A Note on Powers and Proper Circular-Arc Graphs

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All graphs considered are finite. The *distance* $d_G(x,y)$ between two vertices x and y in some given graph $G = (V, E)$ is the length of a shortest x-y path, if there is at least one, and ∞ otherwise. For an integer $k \geq 1$, the kth power G^k of G has the same vertex set as G, and two vertices $x \neq y$ are adjacent in G^k whenever $d_G(x,y) \leq k$.

A *circular-arc graph* is the intersection graph of some family of arcs of some circle. *Proper* circular-arc graphs are intersection graphs of such families, where no arc contains another. *Proper interval graphs* are proper circular-arc graphs with some representation such that the union of the arcs does not cover the whole circle.

We call a class Γ of graphs *closed (under powers)* if every power of any member of Γ lies again in Γ. We call it *strongly closed (under powers)*, if for every integer $k \geq 1$ and every graph G, $G^k \in \Gamma$ implies $G^{k+1} \in \Gamma$.

In [5], RAYCHAUDHURI showed that the class of circular-arc graphs is closed, and asked whether it may be strongly closed. In [1] and [2], FLOTOW investigated some aspects of this question and showed that the class of *proper* circular-arc graphs is closed. In this note we prove that this class of *proper* circular-arc graphs is *strongly* closed (under powers).

We use a characterization of proper circular-arc graphs by means of certain orientations. A digraph $D = (V, A)$ is a *local tournament* if $xz \in A, yz \in A$ implies $xy \in A$ or $yx \in A$, and $zx, zy \in A$ also implies $xy \in A$ or $yx \in A$ [3]. A directed graph $D = (V, A)$ is an *orientation* of an undirected graph $G = (V, E)$ if both have the same vertex set, $xy \in E$ implies $xy \in A$ or $yx \in A$, and if A is antisymmetric.

In [6], SKRIEN showed that a connected graph is a proper circular-arc graph if and only if it can be oriented as a local tournament. Moreover he showed that a graph is a proper interval graph if and only if it has some *acyclic* orientation as a local tournament. The method to prove that

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proper circular-arc graphs are strongly closed gives also another proof of the strong closedness of the class of proper interval graphs [4]:

Theorem: Let $k \geq 1$ be an integer, and $G = (V, E)$ a graph.
a) If G^k is a proper circular-arc graph, then G^{k+1} is also a proper circular-arc graph.
b) [4] If G^k is a proper interval graph, then G^{k+1} is also a proper interval graph.

Proof: Note that G is connected if and only if G^k is connected. We show the results for connected graphs. If G^k is not connected, it must be a proper interval graph. But disjoint unions of proper interval graphs are again proper interval graphs, thus the result follows from the treatment of the connected case.

So let G^k be connected. According to SKRIEN's theorem, we can find some local tournament orientation (V, A) of the edges of G^k. We extend this orientation to some orientation $(V, A \cup A')$ of G^{k+1} as follows: For every pair x, y of vertices of distance $d_G(x, y) = k + 1$ in G, we fix some shortest path $P_{xy} : x = x_0, x_1, \ldots, x_k, x_{k+1} = y$. Note that except x_0 and x_{k+1}, all pairs of vertices of this path are adjacent in G^k. Thus, since (V, A) is a local tournament orientation of G^k:

(*) For every $0 < i < k+1$, $x_0x_i \in A$ if and only if $x_ix_{k+1} \in A$.

We distinguish four cases, depending on this path. In any case we assign some orientation to the edge $xy = x_0x_{k+1}$ of G^{k+1}:

($\alpha 1$) Case $x_0x_1 \in A$ and $x_{k+1}x_k \in A$. If $x_1x_k \in A$, then we choose $yx \in A'$, otherwise (if $x_kx_1 \in A$, note that x_1 and x_k are adjacent in G^k) we choose $xy \in A'$. By (*), $x_1x_{k+1}, x_kx_0 \in A$.

($\alpha 2$) Case $x_1x_0 \in A$ and $x_kx_{k+1} \in A$. Again $d_G(x_1, x_k) < k$, and we choose again $yx \in A'$ if $x_1x_k \in A$, and $xy \in A'$ otherwise. It follows that $x_0x_k, x_{k+1}x_1 \in A$.

($\beta 1$) If $x_0x_1, x_kx_{k+1} \in A$, then we choose $xy \in A'$. Then x_1x_{k+1}, $x_0x_k \in A$.

($\beta 2$) Finally, if $x_1x_0, x_{k+1}x_k \in A$, then we choose $yx \in A'$. Here we obtain $x_{k+1}x_1, x_kx_0 \in A$. Interchanging x and y, this case is equivalent to case (β_1), so we may subsume both cases under the common label (β).

Note that for $k = 1$, only cases (β) are possible.

The digraph $(V, A \cup A')$ is constructed as an orientation of G^{k+1}. To prove (a), it remains the somewhat tedious task to show that it is a local tournament.

(1) If $xz \in A'$ and $yz \in A$, let $x = x_0, x_1, \ldots, x_k, x_{k+1} = z$ be P_{xz}. We may assume $y \notin P_{xz}$, since otherwise $d_G(x,y) \leq k+1$ would be obvious. In cases (α1) or (β) holds $x_1 z \in A$, whence $d_G(x_1, y) \leq k$ by the local tournament property of (V, A). In case (α_2), $d_G(x_k, y) \leq k$, since $x_k z \in A$. Then either $x_k y \in A$ or $y x_k \in A$, but the first implies $d_G(x_1, y) \leq k$ (since $x_k x_1 \in A$), and the latter implies even $d_G(x,y) \leq k$ (since $x x_k \in A$). In any case we obtain $d_G(x,y) \leq k+1$.

In the same way, the case $zx \in A', zy \in A$ is treated. Let $z = z_0, z_1, \ldots,$ $z_k, z_{k+1} = x$ be P_{zx}. As above, it suffices to treat the case $y \notin P_{zx}$. In cases (α2) or (β) we have $z z_k \in A$, thus $d_G(z_k, y) \leq k$. In case (α1), $z z_1 \in A$ implies $d_G(z_1, y) \leq k$. If $y z_1 \in A$, then $d_G(z_k, y) \leq k$, since $z_k z_1 \in A$. If $z_1 y \in A$, then $d_G(x,y) \leq k$, since $z_1 x \in A$. Again $d_G(x,y) \leq k+1$ in every subcase.

(2) Now let $d_G(x,z) = k+1 = d_G(z,y)$. Let $x = x_0, x_1, \ldots, x_k, x_{k+1} = z$ and $y = y_0, y_1, \ldots, y_k, y_{k+1} = z$ be P_{xz} and P_{yz} respectively. In what follows, we want to derive $d_G(x,y) \leq k+1$ from either $xz, yz \in A'$ or $zx, zy \in A'$.

For $(i,j) \in \{0,1,k\} \times \{0,1,k\} \setminus \{(k,k)\}$, $x_i = y_j$ implies $d_G(x,y) \leq k+1$. But the case $x_k = y_k$ is also easy to treat: If then $z x_k \in A$, then $x_k x, x_k y \in A$ by (∗). Then $d_G(x,y) \leq k$. In the same way, $x_k z \in A$ implies $x x_k, y x_k \in A$ and $d_G(x,y) \leq k$ under the hypothesis $x_k = y_k$.

So assume in the following that all seven vertices $x, x_1, x_k, z, y_k, y_1, y$ are distinct. Two special cases can now be treated: The first is $x_1 z, y_1 z \in A$. Then $x_1 y_1 \in A$ or conversely $y_1 x_1 \in A$, since (V, A) is a local tournament. But $x x_1, y y_1 \in A$ by (∗), implying either $d_G(x_1, y) \leq k$ if $x_1 y_1 \in A$ or $d_G(x, y_1) \leq k$ if $y_1 x_1 \in A$. The second case is $z x_1 \in A$ and $z y_1 \in A$. Similiarly, $x_1 y_1 \in A$ or $y_1 x_1 \in A$, but now $x_1 x, y_1 y \in A$, whence $d_G(x, y_1) \leq k$ or $d_G(x_1, y) \leq k$.

Thus we may assume in what follows without loss of generality $x_1 z \in A$ and $z y_1 \in A$ (implying also $x x_1, y_1 y \in A$ by (∗)).

(2i) Let us first treat the case $xz, yz \in A'$. Then yz has been drawn under rule (α2). So $y y_k, y_k z, y_k y_1 \in A$. Thus $x_1 y_k \in A$ or reversely $y_k x_1 \in A$. In the first case, there follows $d_G(x_1, y) \leq k$ and we are done. In the second case, we use $x x_1 \in A$ to obtain either $x y_k \in A$ and $d_G(x,y) \leq k$, or $y_k x \in A$, implying $d_G(x, y_1) \leq k$. In any case we obtain $d_G(x,y) \leq k+1$.

(2ii) The other case is $zx, zy \in A'$. Here zx is an arc of type (α1), and so $z x_k, x_k x, x_1 x_k \in A$. Now either $x_k y_1 \in A$, which implies $d_G(x, y_1) \leq k$, or $y_1 x_k \in A$. In this latter case we obtain $d_G(x_k, y) \leq k$. Then $x_k y \in$

A implies $d_G(x, y) \leq k$, and $yx_k \in A$ implies $d_G(x_1, y) \leq k$, and again $d_G(x, y) \leq k + 1$ in every subcase.

To prove (b), it suffices to note that cases (α1) or (α2) may not occur for acyclic (V, A). As noted above, these cases cannot occur for $k = 1$, but for $k > 1$ there would result directed cycles. But the (β)-arcs are taken from the transitive closure of (V, A), so $(V, A \cup A')$ is acyclic if and only if (V, A) is. □

References

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