{ux_i, y_i u \mid u \in V(D), 2 \leq i \leq t+2\} \\ &\cup \{w_1 x_i, y_i w_2 \mid 2 \leq i \leq t+2\} \\ &\cup \{x_i y_j \mid 2 \leq i, j \leq t+2\}. \end{aligned}$$

3 Intersecting Medians and Peripheries

We now consider the other extreme, namely where the m-median and m-periphery of an asymmetric digraph overlap on any common part of them.

Lemma 4. *Let D be a strong asymmetric digraph with m-diam$D \leq 4$. Let K be an induced subdigraph of D. Then there exists a strong asymmetric digraph H containing D as a proper induced subdigraph such that*

(i) $ms_H(u) = ms_H(v)$, *for all* $u, v \in V(K)$.

(ii) $me_H(u) = \max\{me_D(u), 3\}$, *for all* $u \in V(D)$, *and*

(iii) $me(u) \leq 3$, *for all* $u \in V(H) - V(D)$.

Proof: Let $m_1(D) = \max\{ms_D(x) \mid x \in V(K)\}$, $m_2(D) = \max\{ms_D(x) \mid x \in V(K)\}$, and $n = m_1(D) - m_2(D)$. We consider two cases.

Case 1. Assume that $n \geq 1$.

Let $S(D) = \{x \in V(K) \mid ms_D(x) = m_2(D)\}$. Define H_1 by

$$V(H_1) = V(D) \cup \{u_1, v_1, w_1, x_1, y_1\}$$

and

$$\begin{aligned} E(H_1) = E(D) &\cup \{u_1x_1, v_1w_1, v_1x_1, v_1y_1, w_1x_1, x_1y_1, y_1u_1, y_1w_1\} \\ &\cup \{zu_1, w_1z \mid z \in S(D)\} \\ &\cup \{zu_1, zx_1, w_1z \mid z \in V(D) - S(D)\} \end{aligned}$$

(See Figure 4).

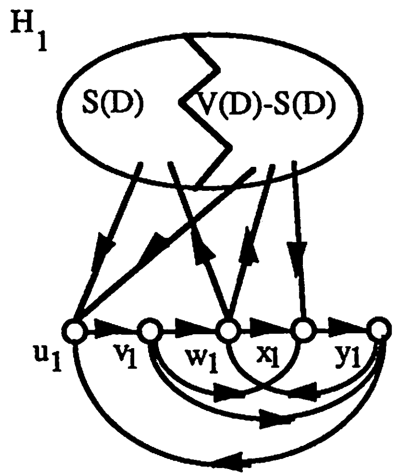

Figure 4

Clearly, H_1 is strong, and D is a proper induced subdigraph of H_1. Since m-diam$D \le 4$, it follows that $md_D(u,v) \le 4$ for $u, v \in V(D)$. Thus, $md_{H_1}(z,t) = md_D(z,t)$ for $z, t \in V(D)$. Therefore, for $z \in S(D)$,

$$\begin{aligned} ms_{H_1}(z) &= md_{H_1}(z,u_1) + md_{H_1}(z,v_1) + md_{H_1}(z,w_1) + md_{H_1}(z,x_1) \\ &\quad + md_{H_1}(z,y_1) + \sum_{t \in V(D)} md_{H_1}(z,t) \\ &= 3 + 2 + 3 + 3 + 3 + \sum_{t \in V(D)} md_{H_1}(z,t) \\ &= 14 + ms_D(z) \\ &= m_2(D) + 14. \end{aligned}$$

Similarly, for $x \in V(K) - S(D)$, $ms_{H_1}(z) = ms_D(z) + 13 \le m_1(D) + 13$.

We define $m_1(H_1) = \max\{ms_{H_1}(x) \mid x \in V(K)\}$ and $m_2(H_1) = \min\{ms_{H_1}(x) \mid x \in V(K)\}$. Then, $m_1(H_1) = m_1(D) + 13$ and $m_2(H_1) = m_2(D) + 14$. Therefore,

$$\begin{aligned} m_1(H_1) - m_2(H_1) &= (m_1(D) + 13) - (m_2(D) + 14) \\ &= m_1(D) - m_2(D) - 1 \\ &= n - 1. \end{aligned}$$

Let $S(H_1) = \{x \in V(K) \mid ms_{H_1}(x) = m_2(H_1)\}$. We define a strong asymmetric digraph H_2 by

$$V(H_2) = V(H_1) \cup \{u_2, v_2, w_2, x_2, y_2\}$$

and

$$\begin{aligned} E(H_2) = E(H_1) &\cup \{u_2x_2, v_2w_2, v_2x_2, v_2y_2, w_2x_2, x_2y_2, y_2u_2, y_2w_2\} \\ &\cup \{zu_2, w_2z \mid z \in S(H_1)\} \\ &\cup \{zu_2, zx_2, w_2z \mid z \in V(H_1) - S(H_1)\}. \end{aligned}$$

By a similar argument, we see that $m_1(H_2) - m_2(H_2) = m_1(D) - m_2(D) - 2 = n - 2$. We repeat this process $n - 2$ times. Let $H = H_n$. Then $m_1(H) = m_2(H)$, i.e. $ms_H(u) = ms_H(v)$ for all $u, v \in V(K)$. Clearly by the construction of H_n, H is strong, D is an induced subdigraph of H, and $\max\{md_H(u, v) \mid u \in V(K), v \in V(H) - V(D)\} = 3$.

Case 2. Assume that $n = 0$.

If D consists of one vertex, then we define H by adding three new vertices to form a directed 4-cycle. It can be easily seen that H has the desired properties. If D contains at least two vertices, we partition $V(D)$ into two nonempty subsets $S_1(D)$ and $S_2(D) = V(D) - S_1(D)$. We then define H by

$$V(H) = V(D) \cup \{u_1, v_1, w_1, x_1, y_1, u_2, v_2, w_2, x_2, y_2\}$$

and

$$\begin{aligned} E(H_1) = E(D) &\cup \{u_1x_1, v_1w_1, v_1x_1, v_1y_1, w_1x_1, x_1y_1, y_1u_1, y_1w_1\} \\ &\cup \{zu_1, w_1z \mid z \in S_1(D)\} \\ &\cup \{zu_1, zx_1, w_1z \mid z \in S_2(D)\} \\ &\cup \{u_2x_2, v_2w_2, v_2x_2, v_2y_2, w_2y_2, w_2x_2, x_2y_2, y_2u_2, y_2w_2\} \\ &\cup \{zu_2, w_2z \mid z \in S_2(D)\} \\ &\cup \{zu_2, zx_2, w_2z \mid z \in S_1(D)\}. \end{aligned}$$

By an argument similar to that in Case 1, we have

$$ms_H(x) = ms_D(x) + 13 + 14 \text{ for } x \in S_1(D)$$

and

$$ms_H(v) = ms_D(v) + 13 + 14 \text{ for } v \in S_2(D).$$

Thus, $ms_H(x) = ms_H(v)$, H is strong, D is an induced subdigraph of H, and $\max\{md_H(u,v) \mid u \in V(K), v \in V(H) - V(D)\} = 3$.

Let D and F be asymmetric digraphs. Let K be a nonempty asymmetric digraph isomorphic to an induced subdigraph of both D and F. Suppose that $V(D) = \{u_1, u_2, \ldots, u_{p1}\}$ and $V(F) = \{v_1, v_2, \ldots, v_{p2}\}$. Without loss of generality, we assume that $\langle\{u_1, u_2, \ldots, u_k\}\rangle \cong \langle\{v_1, v_2, \ldots, v_k\}\rangle \cong K$, and that $u_i \to v_i$, $(i = 1, 2, \ldots, k)$ is an isomorphism. We denote by $(D \cup F)_K$ the digraph obtained from D and F by identifying vertices u_i and v_i, $1 \leq i \leq k$. For the sake of convenience, we consider D, F, and K as induced subdigraphs of $(D \cup F)_K$ in the remainder of the paper. We denote $m\text{-rad}_D K = \min\{\max\{md_D(v,w) \mid w \in V(K)\} \mid v \in V(K)\}$.

Theorem 5. *Let H be a strong asymmetric digraph containing $H_0 = (D \cup F)_K$ as an induced subdigraph such that $mM(H) = D$ and $mP(H) = F$. If m-rad$H_0F \leq 3$, then $H = H_0 = F$.*

Proof: Since $mP(H) = F$, we have $m\text{-diam}H = \min\{me_H(v) \mid v \in V(F)\} = m\text{-rad}_D F$. Note that F is an induced subdigraph of H_0, and that H_0 is an induced subdigraph of H, it follows that $md_H(v,w) \leq md_{H_0}(v,w)$. Therefore,

$$\begin{aligned} m\text{ - diam}H &= \min\{\max\{md_H(v,w) \mid w \in V(F)\} \mid v \in V(F)\} \\ &\leq \min\{\max\{md_{H_0}(v,w) \mid w \in V(F)\} \mid v \in V(F)\} \leq 3. \end{aligned}$$

If $m\text{-diam}H = 2$, then $mM(H) = mP(H) = H$. Thus, $H = H_0 = F$. If $m\text{-diam}H = 3$, then we will prove that $V(H) - V(F) = \emptyset$. Otherwise, $ms(v) = 2(p(H) - 1)$ for $v \in V(H) - V(F)$. Note that $2(p(H) - 1)$ is the minimum possible value for the m-status of a vertex in H. Let w be a vertex in $D \cap F$. Then, $ms(w) = 2(p(H) - 1)$ implying that $Me(w) = 2$ which is a contradiction to $m\text{-diam}H = me_H(w) = 3$. Therefore, $V(H) - V(F) = \emptyset$, i.e. $H = H_0 = F$.

We now define a family of asymmetric digraphs that will be used in our next construction. Let C_4^1 be a directed 4-cycle. We define C_4^n $(n \geq 2)$ inductively from four copies of C_4^{n-1}, say D_0, D_1, D_2, and D_3. Each vertex in D_i is joined to every vertex of D_{i+1} for $0 \leq i \leq 3$ (the subscribes are module 4). It is easily observed that C_4^n is a strong asymmetric digraph of order 4^n. Clearly, all the vertices of C_4^n are similar. Observe also that $m\text{-rad}C_4^n = m\text{-diam}C_4^n = 3$, $n > 1$. We denote the m-status of a vertex in C_4^n by S_n. Then,

$$\begin{aligned} S_n &= 3 \cdot p(C_4^{n-1}) + 3 \cdot p(C_4^{n-1}) + 2 \cdot p(C_4^{n-1}) + S_{n-1} \\ &= 3 \cdot 4^{n-1} + 3 \cdot 4^{n-1} + 2 \cdot 4^{n-1} + S_{n-1} \\ &= 8 \cdot 4^{n-1} + S_{n-1}. \end{aligned}$$

Since $S_1 = 8$, it follows that $Sn = 8 \cdot 4^{n-1} + 8 \cdot 4^{n-2} + \cdots + 8 \cdot 1 = 8(4^n - 1)/3$.

Theorem 6. *Let D and F be asymmetric digraphs. Let K be a nonempty asymmetric digraph isomorphic to an induced subdigraph of both D and F. Let $H_0 = (D \cup F)_K$. If $m\text{-rad}_{H_0} F \geq 4$, then there exists a strong asymmetric digraph H such that $mM(H) \cong D$, $mP(H) \cong F$, and $mM(H) \cap mP(H) \cong K$.*

Proof: Suppose that $V(D) = \{u_1, u_2, \ldots, u_{p1}\}$ and $V(F) = \{v_1, v_2, \ldots, v_{p2}\}$. Without loss of generality, we assume that $\langle\{u_1, u_2, \ldots, u_k\}\rangle \cong \langle\{v_1, v_2, \ldots, v_k\}\rangle \cong K$, and that $u_i \to v_i$, $(i = 1, 2, \ldots, k)$ is an isomorphism. We identify vertices u_i and v_i, $(1 \leq i \leq k)$, and label the new vertices u_i, $1 \leq i \leq k$. Thus, $V(K) = \{u_1, u_2, \ldots, u_k\}$, $V(D) - V(K) = \{u_{k+1}, u_{k+2}, \ldots, u_{p1}\}$, $V(F) - V(K) = \{v_{k+1}, v_{k+2}, \ldots, v_{p2}\}$, and $V(H_0) = \{u_1, u_2, \ldots, u_{p1}, v_{k+1}, v_{k+2}, \ldots, v_{p2}\}$. We define an asymmetric digraph H_1 by

$$V(H_1) = V(H_0) \cup \{w_1, w_2, w_3, w_4\}$$

and

$$\begin{aligned} E(H_1) = E(H_0) &\cup \{w_1w_2, w_1w_3, w_2w_3, w_3w_4, w_4w_1, w_4w_2\} \\ &\cup \{w_2x, xw_3 \mid x \in V(H_0)\} \\ &\cup \{u_iw_1, w_4u_i \mid k+1 \leq i \leq p_1\} \end{aligned}$$

(See Figure 5).

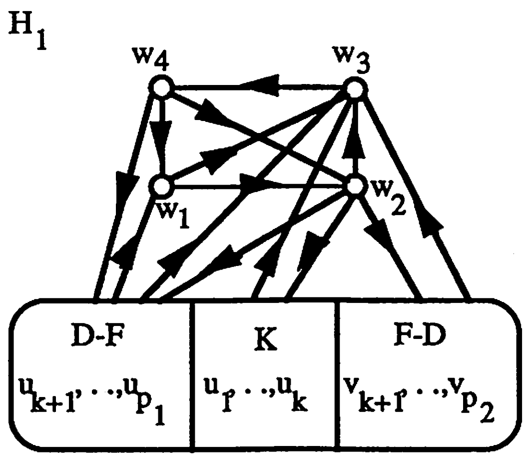

Figure 5

It is clear that $me(w_i) = 3$ $(1 \leq i \leq 4)$ and $meH_1(u_i) = 3$ $(k+1 \leq i \leq p_1)$. Since $m\text{-rad}_{H_0} F \geq 4$, it follows that $me_{H_1}(v) = 4$, $v \in V(F)$. Thus, $m\text{-diam}H_1 = 4$ and $mP(H_1) = F$.

By Lemma 4, there exists an asymmetric digraph H_2 containing H_1 as an induced subdigraph such that

(i) $ms_{H_2}(x) = ms_{H_2}(y)$ for all $x, y \in V(H_1)$,

(ii) $me_{H_2}(x) = \max\{me_{H_1}(x), 3\}$ for all $x \in V(H_1)$, and

(iii) $me(x) \leq 3$ for all $x \in V(H_2) - V(H_1)$.

Therefore, m-diam$H_2 = 4$ and $mP(H_2) = mP(H_1) = F$.

We define H_3 by a similar construction as in Lemma 4.

$$V(H3) = V(H_2) \cup \{u, v, w, x, y\}$$

and

$$\begin{aligned} E(H3) = E(H_2) &\cup \{uv, vw, vx, vy, wx, xy, yu, yw\} \\ &\cup \{zu, wz \mid z \in V(H_1) - V(D)\} \\ &\cup \{zu, zx, wz \mid z \in V(D)\}. \end{aligned}$$

Note that all the vertices of H_1 have the same m-status in H_2. By the construction of H_3, the m-status of those vertices in D is increased by 13, meanwhile, the m-status of vertices in $V(H_1) - V(D)$ is increased by 14. Furthermore, for every $v \in V(F)$ there exists $y \in V(F)$ such that $md_{H_3}(x, y) = 4$.

We now construct $_4$ by adding two vertices u and v to H_3. All of the vertices in $V(H_3) - V(H_2)$ are then joined to u and from v. Accordingly, all vertices in $V(H_2) - V(H_1)$ are joined to v and from u (See Figure 6).

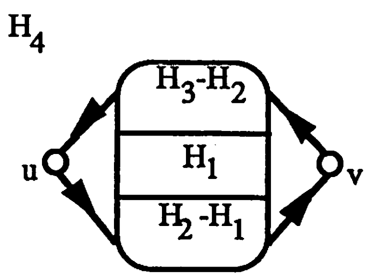

Figure 6

It is easy to see that for every $x \in V(H_1)$ there exists vertices $z_1, z_2 \in V(H_2) - V(H_1)$, and $z_3, z4 \in V(H_3) - V(H_2)$ such that $zz_1, z_2z, zz_3, z_4z \in E(H_3)$. Therefore, $md(u, x) = 2$ for $x \in V(H_1)$. If $y \in V(H_3) - V(H_1)$, then $md(u, y) = 3$. Clearly, $md(u, v) = 2$. Thus, $me(u) = 3$. Since u and v are similar in H_4, it follows that $me(v) = 3$. Let x be a vertex in

$V(H_2) - V(H_1)$. Then, $md(x,y) = 2$, for all $y \in V(H_3) - (H_2)$. Note that $md(x,u) = md(x,v) = 3$, and $md_{H_4}(x,y) \leq md_{H_2}(x,y) \leq me_{H_2}(x) \leq 3$, for all $y \in V(H_2)$. Therefore, $me(x) = 3$, for $x \in V(H_3) - V(H_2)$, $md(x,y) = 2$, for $y \in V(H_2) - V(H_1)$, and $md(x,u) = md(x,v) = 3$. By construction, $md(x,y) \leq 3$ for $y \in (V(H_3) - V(H_2)) \cup V(H_1)$. Then $me(x) \leq 3$ for $x \in V(H_3) - V(H_2)$.

Consider two vertices x and y in H_1. Then, $md_{H_4}(x,y) \leq md_{H_2}(x,y) \leq m\text{-diam}H_2 = 4$. Moreover, if $x \in V(H_1) - V(F)$, then $md_{H_4}(x,y) < 4$. For a vertex $x \in V(F)$, let y be the vertex such that $md_{H_2}(x,y) = 4$. By construction, there is no shorter $x-y$ path in H_4. Therefore, $md_{H_4}(x,y) = 4$. Hence, $me_{H_4}(x) = 4$ for $x \in V(F)$. Combining the above arguments, $m\text{-diam}H_4 = 4$ and $mP(H_4) = F$.

Clearly, all the vertices in $V(D)$ have the same m-status in H_4. Let m be the m-status of a vertex in $V(D)$. Let

$$t = \max\{\lceil \log_4((m - 2p(H_3) - 2)/2)\rceil, \lceil \log_4(3m + 3p(H_1) - 9p(H_3) - 4)/2)\rceil\} + 1.$$

In H_4 replace u with a copy of C_4^n and join each vertex of C_4^n to (from) all vertices adjacent from (to) u. In the resulting digraph, we replace v with another copy of C_4^n and join the vertices in similar fashion. Let this digraph be H. Since every vertex in the two copies of C_4^n has an m-eccentricity of 3, it follows that $m\text{-diam}H = 4$, and $mP(H) = mP(H_4) \cong F$. Let x be a vertex in $V(H_3)$. Then,

$$ms_H(x) = ms_{H_4}(x) - 4 + 2 \cdot \sum_{y \in C_4^t} md(x,y).$$

If $x \in V(H_1)$, then $md(x,y) = 2$ for $y \in V(C_4^n)$. Therefore,

$$ms_H(x) = ms_{H_4}(x) + 2 \cdot 2 \cdot 4^t - 4 = ms_{H_4}(x) + 4 \cdot 4^t - 4 \text{ for } x \in V(H_1).$$

If $x \in V(D)$, then $ms_{H_4}(x) = m$ implying that $ms_H(x) = m + 4 \cdot 4^t - 4$. If $x \in V(H_1) - V(D)$, then $ms_{H_4}(x) = 4 + ms_{H_3}(x) = 4 + (ms_{H_3}(y) + 1) = ms_{H_4}(y) + 1 = m + 1$, where y is an arbitrary vertex in $V(D)$. Therefore,

$$ms_H(x) = m + 1 + 4 \cdot 4^t - 4 = m + 4 \cdot 4^t - 3, \text{ for } x \in V(H_1) - V(D).$$

If $x \in V(H_3) - V(H_1)$, then $md(x,y) = 3$ for $y \in V(C_4^n)$. Thus,

$$ms_H(x) = ms_{H_4}(x) + 2 \cdot 3 \cdot 4^t - 4 \geq 2(p(H_4) - 1) + 6 \cdot 4^t - 4 = 2p(H_3) + 6 \cdot 4^t - 2.$$

By the choice of t, $4^t > (m - 2p(H_3) - 2)/2$, i.e. $2p(H_3) + 6 \cdot 4^t - 2 > m + 4 \cdot 4^t - 4$. Therefore,

$$ms_H(x) \geq 2p(H_3) + 6 \cdot 4^t - 2 > m + 4 \cdot 4^t - 4 = ms_H(y), \text{ where } y \in V(D).$$

To complete the proof, it suffice to show that $ms(x) > ms(y)$ for $x \in V(C_4^t)$ and $y \in V(D)$. If $x \in V(C_4^n)$, then

$$\begin{aligned} ms(x) &= ms_{C_4^n}(x) + 2p(C_4^t) + \sum_{y \in V(D)} md(x,y) + \sum_{y \in V(H_3) - V(H_1)} md(x,y). \\ &= \frac{8}{3} \cdot (4^t - 1) + 2 \cdot 4^t + 2p(H_1) + 2(p(H_3) - p(H_1)) \\ &= \frac{14}{3} \cdot 4^t + 3p(H_3) - p(H_1) - 8/3. \end{aligned}$$

By the choice of t, $4^t > (3m + 3p(H_1) - 9p(H_3) - 4)/2$, i.e. $\frac{14}{3} \cdot 4^t + 3p(H_3) - p(H_1) - 8/3 > m + 4 \cdot 4^t - 4$. Therefore, $ms(x) > ms(y)$ for $x \in V(C_4^n)$ and $y \in V(D)$. Hence, $mM(H) \cong D$.

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