λ-Designs on $4p + 1$ points

Yury J. Ionin and Mohan S. Shrikhande*

Department of Mathematics
Central Michigan University
Mt. Pleasant, MI 48859
3aztpfs@cmuvm.csv.cmich.edu
3iamxmz@cmuvm.csv.cmich.edu

ABSTRACT. A λ-design on v points is a family of v subsets (blocks) of a v-set such that any two distinct blocks intersect in λ points and not all blocks have the same cardinality. Ryser's and Woodall's λ-design conjecture states that each λ-design can be obtained from a symmetric design by complementing with respect to a fixed block. In a recent paper we proved this conjecture for $v = p + 1$, $2p + 1$, $3p + 1$, where p is prime, and remarked that similar methods might work for $v = 4p + 1$. In the present paper we prove the conjecture for λ-designs having replication numbers r and r^* such that $(r-1, r^*-1) = 4$ and as a consequence the λ-design conjecture is proved for $v = 4p+1$, where p is prime.

1 Introduction

Let v and λ be fixed integers, $0 < \lambda < v$. A λ-design $\mathcal{B}$ on v points is a family of v subsets (blocks) of the set $[v] = \{1, 2, \ldots, v\}$ such that any two blocks meet in λ points, all blocks have cardinality greater than λ, and not all blocks have the same cardinality. The notion of λ-design was introduced independently by Ryser [7] and Woodall [10]. They proved that in any λ-design $\mathcal{B}$ on v points, there are two distinct (point) replication numbers r and r^* ($r > 1, r^* > 1$) such that $r + r^* = v + 1$. The only known examples of λ-designs are those obtained in the following way: Let $\mathcal{A}$ be the block set of a symmetric 2-$(v, k, k - \lambda)$ design with the point set

*Acknowledges hospitality of the Mathematics Departments of Michigan Technological University and Ohio State University during sabbatical leave and support of Central Michigan University FRCE Grant #42493.

$[v]$ and having $k \neq 2\lambda$. Let A be a fixed block in $\mathcal{A}$. Form the family $\mathcal{B} = \{A\} \cup \{A\Delta B \colon B \in \mathcal{A}, B \neq A\}$, where $A\Delta B$ is the usual symmetric difference of A and B. Then $\mathcal{B}$ is a λ-design on v points. Any λ-design obtained in this manner is called type-1.

The λ-design conjecture due to Ryser [7] and Woodall [10] states that every λ-design is type-1. This conjecture has been proved for $\lambda = 1$ (de Bruijn and Erdös [3]), $\lambda = 2$ (Ryser [7]), $\lambda = 3$ (Bridges and Kramer [2]), $\lambda = 4$ (Bridges [1]), $5 \leq \lambda \leq 9$ (Kramer [5], [6]), $\lambda = 10$ (Seress [8]), and for any prime λ (Singhi and S.S. Shrikhande [9]).

In a recent paper, Ionin and Shrikhande [4] investigated the truthfulness of the λ-design conjecture as a function of v rather than λ. Let $g = (r - 1, r^* - 1)$ be the greatest common divisor of $r - 1$ and $r^* - 1$. The main result of [4] is the assertion that the λ-design conjecture is true for $g = 1$, 2, and 3. As a consequence of this, it was shown in [4] that the conjecture is true for $v = p+1$, $2p+1$, and $3p+1$, where p is prime. It was remarked in [4] that the techniques developed in the proof of these results might work in some other cases, particularly $v = 4p + 1$, where p is prime.

The main results of the present paper are the following two theorems.

Theorem 2.1. *Let $\mathcal{B}$ be a λ-design with replication numbers r and r^*. If $(r - 1, r^* - 1) = 4$, then $\mathcal{B}$ is a type-1 design.*

Theorem 2.2. *Let p be a prime. Then any λ-design on $4p + 1$ points is type-1.*

2 Preliminaries

We begin with the following two useful results.

Theorem 1.1. (Ryser [7], Woodall [10]) *Let a λ-design $\mathcal{B}$ on v points have replication numbers r and r^*. Then*

$$\frac{1}{\lambda} + \sum_{A \in \mathcal{B}} \frac{1}{|A| - \lambda} = \frac{(v-1)^2}{(r-1)(r^*-1)}. \tag{1}$$

Theorem 1.2. (Woodall [11]) *A λ-design $\mathcal{B}$ on v points with replication numbers r and r^* is type-1 if and only if $r(r-1)/(v-1)$ or $r^*(r^*-1)/(v-1)$ is an integer.*

If $\mathcal{A}$ is the family of blocks of a symmetric $(v, k, k-\lambda)$-design, where $k \neq 2\lambda$, with a fixed block A, then the family $\mathcal{B} = \{A\} \cup \{A\Delta B \colon B \in \mathcal{A}, B \neq A\}$ is a λ-design. This is the only known construction of λ-designs. Any λ-design which can be obtained by this construction is said to be type-1. The main conjecture of Ryser [7] and Woodall [10] in the area of λ-designs is *the λ-design conjecture*:

every λ-design is a type-1 design.

The following theorem summarizes results of de Bruijn and Erdös, Ryser, Bridges, and Kramer ([1], [2], [3], [5], [6], [7]) which will be needed in the present paper.

Theorem 1.3. *Any λ-design with $\lambda \leq 6$ is a type-1 design.*

The following result from Ionin and Shrikhande [4, Theorems 4.1 and 5.1] will also be used in the sequel.

Theorem 1.4. *Let $\mathcal{B}$ be a λ-design on v points and with replication numbers r and r^*. Let $g = (r-1, r^*-1)$ be the greatest common divisor of $r-1$ and r^*-1. Then*

(i) *if $g = 1$, then $\lambda = 1$ and hence $\mathcal{B}$ is type-1;*

(ii) *if $g = 2$, then $\mathcal{B}$ is type-1.*

We now collect all relevant notions about λ-designs and results from Ionin and Shrikhande [4] which will be needed in the sequel. The set $[v]$ is partitioned into subsets E and E^* of points having replication numbers r and r^* respectively. Let $e = |E|$ and $e^* = |E^*|$, so $e + e^* = v$. For any block A, let $\tau_A = |A \cap E|$ and $\tau_A^* = |A \cap E^*|$ and hence $\tau_A + \tau_A^* = |A|$.

The following result is used often in [4].

Proposition 1.5. *For any block A of a λ-design $\mathcal{B}$ on v points, the relation below holds:*

$$(r-1)(|A| - 2\tau_A) = (v-1)(|A| - \tau_A - \lambda). \tag{2}$$

Proof: Let $\mathcal{B}$ be a λ-design on v points. Fix a block A in $\mathcal{B}$ and count in two ways pairs (i, B), where B is a block in $\mathcal{B}$, $B \neq A$, and $i \in A \cap B$. This yields the equality $\tau_A(r-1) + \tau_A^*(r^*-1) = (v-1)\lambda$, which can be transformed into (2) by routine manipulations.

Next, since $g = (r-1, r^*-1)$ and $(r-1) + (r^*-1) = v-1$, hence $g = (r-1, v-1) = (r^*-1, v-1)$. We let

$$q = (v-1)/g. \tag{3}$$

Since $((r-1)/g, q) = 1$, eq. (2) implies $|A| - 2\tau_A \equiv 0 \pmod{q}$ for any block A. We now define integers σ_A and s by

$$|A| - 2\tau_A = \sigma_A q \tag{4}$$

and

$$s = \sum_{A \in \mathcal{B}} \sigma_A. \tag{5}$$

Set $\sigma_A^* = -\sigma_A$. The following three equations are easily verified:

$$\tau_A = \lambda - \frac{r^* - 1}{g} \sigma_A. \tag{6}$$

$$\tau_A^* = \lambda - \frac{r - 1}{g} \sigma_A^*. \tag{7}$$

$$|A| = 2\lambda + \frac{r - r^*}{g} \sigma_A. \tag{8}$$

The next series of identities is easily verified:

$$\sum_{A \in \mathcal{B}} |A| = er + e^* r^*. \tag{9}$$

$$\sum_{A \in \mathcal{B}} \tau_A = er. \tag{10}$$

$$sq = gq(gq - e - r + 3) - (2e + r - 2). \tag{11}$$

Eq. (11) implies $2e + r - 2 \equiv 0 \pmod q$. We therefore define integers m and m^* by

$$2e + r - 2 = mq \tag{12}$$

and

$$2e^* + r^* - 2 = m^* q. \tag{13}$$

Then it follows that

$$m + m^* = 3g \tag{14}$$

and

$$s = g^2 q - g(e + r) + m^*. \tag{15}$$

Thus, to each λ-design $\mathcal{B}$ on v points we associate integer parameters r, e, g, q, m (and "dual" parameters r^*, e^*, m^*). When necessary, we will denote these parameters as $r(\mathcal{B})$, $e(\mathcal{B})$, etc. The following lemma from Ionin and Shrikhande [4, Lemma 3.5] establishes relations between these parameters.

Lemma 1.6. *If $v \neq 4\lambda - 1$, then*

$$r = [(2g - m)(gq + 2) - 2\lambda g]/(3g - 2m) \tag{16}$$

and

$$e = [\lambda g - (g - m)^2 q + g - m]/(3g - 2m). \tag{17}$$

The next result is folklore in the area of λ-designs. A proof can be found in Ionin and Shrikhande [4, Proposition 3.7].

Lemma 1.7. *Let $\mathcal{B}$ be a λ-design on v points and let A be a fixed block in $\mathcal{B}$. Define*

$$\mathcal{B}(A) = \{A\} \cup \{A \Delta B \colon B \in \mathcal{B}, B \neq A\}.$$

Then

(i) $\mathcal{B} = \mathcal{B}(A)(A)$;

(ii) *if $A = E$ or $A = E^*$, then $\mathcal{B}(A)$ is a symmetric $(v, |A|, |A| - \lambda)$-design;*

(iii) *if $A \neq E$ and $A \neq E^*$, then $\mathcal{B}(A)$ is a $\lambda(A)$-design on v points, where $\lambda(A) = |A| - \lambda$, and $\mathcal{B}(A)$ has the same replication numbers r and r^* as $\mathcal{B}$;*

(iv) *if $A \neq E$ and $A \neq E^*$, then $e(\mathcal{B}(A)) = e + q\sigma_A$ and $m(\mathcal{B}(A)) = m + 2\sigma_A$;*

(v) *if $A \neq E$, $A \neq E^*$, and $\mathcal{B}$ is type-1, then $\mathcal{B}(A)$ is type-1 too.*

We will conclude this section with a simple inequality for the integers σ_A.

Lemma 1.8. *If $\lambda \neq 1$, then for any block A, $|\sigma_A| \leq g - 1$.*

Proof: Since $-|A| \leq |A| - 2\tau_A \leq |A|$ and $|A| \leq v - 1 = gq$, (4) implies that $|\sigma_A| \leq g$. If $|\sigma_A| = g$ for a block A, then $|A| = v - 1$ and $\tau_A = 0$ or $|A|$, so $A = E^*$ or $A = E$. Therefore, Lemma 1.7 implies that $\mathcal{B}(A)$ is a symmetric $(v, v-1, v-1-\lambda)$-design and then the basic symmetric design equation $(v-1)(v-1-\lambda) = (v-1)(v-2)$ implies $\lambda = 1$.

3 Main Results

Theorem 2.1. *Let $\mathcal{B}$ be a λ-design with replication numbers r and r^*. If $(r-1, r^*-1) = 4$, then $\mathcal{B}$ is a type-1 design.*

Proof: Let a λ-design $\mathcal{B}$ on v points have replication numbers r and r^* and let $g = (r-1, r^*-1) = 4$. By (3), $v = 4q+1$. Then eq. (1) reads

$$\frac{1}{\lambda} + \sum_{A \in \mathcal{B}} \frac{1}{|A| - \lambda} = \frac{16q^2}{(r-1)(r^*-1)}. \tag{18}$$

If $\lambda = 1$, then $\mathcal{B}$ is type-1, so from now on we assume that $\lambda \neq 1$. Then Lemma 1.8 implies that for any block A, $|A| = 2\tau_A + \sigma_A q$, where $|\sigma_A| \leq 3$. For $-3 \leq i \leq 3$, we denote by a_i the number of blocks A for which $\sigma_A = i$. Using (8), we rewrite (18) as follows:

$$\frac{1}{\lambda} + \sum_{i=-3}^{3} \frac{4a_i}{4\lambda + (r - r^*)i} = \frac{16q^2}{(r-1)(r^*-1)}. \tag{19}$$

Since $r - 1 \equiv 0 \pmod 4$, r is odd, and then (12) implies that q and m are odd. Without loss of generality, we assume that $m \leq m^*$, and then (14) implies that $m \in \{1, 3, 5\}$.

Case 1. $m = 1$.

In this case, Lemma 1.6 implies that $r^* = (6q + 4\lambda + 3)/5$ and $e = (4\lambda - 9q + 3)/10$. Eq. (6) implies that $\tau_A = \lambda - \sigma_A(3q + 2\lambda - 1)/10$, for any block A. Since $\tau_A \leq e$, we obtain that $\sigma_A \geq 3$. By Lemma 1.8, $\sigma_A = 3$ for any block A, and therefore all blocks have the same cardinality. Since this contradicts the definition of a λ-design, the case $m = 1$ is impossible.

Case 2. $m = 3$.

In this case, Lemma 1.6 implies that $r = (10q - 4\lambda + 5)/3$, $r^* = (2q + 4\lambda + 1)/3$, $e = (4\lambda - q + 1)/6$. Eq. (6) implies that $\tau_A = \lambda - \sigma_A(q + 2\lambda - 1)/6$ for any block A. Inequalities $0 \leq \tau_A \leq e$ imply that $1 \leq \sigma_A \leq 2$. Therefore, $a_i = 0$ for all i except 1 and 2 and $a_1 + a_2 = 4q + 1$. Using (5) and (15), we obtain that $a_1 + 2a_2 = (10q + 8\lambda + 5)/3$. Therefore, $a_1 = (14q - 8\lambda + 1)/3$, $a_2 = (8\lambda - 2q + 2)/3$. Substituting the values of the a_i's in (19), we obtain by routine manipulations the following equation:

$$e(2\lambda - 2q - 1)^2[4\lambda^2 - 3\lambda + 2 - q(7\lambda - 10)] = 0.$$

Obviously, $2\lambda - 2q - 1 \neq 0$. Therefore, $q = (4\lambda^2 - 3\lambda + 2)/(7\lambda - 10)$ which can be transformed into

$$(7q - 4\lambda - 2)(7\lambda - 10) = 5\lambda + 34. \tag{20}$$

If $7\lambda - 10 = 5\lambda + 34$, then $\lambda = 22$ and $q = 13$. In this case, $r = 47/3$, a contradiction. Therefore, $2(7\lambda - 10) \leq 5\lambda + 34$, which implies $\lambda \leq 6$, and we refer to Theorem 1.3 to conclude that $\mathcal{B}$ is type-1.

Case 3. $m = 5$.

In this case, Lemma 1.6 implies that $r = 6q - 4\lambda + 3$, $r^* = 4\lambda - 2q - 1$, $e = (4\lambda - q - 1)/2$. Eq. (6) implies $\tau_A = \lambda - \sigma_A(2\lambda - q - 1)/2$ for any block A. Using $\tau_A \leq e$, we obtain $\sigma_A \geq -1$.

Suppose $\mathcal{B}$ contains a block A with $\sigma_A \geq 2$. Consider the family $\mathcal{B}(A)$. If it is a symmetric design, then $\mathcal{B}$ is type-1. Otherwise, by Lemma 1.7, $m(\mathcal{B}(A)) = 5 + 2\sigma_A \geq 9$, so $m^*(\mathcal{B}(A)) = 12 - m(\mathcal{B}(A)) \leq 3$. Cases 1 and 2 imply that $\mathcal{B}(A)$ is type-1, so $\mathcal{B}$ is type-1 too.

Suppose $\mathcal{B}$ contains a block A with $\sigma_A = -1$. Then $m(\mathcal{B}(A)) = 3$ and again $\mathcal{B}$ is type-1. Thus we can assume that $a_i = 0$ for all i except 0 and 1. Then, by (5), $s = a_1$. Using (15), we obtain that $a_1 = 8\lambda - 6q - 3$ and then $a_0 = 4q + 1 - a_1 = 10q - 8\lambda + 4$. By routine manipulations, (19) can be transformed into the following equation:

$$(2\lambda - 2q - 1)^2[15q^2 - 2(16\lambda - 10)q + (16\lambda^2 - 16\lambda + 5)] = 0.$$

Since $2\lambda - 2q - 1 \neq 0$, the discriminant of the quadratic (in q) equation

$$15q^2 - 2(16\lambda - 10)q + (16\lambda^2 - 16\lambda + 5) = 0 \qquad (21)$$

must be a perfect square. This condition yields the equation $(4\lambda - 10)^2 - x^2 = 75$ with a positive integer x. The solutions of this equation are $\lambda = x = 5$; $\lambda = 6, x = 11$; $\lambda = 12, x = 37$. The last solution does not yield an integer q in (21), so $\lambda \leq 6$, and we again refer to Theorem 1.3 to conclude that $\mathcal{B}$ is type-1. This completes the proof of Theorem 2.1.

Theorem 2.2. *If p is prime, then any λ-design on $4p+1$ points is type-1.*

Proof: Let $\mathcal{B}$ be a λ-design on $4p + 1$ points, where p is prime. Let r and r^* be replication numbers of $\mathcal{B}$ and let $g = (r - 1, r^* - 1)$. If $p = 2$, then $v - 1 = 8$, so $g \in \{1, 2, 4\}$ and we apply Theorem 1.4 and Theorem 2.1 to conclude that $\mathcal{B}$ is type-1. Suppose that p is an odd prime. If $g \equiv 0 \pmod{p}$, then $r \equiv r^* \equiv 1 \pmod{p}$. Assuming $r > r^*$, we obtain $r = 3p+1$, $r^* = p + 1$. If $p \equiv 1 \pmod{4}$, then $r(r-1)/(v-1)$ is an integer; if $p \equiv 3 \pmod{4}$, then $r^*(r^* - 1)/(v - 1)$ is an integer. In either case, Theorem 1.2 implies that $\mathcal{B}$ is type-1. If p does not divide g, then $g \in \{1, 2, 4\}$, and we again apply Theorem 1.4 and Theorem 2.1 to conclude that $\mathcal{B}$ is type-1.

References

[1] W.G. Bridges, Some results on λ-designs, *J. Combin. Theory* **8** (1970), 350–360.

[2] W.G. Bridges and E.S. Kramer, The determination of all λ-designs with $\lambda = 3$, *J. Combin. Theory* **8** (1970), 343–349.

[3] N.G. De Bruijn and P. Erdös, On a combinatorial problem, *Indag. Math.* **10** (1948), 421–423.

[4] Y.J. Ionin and M.S. Shrikhande, On the λ-design conjecture, *J. Combin. Theory Ser A*, (in press).

[5] E.S. Kramer, "On λ-designs", Ph.D. dissertation, Univ. of Michigan, 1969.

[6] E.S. Kramer, On λ-designs, *J. Combin. Theory Ser. A* **16** (1974), 57–75.

[7] H.J. Ryser, An extension of a theorem of de Bruijn and Erdös on combinatorial designs, *J. Algebra* **10** (1968), 246–261.

[8] A. Seress, On λ-designs with $\lambda = 2p$, in *Coding Theory and Design Theory*, Part II, *Design Theory*, The *IMA Volumes in Mathematics and its Applications* **21**, edited by D.K. Ray-Chauduri, pp. 290–303, Springer-Verlag, New York, 1990.

[9] N.M. Singhi and S.S. Shrikhande, On the λ-design conjecture, *Utilitas Math.* **9** (1976), 301–318.

[10] D.R. Woodall, Square λ-linked designs, *Proc. London Math. Soc.* **20** (1970), 669–687.

[11] D.R. Woodall, Square λ-linked designs: A survey, in *Combinatorial Mathematics and Its Applications*, pp. 349–355, Academic Press, New York/London, 1971.