The higher-order edge toughness of a graph and truncated uniformly dense matroids

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ABSTRACT. In [Discrete Math. 111 (1993), 113–123], the cth-order edge toughness of a graph G is defined as

$$\tau_c(G) = \min_{X \subseteq E(G), \& \omega(G-X)>c} \left\{ \frac{|X|}{\omega(G-X)-c} \right\},$$

for any $1 \leq c \leq |V(G)| - 1$.

It is proved that $\tau_c(G) \geq k$ if and only if G has k edge-disjoint spanning forests with exactly c components and that for a given graph G with $s = |E(G)|/(|V(G)| - c)$ and $1 \leq c \leq |E(G)|$, $\tau_c(G) = s$ if and only if $|E(H)| \leq s(|V(H)| - 1)$ for any subgraph H of G. In this note, we shall present short proofs of the abovementioned theorems and shall indicate that these results can be extended to matroids.

We use the notation in [2] for graphs, and [1] for matroids. Please refer to [2] and [1] for the literature. In [2], Chen *et al* proved these results:

Theorem 1. (Chen, Koh and Peng [2]) *A graph G has k edge-disjoint spanning c-forests if and only if $\tau_c(G) \geq k$, where $c = 1, 2, \ldots, |V(G)| - 1$ and k is a nonnegative integer.*

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Theorem 2. (Chen, Koh and Peng [2]) *Let G be a graph with p vertices and q edges, and let $s = q/(p-c)$, where c is an integer satisfying $1 \le c \le p-1$. Then $\tau_c(G) = s$ if and only if $|E(H)| \le s(|V(H)|-1)$ for every subgraph of G.*

In this note, we shall present short proofs of Theorems 1 and 2, and shall indicate that these results can be extended to matroids.

For a matroid M, $M|X$ denotes the loopless contraction, and ρ denotes the rank func-tion. The density of a subset X with $\rho(X) > 0$ is $g(X) = \frac{|X|}{\rho(X)}$. In [1], the *fractional arboricity* and the *strength* of M are respectively defined as:

$$\gamma(M) = \max_{X \subseteq S, \rho(X)>0} g(X)$$

and

$$\eta(M) = \min_{X \subset S, \rho(X)<\rho(S)} g(M|X).$$

Note that the strength of M can be alternatively expressed as:

$$\eta(M) = \min_{X \subset S, \rho(X)<\rho(S)} \frac{|S-X|}{\rho(S)-\rho(X)}. \tag{1}$$

A matroid M on S is *uniformly dense* if $\eta(M) = \gamma(M)$. For a graph G, $\eta(G)$ and $\gamma(G)$ are defined as $\eta(M(G))$ and $\gamma(M(G))$, respectively, where $M(G)$ is the cycle matroid of G. By a *family* we mean a multiset in which an element may occur more than once.

Theorem 3. (Theorem 4 and Theorem 6 of [1]) *Let M be a loopless matroid on a set S and let h and k be two positive integers. Each of the following holds.*

(i) *$\eta(M) \ge h/k$ if and only if M has a family $\mathcal{F}$ of h bases such that every element in S lies in at least k bases in $\mathcal{F}$.*

(ii) *$\eta(M)\rho(S) = |S|$ if and only if $\gamma(M)\rho(S) = |S|$.*

Note that the *truncation* of M at k (see [3], Chapter 4), denoted by M_k, has rank

$$\rho_k(X) = \min\{k, \rho(X)\} \text{ for any } X \subseteq S.$$

In Lemmas 4 and 5 below, let G be a graph with p vertices and without isolated vertices, let $M = M(G)$ be the cycle matroid of G, and let M_{p-c} denote the truncation of M at $p-c$, where c is an integer with $1 \le c \le p-1$. For an edge subset $X \subseteq E(G)$, $G(X)$ denotes the spanning subgraph of G with edge set X.

Lemma 4. *Let B be a subset of $E(G)$. The following are equivalent:*

(a) *B is a basis in M_{p-c}.*

(b) *$G(B)$ is a forest with exactly $p-c$ edges.*

(c) *$G(B)$ is a c-forest.*

Proof: Note that the rank of M_{p-c} is $p-c$ and that an edge subset $X \subseteq E(G)$ is independent in M if and only if $G(X)$ is a forest. These give (a) $\iff$ (b). Since $G(B)$ is a forest with p vertices and with $p-c$ edges if and only if $G(B)$ is a forest with p vertices and with c components, (b) $\iff$ (c). □

Lemma 5. $\eta(M_{p-c}) = \tau_c(G)$.

Proof: Let ρ_{p-c} denote the rank function of M_{p-c}. Let $X \subsetneq E(G)$ be such that $\rho_{p-c}(X) < \rho_{p-c}(E(G))$. Then we have

$$\rho_{p-c}(E(G)) = p - c \text{ and } \rho_{p-c}(X) = \rho(X) = p - \omega(G(X)). \qquad (2)$$

Note that if $Y = E(G) - X$ for the subset X in (2), then $G(X) = G - Y$. Thus by (1) and (2), we have

$$\begin{aligned}
\eta(M_{p-c}) &= \min_{X \subset E(G), \rho_{p-c}(X) < \rho_{p-c}(E(G))} \frac{|E(G) - X|}{\rho_{p-c}(E(G)) - \rho_{p-c}(X)} \\
&= \min_{X \subset E(G), \rho_{p-c}(X) < p-c} \frac{|E(G) - X|}{\omega(G(X)) - c} \\
&= \min_{Y \subset E(G), \omega(G-Y) > c} \frac{|Y|}{\omega(G-Y) - c} = \tau_c(G).
\end{aligned}$$

□

Proof of Theorem 1: Let $k \geq 1$ be an integer, let G be a graph with p vertices and let c be an integer such that $c \in \{1, 2, \ldots, |V(G)| - 1\}$. Thus G has k edge-disjoint spanning c-forests if and only if M_{p-c} has k disjoint bases (by Lemma 4), if and only if $\eta(M_{p-c}) \geq k$ (by Theorem 3(i)), if and only if $\tau_c(G) \geq k$ (by Lemma 5). □

Theorem 2 can have the following variation.

Theorem 6. *Let G be a graph with p vertices and q edges, and let $s = q/(p-c)$, where c is an integer satisfying $1 \leq c \leq p-1$. The following are equivalent:*

(i) $\tau_c(G) = s$.

(ii) *$|E(H)| \leq s(|V(H)| - c)$ for every subgraph H of G.*

(iii) *$|E(H)| \leq s(|V(H)| - 1)$ for every subgraph H of G.*

Proof: (i) of Theorem 6 $\Longleftrightarrow \eta(M_{p-c}) = s$ (by Lemma 5) $\Longleftrightarrow \gamma(M_{p-c}) = s$ (by Theorem 3 (ii)) $\Longleftrightarrow$ (ii) of Theorem 6 (by the definition of γ).

Clearly (ii) of Theorem 6 implies (iii) of Theorem 6. Chen *et al* in [2] have a simple proof for (iii) $\Longrightarrow$ (i). We quote their proof here for the sake of completeness.

Let $X \subseteq E(G)$ be such that $G-X$ has components $H_1, H_2, \ldots, H_t$ where $t > c$. Apply (iii) to each H_i to get

$$s(p-c)\sum_{i=1}^{t} |E(H_i)| + |X| \leq \sum_{i=1}^{t} s(|V(H_i)| - 1) + |X| = sp - st + |X|.$$

Thus $s(t-c) \leq |X|$, and so (i) follows by the definition of τ_c. □

References

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