Subsequences of a Multiset

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ABSTRACT. We call a partition $\mu = (\mu_1, \dots, \mu_k)$ of m, $m \leq n$, a constrained induced partition (cip) from a partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n if $\mu_i \leq \lambda_i$ for $i = 1, 2, \dots, k$. In this paper we study the set of *cips* (sections 1-2), determine *cips* of size p (section 4), and give a formula for the number of total subsequences with fixed size chosen from a given multiset such that the multiplicity of each digit in a subsequence is less than or equal to the multiplicity of this digit in the given multiset

1 Introduction

We consider the following

Problem 1.1 *Given a size n multiset $\mathbb{A}_n = a_1^{n_1} \dots a_k^{n_k}, m \leq n$, find out the set of all possible size m subsequences, $\Omega(\mathbb{A}_n, m, n)$, chosen from $\mathbb{A}_n$ such that the multiplicity of each digit in a size m subsequence is less than or equal to the multiplicity of this digit in the given multiset $\mathbb{A}_n$.*

For example we can take size 7 subsequences $5305011, 9101135, \dots$ etc. from a size 9 multiset $1^3 0^2 5^2 3^1 9^1$. An equivalent statement of problem 1.1 is given by (9).

Let $\mathbf{n}$ denote the set of integers $\{1, \dots, n\}$, and n^m denote the set of maps $\alpha : m \to n$. We view α as a sequence $(\alpha(1), \dots, \alpha(m))$. Let $\ell(\alpha)$ denote the size of a sequence α and $|A|$ denote the cardinality of a set A. For $\forall \alpha \in n^m$ and $\forall \beta \in n'^{m'}$, we say β majorizes α, $\alpha \prec \beta$, if $m' \leq m, \sum_{i=1}^{k} \alpha[i] \leq \sum_{i=1}^{k} \beta[i], k = 1, \dots, m'-1$, and $\sum_{i=1}^{m} \alpha[i] = \sum_{i=1}^{m'} \beta[i]$, here $(\alpha[1], \dots, \alpha[m])$ is in the decreasing order of permutation of $\alpha(1), \dots, \alpha(m)$. For $\alpha \in n^m$, let $m_t(\alpha)$ denote the multiplicity of t in α. Let $M_\alpha = (\alpha_1, \dots, \alpha_s)$ denote the decreasing permutation of $(m_1(\alpha), \dots, m_m(\alpha))$ after deleting the zero terms, then M_α is a partition of m. Furthermore we define $\alpha\uparrow$ to be the sequence of increasing permutation of $Im\,\alpha$ and $\alpha\downarrow$ the sequence of decreasing permutation of $Im\,\alpha$. For convenience we give an extended definition of partition:

Definition 1.2 *A sequence of integers $\lambda = (\lambda_1, \dots, \lambda_s, \lambda_{s+1}, \dots, \lambda_k, \dots)$ is called a partition of n, $\lambda \vdash n$, if there exists some integer s such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_s = 1, \lambda_k = 0$ if $k > s$, and $\sum_{i=1}^{s} \lambda_i = n$. We call s the size of λ, denoted as $\ell(\lambda)$. We use λ^* to denote the conjugate partition of λ, which is defined as a sequence of integers $(a_1, \dots, a_n, \dots)$ such that $a_i = |\{j|\lambda_j \geq i\}|$.*

We accept the common notation of writing a partition as a finite sequence by dropping the zeros without ambiguity in the context, and let

$$P_m = \{\text{partitions of } m\}, \tag{1}$$

$$P = \bigcup_{m\in \mathbb{N}} P_m. \tag{2}$$

For $\lambda, \xi \in P$, let $p = min(\ell(\lambda), \ell(\xi)), q = max(\ell(\lambda), \ell(\xi))$ and

$$\lambda \wedge \xi = (min(\lambda_1, \xi_1), \dots, min(\lambda_p, \xi_p), 0, \dots, 0, \dots), \tag{3}$$

$$\lambda \vee \xi = (max(\lambda_1, \xi_1), \dots, max(\lambda_q, \xi_q), 0, \dots, 0, \dots). \tag{4}$$

Any $\lambda \in P$ induces a subset $\lambda\wedge P = \{\lambda\wedge\xi|\xi \in P\}$ of P. The full permutation group S_m acts on n^m in a natural way: $\forall\sigma \in S_m$, $\alpha \in n^m, \sigma.\alpha(i) = \alpha(\sigma^{-1}(i))$ for $i = 1, \dots, m$. The α orbit is denoted by O_α.

Example 1.3 *Take $\alpha = (2,2,2,2) \in 3^4, \beta = (3,2,2,1) \in 3^4$, then $\alpha \prec \beta$, $M_\beta = (2,1,1) \prec (4) = M_\alpha, M_\beta \wedge M_\alpha = (2), M_\beta \vee M_\alpha = (4,1,1)$; $O_\beta = \{(3,2,2,1), (3,2,1,2), (3,1,2,2), (2,3,2,1), (2,3,1,2), (2,2,3,1), (2,2,1,3), (2,1,3,2), (2,1,2,3), (1,3,2,2), (1,2,3,2), (1,2,2,3)\}$.*

Definition 1.4 *Let $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, $m \leq n$, a partition $\mu = (\mu_1, \dots, \mu_k) \vdash m$ is called a constrained induced partition (cip) of m from λ if $\mu \wedge \lambda = \mu$ (i.e., $\mu \leq \lambda$).*

Denote

$$\Omega_\lambda^m = \{\mu|\mu = (\mu_1, \dots, \mu_k) \vdash m, \mu \leq \lambda\} \tag{5}$$

$$\Omega_\lambda^m(p) = \{\mu|\mu \in \Omega_\lambda^m, \ell(\mu) = p\}, \text{ and} \tag{6}$$

$$\Omega_\lambda = \bigcup_{m=0}^{n} \bigcup_{p=0}^{m} \Omega_\lambda^m(p) \text{ with assumption that } |\Omega_\lambda^0(0)| = 1. \quad (7)$$

If $m > n$ we let $\Omega_\lambda^m = \phi$ and $\Omega_\lambda^m(p) = \phi$. It is clear that

$$\mu \in \Omega_\lambda^m \text{ if and only if } \lambda \wedge \mu = \mu \text{ and } \sum_{i=1}^{\ell(\mu)} \mu_i = m. \quad (8)$$

A restatement of problem 1.1 is (see corollary 3.3)

Problem 1.5 *Let $\alpha \in n^n, m \leq n$, determine*

$$\Omega(\alpha, m, n) = \{\beta | \beta \in n^m, Im\beta \subseteq Im\alpha, m_t(\beta) \leq m_t(\alpha), t = 1, \ldots, n\}. (9)$$

Moreover we put

$$\Omega_1(\alpha, m, n) = \{\beta | \beta \in n^m, Im\beta \subseteq Im\alpha, M_\beta = M_\beta \wedge M_\alpha\}. \quad (10)$$

2 Partition Lattice

Propositions 2.1-2.3 and corollary 2.4 are clear.

Proposition 2.1 $\{\Omega_\lambda^m, \prec\}$ *is a partially ordered set.*

Proposition 2.2 $\{P, \vee, \wedge\}$ *is a distributive lattice* (cf. [4] p.27, where the lattice structure is different).

Proposition 2.3 *We have*

1. $\Omega_\lambda = \lambda \wedge P$,
2. $\Omega_\lambda^m = (\lambda \wedge P) \bigcap P_m$.

Corollary 2.4 *We have*

1. $\Omega_\lambda \bigcap \Omega_\xi = \Omega_{\lambda \wedge \xi}$,
2. $\Omega_\lambda^m \bigcap \Omega_\xi^{m'} = \phi$ *if* $m \neq m'$,
3. $\Omega_\lambda^m \bigcap \Omega_\xi^m = \Omega_{\lambda \wedge \xi}^m$,
4. $\Omega_\lambda^m(p) \bigcap \Omega_\xi^m(p') = \phi$ *if* $p \neq p'$,
5. $\Omega_\lambda^m(p) \bigcap \Omega_\xi^m(p) = \Omega_{\lambda \wedge \xi}^m(p)$.

Theorem 2.5 *The map* $*: P \to P$ *given by* $\lambda \to \lambda^*$ *is a lattice isomorphism. i.e., for any* $\lambda, \xi \in P$, *we have*

1. $(\lambda \wedge \xi)^* = \lambda^* \wedge \xi^*$,
2. $(\lambda \vee \xi)^* = \lambda^* \vee \xi^*$. *(cf. [4] p.27 where * is an anti-isomorphism)*

Proof. To show that * is a one-to-one map, we need to show that for any i

$$(\lambda \wedge \xi)^*_i = min(\lambda^*_i, \xi^*_i),$$
$$(\lambda \vee \xi)^*_i = max(\lambda^*_i, \xi^*_i).$$

Let

$$I_1 = \{k | \lambda_k \geq i\},$$
$$I_2 = \{k | \xi_k \geq i\},$$

then

$$(\lambda \wedge \xi)^*_i = \left| I_1 \bigcap I_2 \right|,$$
$$(\lambda \vee \xi)^*_i = \left| I_1 \bigcup I_2 \right|,$$
$$\lambda^*_i = |I_1|, ([1]p.7),$$
$$\xi^*_i = |I_2|, ([1]p.7).$$

We show that at least one of $I_1 \setminus I_2, I_2 \setminus I_1$ is empty. Suppose the contrary we pick up

$$i_1 \in I_1 \setminus I_2,$$
$$i_2 \in I_2 \setminus I_1,$$

then

$$\lambda_{i_1} \geq i, \lambda_{i_2} < i,$$
$$\xi_{i_1} < i, \xi_{i_2} \geq i.$$

Therefore

$$(\lambda_{i_1} - \lambda_{i_2})(\xi_{i_1} - \xi_{i_2}) < 0.$$

Since λ, ξ are partitions, $\lambda_1 \geq \ldots, \lambda_s$ and $\xi_1 \geq \ldots, \xi_s$, so

$$(\lambda_{i_1} - \lambda_{i_2})(\xi_{i_1} - \xi_{i_2}) \geq 0$$

gives a contradiction. Thus

$$\left| I_1 \bigcup I_2 \right| = max\{|I_1|, |I_2|\},$$

$$\left| I_1 \bigcap I_2 \right| = min\{|I_1|, |I_2|\}.$$

Corollary 2.6 *$\mu \in \Omega_\lambda^m$ if and only if $\mu^* \in \Omega_{\lambda^*}^m$.*

Proof. By symmetry we just need to show the "only if" part. It follows from
$\mu^* = (\lambda \wedge \mu)^* = \lambda^* \wedge \mu^*$, and that μ^* is a partition of m.

Corollary 2.7 *For any $\lambda \in P, \lambda \wedge \lambda^*$ and $\lambda \vee \lambda^*$ are self-conjugate partitions.*

Corollary 2.8 *$\Omega_\lambda^m \bigcap \Omega_{\lambda^*}^m \neq \phi$ if and only if $m \leq \sum_i \mu_i = \sum \lambda_i - 1/2 \sum |\lambda_i - \lambda_i^*|$, where $\mu = \lambda \wedge \lambda^*$ is a self-conjugate partition.*

A subset J of P is called an ideal (see Birkhoff [2]) of P if and only if

1. $\lambda \in J, \xi \in P$ and $\lambda \wedge \xi = \xi$, then $\xi \in J$,
2. $\lambda, \xi \in J$, then $\lambda \vee \xi \in J$.

Ω_λ is a principal ideal of P for each $\lambda \in P$. For any two ideals A, B of P, we let

$$A \wedge B = A \bigcap B = \{\lambda | \lambda \in A, \lambda \in B\}, \tag{11}$$

$$A \vee B = \text{minimal ideal of } P_m \text{ which contains both } A \textit{ and } B, \tag{12}$$

and

$$\Omega_P = \{\Omega_\lambda | \lambda \in P\}. \tag{13}$$

Theorem 2.9 *([2]) The map $\lambda \rightarrow \Omega_\lambda$ gives a lattice isomorphism between P and Ω_P.*

For any λ and ξ, if

$$P_m \subseteq \Omega_\lambda, P_m \subseteq \Omega_\xi,$$

then $P_m \subseteq \Omega_\lambda \bigcap \Omega_\xi = \Omega_{\lambda \wedge \xi}$ and $\lambda \wedge \xi$ is smaller than both λ and ξ. We want to find out the minimal λ such that $P_m \subseteq \Omega_\lambda$. We give a clear

Lemma 2.10 *The following statements are equivalent:*

1. *$\lambda \wedge \xi = \xi$,*
2. *$\Omega_\xi \subseteq \Omega_\lambda$,*
3. *for any $m, \Omega_\xi^m \subseteq \Omega_\lambda^m$.*

Theorem 2.11 *For any m, the induced lattice*

$$IP_m = \{\lambda | \lambda \in P, P_m \subseteq \Omega_\lambda\}$$

has a minimal self conjugate element which is given by

$$\lambda = (m, [\frac{m}{2}], [\frac{m}{3}], \dots, [\frac{m}{m-1}], 1).$$

Proof. It's clear that IP_m has an induced lattice structure from P. We first show that $\lambda^* = \lambda$. For $k = 1, \dots, m$ we look at the $k-th$ component of λ^*

$$\lambda_k^* = \left| \{p | [\frac{m}{p}] \geq k\} \right|$$

and

$$[\frac{m}{p}] \geq k \text{ if and only if } [\frac{m}{k}] \geq p,$$

$\lambda_k^* = [\frac{m}{k}]$ follows. Now we show the minimal element is given by λ as above, we finish this in two steps.

1. If $\lambda = (m, [\frac{m}{2}], [\frac{m}{3}], \dots, [\frac{m}{m-1}], 1)$ then $P_m \subseteq \Omega_\lambda$. We need to show for any $\mu = (\mu_1, \mu_2, \dots, \mu_s) \in P_m$,

   $$\mu_i \leq [\frac{m}{i}] \quad i = 1, 2, \dots, s.$$

   Suppose the contrary there is a j such that $\mu_j > [\frac{m}{j}]$, since $[\frac{m}{j}] \leq \frac{m}{j} < [\frac{m}{j}] + 1$, so $\mu_j \geq [\frac{m}{j}] + 1 > \frac{m}{j}$, therefore $\sum_{k=1}^{j} \mu_k > m$, this is a contradiction.

2. The minimal element is $\lambda = (m, [\frac{m}{2}], [\frac{m}{3}], \dots, [\frac{m}{m-1}], 1)$. We want to show that if $P_m \subseteq \Omega_\xi$, then $\lambda \leq \xi$. For any i, let

   $$k = min\{j | m < (j+1)[\frac{m}{i}], j \in \mathbb{N}\}$$

   and

   $$\eta = (\underbrace{[\frac{m}{i}], [\frac{m}{i}], \dots, [\frac{m}{i}]}_{k-copies}, m - k[\frac{m}{i}]) \in P_m.$$

   Then

   $$k \geq i \text{ and } \lambda_i = [\frac{m}{i}] \leq \xi_i,$$

   therefore

   $$\xi \geq \lambda.$$

Proposition 2.12 *If* $\lambda_{(k)} = (m-k+1, [\frac{m-k+2}{2}], \ldots, [\frac{m}{k}])$, *then* $\Omega^m_{\lambda_{(k)}}$ *contains all the partitions of* m *of size* k, *but no partitions of size* $> k$, *i.e.,*

$$\Omega^m_{\lambda_{(k)}} \setminus \Omega^m_{\lambda_{(k-1)}} = \Omega^m_\lambda(k), k = 2, \ldots, m.$$

λ *as in theorem 2.11.*

Proof. For any $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in P_m$ of size k, and $i \leq k$,

$$\sum_{j=1}^{i} \mu_j \leq m - (k-i),$$

thus

$$\mu_i \leq [\frac{m-k+i}{i}].$$

Corollary 2.13 $\Omega^m_\lambda(k) = \{$*partitions of* m *of size* $k\}$ *can be formed by the following steps:*

1. *Let* $\xi = (m-k, [\frac{m-k+2}{2}] - 1, \ldots, [\frac{m}{k}] - 1)$,
2. *Construct* Ω^{m-k}_ξ,
3. *Add* $\underbrace{(1, 1, \ldots, 1)}_{k-\text{copies}}$ *to the elements of* Ω^{m-k}_ξ.

Moreover if $k \geq [\frac{m}{2}]$, *then* ξ *is given by theorem 2.11 with* m *replaced by* $m-k$.

Example 2.14 *For* $m = 7, k = 4, \lambda_{(4)} = (4, 2, 2, 1) \in P_9$, *it is easy to figure out all the possible size 4 partitions of 7 as* $\{(4,1,1,1), (3,2,1,1), (2,2,2,1)\}$. *We can also get these by firstly cutting off all the base "+" from* $\lambda_{(4)}$ *to form a new partition of* $9 - k = 5 : (3,1,1)$, *then construct* $\Omega^3_{(3,1,1)}$ *yielding* $\{(4,1,1,1), (3,2,1,1), (2,2,2,1)\}$.

Young-Ferrars graph shows the procedures below.

$$\lambda_{(4)} = (4,2,2,1) \quad \begin{array}{llll} + & & & \\ + & & & \\ + & + & + & \\ \oplus & \oplus & \oplus & \oplus \end{array} \qquad \overset{\Rightarrow}{\textit{removing}\ \oplus} \quad \xi = (3,1,1) \quad \begin{array}{lll} + & & \\ + & & \\ + & + & + \end{array}$$

We construct $\Omega^3_{(3,1,1)} = \{(3), (2,1), (1,1,1)\}$, *attach* $\oplus\ \oplus\ \oplus\oplus$ *to* $\Omega^3_{(3,1,1)}$ *like*

$$\begin{array}{l} + \\ + \\ + \end{array} \quad \Rightarrow \quad \begin{array}{llll} + & & & \\ + & & & \\ + & & & \\ \oplus & \oplus & \oplus & \oplus \end{array}$$

etc., the results are $\{(4,1,1,1), (3,2,1,1), (2,2,2,1)\}$.

Proposition 2.15 $\{\Omega_\lambda^m, \prec\}$ *has both a unique maximal and a unique minimal element.* [1]

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of n, $m \leq n, \zeta = n - m \geq 0$. If we take off ζ units starting from the last number λ_r of λ, let

$$\lambda^{max} = \begin{cases} (\lambda_1, \dots, \lambda_s, \lambda'_{s+1}) & if\ \lambda'_{s+1} > 0, \\ (\lambda_1, \dots, \lambda_s) & if\ \lambda'_{s+1} = 0. \end{cases}$$

here $\lambda'_{s+1} + \zeta = \lambda_{s+1} + \dots + \lambda_r$. Then λ^{max} is a partition of m, and for each *cip* $\mu = (\mu_1, \dots, \mu_k)$ of m, since $\lambda_1 \geq \mu_1, \dots, \lambda_k \geq \mu_k$ and $\sum_{i=1}^k \mu_i = \sum_{i=1}^s \lambda_i + \lambda'_{s+1}$, therefore μ is majorized by λ^{max}. Let the conjugate partition of λ be λ^*, we construct $(\lambda^*)^{max}$ using the same way as we used in the construction of λ^{max}, then $(\lambda^*)^{max}$ majorizes μ^* since $\mu^* \in \Omega_{\lambda^*}^m$ by corollary 2.6. Let the conjugate of $(\lambda^*)^{max}$ be λ_{min}, then $\lambda_{min} \prec \mu$ (see [3] p.9). The uniqueness follows from the definition of majorization.

Corollary 2.16 *Between the two sets* Ω_λ^m *and* $\Omega_{\lambda^*}^m$, *we have*

1. $(\lambda^{max})^* = (\lambda^*)_{min}$,
2. $(\lambda^*)^{max} = (\lambda_{min})^*$.

Corollary 2.17 *We have*

1. *If* $\mu \in \Omega_\lambda^m$, *then* $\lambda_{min} \prec \mu \prec \lambda^{max}$. *The inverse is not true by taking* $\lambda = (6, 2, 1), m = 6, \mu = (3, 3)$ *for example.*
2. *If* $\mu \in \Omega_\lambda^m \bigcap \Omega_{\lambda^*}^m$, *then* $\lambda_{min} \prec \mu, \mu^* \prec \lambda^{max}$. *The inverse is not true by taking* $\lambda = (6, 3, 2, 2, 1), m = 12, \mu = (4, 3, 3, 2)$ *for example.*

Example 2.18 *Let* $\lambda = (3, 2, 2, 1, 1)$ *be a partition of* $9, m = 7, \zeta = 2$, *using Young-Ferrars graph, we can readily verify the following procedures.*

```
        +                                                   +
λ       + + +          ⇒ removing ⊕   λ^max = (3,2,2)       + + +
        + + + ⊕ ⊕                                           + + +
⇓
        +                                                   +
        +                                                   +
λ*      + ⊕            ⇒ removing ⊕   (λ*)^max = (5,2)      +
        + +                                                 + +
        + + ⊕                                               + +
                                      ⇓ conjugate
                                                            + +
                                      λ_min                 + + + + +
```

[1]This maximality (minimality) is not going to be used in subsequent sections.

Moreover we have

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              (3,2,2)
                 |
             (3,2,1,1)
              /     \
(3,1,1,1,1)             (2,2,2,1)
              \     /
            (2,2,1,1,1)
```

3 $|\Omega(\alpha, m, n)|$

Lemma 3.1 *For $\alpha \in n^m$, the number of all the possible size m sequences chosen from $\alpha(1)\ldots\alpha(m)$ is $\frac{m!}{\alpha_1!\ldots\alpha_s!}$, here $M_\alpha = (\alpha_1, \ldots, \alpha_s)$.*

Proof. Consider the α orbit O_α, the number of all the possible size m sequences chosen from $\alpha(1)\ldots\alpha(m)$ is $|O_\alpha|$, and

$$|O_\alpha| = [S_m : (S_m)_\alpha]$$

here $(S_m)_\alpha = \{\sigma | \sigma \in S_m, \sigma.\alpha = \alpha\} \cong S_{\alpha_1} \times \ldots \times S_{\alpha_s}$, therefore

$$|O_\alpha| = \frac{m!}{\alpha_1!\ldots\alpha_s!}.$$

Lemma 3.2 *Given an $\alpha \in n^m, m \le n$, for any $\sigma \in S_n$*

$$\Omega(\alpha, m, n) = \Omega(\sigma.\alpha, m, n).$$

Proof. It follows from $Im\alpha = Im\sigma.\alpha$ and $m_t(\alpha) = m_t(\sigma.\alpha)$ for $t = 1, \ldots, n$.

Corollary 3.3 *Let $\mathbb{A}_n = a_1^{n_1}\ldots a_k^{n_k}, m \le n$. If*

$$\alpha = (\underbrace{a_1, \ldots, a_1}_{n_1}, \ldots, \underbrace{a_k, \ldots, a_k}_{n_k})$$

then $\Omega(\mathbb{A}_n, m, n) = \Omega(\sigma.\alpha, m, n)$ for all $\sigma \in S_n$.

Now we give an enumeration formula for problem 1.1.

Theorem 3.4 *Let $\alpha \in n^m, m \le n$ and $M_\alpha = (\alpha_1, \ldots, \alpha_k)$. Then*

$$|\Omega(\alpha, m, n)| = \sum_{\mu \in \Omega^m_{M_\alpha}} \prod_{j=1}^{\ell(\mu)} \left\{ \sum_{p=1}^{\ell(M_\alpha)} sgn^+(\alpha_p - \mu_j) - j + 1 \right\} \frac{m!}{\mu_1!\ldots\mu_{\ell(\mu)}!},$$

where $sgn^+(x) = 1$ if $x \ge 0$, 0 if $x < 0$.

Proof. Let $\mu = (m_1, \dots, m_s) \in \Omega_\lambda^m$ be a cip of m from $\lambda = (\alpha_1, \dots, \alpha_k)$ and $l_j^\mu = |\{p|\alpha_p \geq m_j, p = 1, \dots, k\}|$, then $l_1^\mu \leq l_2^\mu \dots \leq l_s^\mu$ and $l_j^\mu = \sum_{p=1}^k sgn^+(\alpha_p - m_j)$ for $j = 1, \dots, s$. For a typical pattern $Y_1^{m_1} \dots Y_s^{m_s}$ where $Y_1, \dots, Y_s$ are chosen from $Im\alpha = \{X_1, \dots, X_k\}$, in order to keep it, the number of total choices for $Y_1, \dots, Y_s$ is $\prod_{j=1}^s \{l_j^\mu - j + 1\}$. By lemma 3.1, for $\mu = (m_1, \dots, m_s) \in \Omega_\lambda^m$, the number of total size m sequences which keep the pattern like $Y_1^{m_1} \dots Y_s^{m_s}$ for $Y_1, \dots, Y_s$ chosen from $X_1, \dots, X_k$ is

$$\prod_{j=1}^{s} \left\{ \sum_{p=1}^{k} sgn^+(\alpha_p - m_j) - j + 1 \right\} \frac{m!}{m_1! \dots m_s!}$$

Since different cip produces different set of size m sequences, therefore the number of total size m sequences chosen from $Im\alpha$ such that the multiplicity of each digit in a size m sequence is less than or equal to the multiplicity of this digit in the given sequence $\alpha(1) \dots \alpha(n)$ is

$$\sum_{\mu=(m_1,\dots,m_s)\in\Omega_\lambda^m} \prod_{j=1}^{s} \left\{ \sum_{p=1}^{k} sgn^+(\alpha_p - m_j) - j + 1 \right\} \frac{m!}{m_1! \dots m_s!}$$

Corollary 3.5 *Let $\alpha \in n^n, m \leq n$ and $M_\alpha = (\alpha_1, \dots, \alpha_k)$, then*

$$|\Omega_1(\alpha, m, n)| = \sum_{\mu\in\Omega_{M_\alpha}^m} \prod_{j=1}^{\ell(\mu)} \left\{ \sum_{p=1}^{\ell(M_\alpha)} sgn^+(\alpha_p - \mu_j) - j + 1 \right\} \frac{m!}{\mu_1! \dots \mu_{\ell(\mu)}!} |O_\mu|,$$

where $sgn^+(x) = 1$ if $x \geq 0$, 0 if $x < 0$, O_μ is the μ-orbit under $S_{\ell(\mu)}$.

4 Cips of Size p

Now we proceed to evaluate $|\Omega_\lambda^m(p)|$, where $\lambda = (n_1, \dots, n_r) \vdash n, m \leq n, p$ is an integer. Let

$$S_\lambda(m) = \begin{cases} t & if\ \sum_{i=1}^t n_i = m, \\ t+1 & if\ \sum_{i=1}^t n_i < m < \sum_{i=1}^{t+1} n_i, 1 \leq t \leq r, \\ o & if\ m = 0. \end{cases} \tag{14}$$

and

$$S^\lambda(m) = min(m, r). \tag{15}$$

Lemma 4.1 *We have*

1. $S_\lambda(m) \leq S^\lambda(m)$,
2. $\Omega_\lambda^m(p) = \phi$ *if* $p > S^\lambda(m)$ *or* $p < S_\lambda(m)$,

3. $|\Omega_\lambda| = \sum_{m=0}^{n} \sum_{p=0}^{m} |\Omega_\lambda^m(p)| = \sum_{m=0}^{n} \sum_{p=S_\lambda(m)}^{S^\lambda(m)} |\Omega_\lambda^m(p)|$.

Lemma 4.1 3) states that the following two-variable generating function

$$\begin{aligned}\sum_{m=0}^{\infty} \sum_{p=0}^{\infty} |\Omega_\lambda^m(p)|\, z^p q^m &= \sum_{m=0}^{n} \sum_{p=S_\lambda(m)}^{S^\lambda(m)} |\Omega_\lambda^m(p)|\, z^p q^m \\ &= \sum_{m=0}^{n} \left\{ \sum_{\mu=(m_1,\dots,m_s)\in\Omega_\lambda^m} z^s q^m \right\} \\ &= \sum_{\mu=(m_1,\dots,m_s)\in\Omega_\lambda^m} z^s q^{\sum_{i=1}^{s} m_i} \quad ([1]\ p.16)\end{aligned}$$

Now let $H = \{h_1, \dots, h_k, \dots\}$ be a set of non-negative integers, we consider the following formal calculations

$$\begin{aligned}\prod_{h\in H}(1 - zq^h)^{-1} &= \{1 + zq^{h_1} + z^2q^{2h_1} + \dots + z^{i_1}q^{i_1h_1} + \dots\} * \\ &\quad \{1 + zq^{h_2} + z^2q^{2h_2} + \dots + z^{i_2}q^{i_2h_2} + \dots\} * \\ &\quad \dots \\ &\quad \{1 + zq^{h_k} + z^2q^{2h_k} + \dots + z^{i_k}q^{i_kh_k} + \dots\} * \dots \\ &= \sum_{i_1\ge 0} \cdots \sum_{i_k \ge 0} \cdots z^{i_1+\dots+i_k+\dots} q^{i_1h_1+\dots+i_kh_k+\dots} \\ &= \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \rho(H,p,m) z^p q^m,\end{aligned}$$

here $\rho(H,p,m)$ is the number of total partitions of m with components in H and of size p. We choose a special finite set $H = \{1, 2, \dots, n_1\}$, where n_1 is from $\lambda = (n_1, n_2, \dots, n_r)$, which is a partition of n. Let $d_i^H = |\{p | n_p \ge i\}|\, i = 1, \dots, n_1$, i.e., $\lambda^* = (d_1^H, \dots, d_{n_1}^H)$ is a partition of n conjugate to λ. Using lemma 4.1, let's look at the summation

$$\sum_{0\le i_1 \le d_1^H} \cdots \sum_{0 \le i_{n_1} \le d_{n_1}^H} z^{i_1+\dots+i_{n_1}} q^{i_1 1+\dots+i_{n_1} n_1}$$

$$= \sum_{m=0}^{n} \left\{ \sum_{\substack{0\le i_k \le d_k^H, k=1,\dots,n_1 \\ i_1 1+\dots+i_{n_1}n_1 = m}} z^{\sum_{k=1}^{n_1} i_k} \right\} q^m + \text{ other terms}$$

$$= \sum_{m=0}^{n} \left\{ \sum_{\substack{0\le i_k \le d_k^H, k=1,\dots,n_1 \\ i_1 1+\dots+i_{n_1}n_1 = m \\ S_\lambda(m) \le i_1+\dots+i_{n_1} \le S^\lambda(m)}} z^{\sum_{k=1}^{n_1} i_k} + \text{ other terms} \right\} q^m + other terms$$

$$= \sum_{m=0}^{n} \left\{ \sum_{S_\lambda(m) \le \zeta \le S^\lambda(m)} \sum_{\substack{0 \le i_k \le d_k^H, k=1,\dots,n_1 \\ i_1 1+\dots+i_{n_1} n_1 = m \\ \zeta = i_1+\dots+i_{n_1}}} z^\zeta q^m \right\} + \textit{ other terms.} \qquad (16)$$

Now we take into account of the multiplicities of the components in a partition λ of n, assume that

$$\lambda = ((n_1')^{r_1} \dots (n_k')^{r_k})$$

where $r_1, \dots, r_k$ represent the multiplicities of $n_1', \dots, n_k'$ respectively, and $n_1 = n_1' > n_2' > \dots > n_k' \ge 1$. Now the conjugate partition is

$$\lambda^* = ((r_1 + \dots + r_k)^{n_k'} (r_1 + \dots + r_{k-1})^{n_{k-1}' - n_k'} \dots (r_1)^{n_1 - n_2'}).$$

We introduce

Condition (*) : A set of non-negative integers $I_1, I_2, \dots, I_{n_1}$ satisfy condition (*) if and only if

$$\sum_{j=1+n_{i+1}'}^{n_i'} I_j \le r_1 + \dots + r_i, i = 1, \dots k.$$

Assuming $n_{k+1}' = 0$.

Let's go on a finer summation of equation (16)

$$\sum_{S_\lambda(m) \le \zeta \le S^\lambda(m)} \sum_{\substack{0 \le i_k \le d_k^H, k=1,\dots,n_1 \\ i_1 1+\dots+i_{n_1} n_1 = m \\ \zeta = i_1+\dots+i_{n_1}}} z^\zeta q^m$$

$$= \sum_{S_\lambda(m) \le \zeta \le S^\lambda(m)} \left\{ \sum_{\substack{i_1,\dots,i_{n_1} \text{ satisfy condition } (*) \\ \zeta = i_1+\dots+i_{n_1}, m = i_1 1+\dots+i_{n_1} n_1}} z^\zeta q^m \right\} + \textit{other terms.}$$

$$= |\Omega_\lambda^m(p)| \, z^p q^m + \textit{other terms.}$$

which give us the following

Theorem 4.2 *Let $\lambda \vdash n$ be a partition of n, $m \leq n$, then the number, $|\Omega_\lambda^m(p)|$, of size p cips $\Omega_\lambda^m(p)$ equals*

$$\sum_{\substack{i_{1+(\lambda\downarrow)_{k+1}}+\cdots+i_{(\lambda\downarrow)_k}\leq(\lambda^*\uparrow)_k, k=1,\ldots,\ell(\lambda)\\ p=i_1+\ldots+i_{n_1}, m=i_1 1+\ldots+i_{n_1}n_1}} 1$$

$$=\left|\left\{(i_1,\ldots,i_{n_1}) \mid i_{1+(\lambda\downarrow)_{k+1}}+\cdots+i_{(\lambda\downarrow)_k}\leq(\lambda^*\uparrow)_k,\right.\right.$$
$$\left.\left. k=1,\ldots,\ell(\lambda), p=\sum_{k=1}^{\lambda_1} i_k, m=\sum_{k=1}^{\lambda_1} k i_k\right\}\right|.$$

Proof. If we specify $\lambda = (n_1^{r_1}\ldots n_k^{r_k})$, then by the above arguments, $|\Omega_\lambda^m(p)|$ equals

$$\sum_{\substack{i_1,\ldots,i_{n_1} \text{ satisfy condition } (*)\\ p=i_1+\ldots+i_{n_1}, m=i_1 1+\ldots+i_{n_1}n_1}} 1 .$$

Now it is easy to check out that condition (*) is equivalent to

$$i_{1+(\lambda\downarrow)_{k+1}}+\cdots+i_{(\lambda\downarrow)_k}\leq(\lambda^*\uparrow)_k, k=1,\ldots,\ell(\lambda)$$

where we let $(\lambda\downarrow)_{k+1}=0$ if $k=\ell(\lambda\downarrow)$.

Corollary 4.3 *Given a partition $\lambda^* = (d_1^{s_1}\ldots d_k^{s_k})$ of n, $d_1 > d_2 > \ldots > d_k \geq 1$, the following integer programming*

$$Z = x_1 + \ldots + x_{s_1+\ldots+s_k}$$

subject to

$$\begin{cases} x_j \geq 0, & j=1,\ldots,s_1+\ldots+s_k,\\ \sum_{\xi=1+s_1+\ldots+s_{k-i}}^{s_1+\ldots+s_{k-i+1}} x_\xi \geq (\leq) d_{k-i+1}, & i=1,\ldots,k, s_o=0,\\ x_1+2x_2+\ldots+(s_1+\ldots+s_k)x_{s_1+\ldots+s_k} \leq (\geq) m, & \end{cases}$$

has a max (min) feasible solution if and only if $n \geq m$, moreover $z = S^\lambda(m)$ ($S_\lambda(m)$) is the max (min) feasible solution respectively, where $\lambda = (\lambda^*)^*$.

Proof. Consider the conjugate partition of $\lambda^* = (d_1^{s_1}\ldots d_k^{s_k})$,

$$\lambda = ((s_1+\ldots+s_k)^{d_k}(s_1+\ldots+s_{k-1})^{d_{k-1}-d_k}\ldots(s_1)^{d_1-d_2}).$$

Then apply theorem 4.2, the maximum size of cips of m from λ is $S^\lambda(m)$, and the minimum size of cips of m from λ is $S_\lambda(m)$.

Example 4.4 *Given a partition* $\lambda = (3,2,2,1,1)$ *of* 9, $m = 7, n_1 = 3, H = \{1,2,3\}$, $d_1^H = 5$, $d_2^H = 3$, $d_3^H = 1$, $S^\lambda(m) = 5, S_\lambda(m) = 3$. *By choosing from all the possible* 48 *different outcomes of* (i_1, i_2, i_3) *subject to* $0 \le i_1 \le 5, 0 \le i_2 \le 3$ *and* $0 \le i_3 \le 1$, *we have*

$$\{(i_1,i_2,i_3)|0 \le i_1 \le 5, 0 \le i_2 \le 3, 0 \le i_3 \le 1, 3 \le i_1+i_2+i_3 \le 5, i_1+2i_2+3i_3 = 7\}$$

$$= \{(0,2,1),(1,3,0),(2,1,1),(3,2,0),(4,0,1)\},$$

and

(i_1,i_2,i_3)	Ω_λ^m	*size* p	$\lvert\Omega_\lambda^m(p)\rvert$
(0,2,1)	(3,2,2)	3	1
(1,3,0)	(2,2,2,1)	4	
(2,1,1)	(3,2,1,1)	4	2
(3,2,0)	(2,2,1,1,1)	5	
(4,0,1)	(3,1,1,1,1)	5	2

Which are the same results as we got in example 2.18.

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