On the Minimum Co-operative Guard Problem

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Abstract

In this paper, we study the minimum co-operative guards problem, a variation of the art gallery problem. First, we show that the minimum number of co-operative guards required for a k-spiral polygon is at most N_k, the total number of reflex vertices in the k-spiral. Then we classify 2-spirals into seven different types based on the structure. Finally, we present a minimum co-operative guard placement algorithm for general 2-spirals.

1 Introduction

The art gallery problem deals with the placement of minimum number of stationary guards in an n–walled gallery room such that every point in the room is visible to at least one guard. The floor plan of the art gallery can be modeled as an n-vertex polygon and the guards as points in the polygon. The problem now translates to finding a set of minimum number of points in the polygon, where guards can be posted such that every point in the polygon is visible to some guard. Several variations of the art gallery problem have been studied [2] such as mobile guards and co-operative guards. In the mobile guards problem the guards are allowed to move and in the co-operative guards problem, guards are supposed to co-operate by watching each other. An extensive list of references and results on the variations of art gallery problem can be found in [2] and [3]. In this paper we study the minimum co-operative guard

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problem for k-spirals. This problem was proposed by Bern–Cherng Liaw *et al* [1].

The Minimum Co-operative Guard (MCG) problem addresses the issue of placing the minimum number of guards in an art gallery such that every point is seen by at least one guard and each guard is visible to at least one other guard. The need for reliability in the safeguard of the objects and for the security of the guards themselves has inspired this problem. This problem has application in the safeguard of equipments in military establishments, shopping complex or any other installations. The MCG problem is proved to be an NP-hard problem [1, 5]. Bern–Cherng *et al* solved the minimum co-operative guard problem for 1-spirals by a greedy algorithm [1]. They have also given a partial solution for the MCG problem in 2-spirals. In this paper, we present a complete solution for the MCG problem in 2-spirals. We also establish an upper bound on the minimum number of co-operative guards required for a k-spiral.

Section 2 is devoted to preliminary discussions, definitions and notations used. In Section 3, we show that the minimum number of guards for the MCG problem is at most N_k where N_k denotes the total number of reflex vertices in the given k-spiral. In Section 4, we give a classification of the 2-spirals and then present the MCG placement algorithm for general 2-spirals.

2 Preliminaries and Definitions

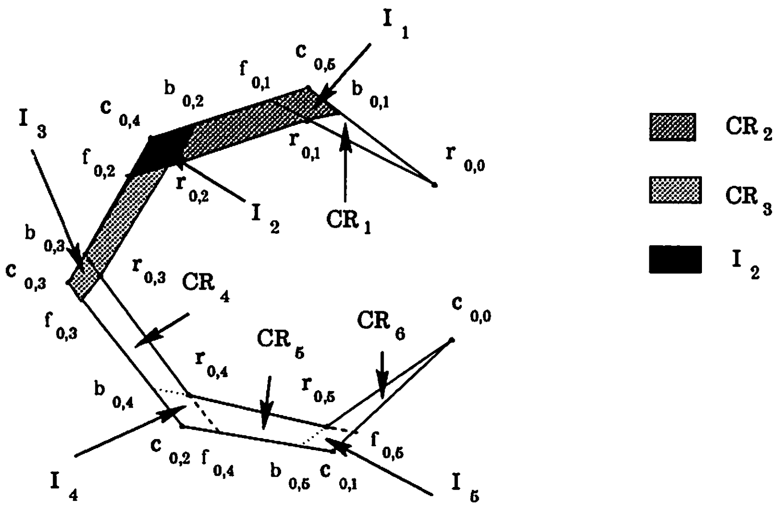
A boundary chain of a simple polygon P is an ordered sequence of its vertices, v_1, v_2, ... , v_n. Let $G(P) = (V, E)$ denote the visibility graph of a simple polygon P, where V is the set of vertices corresponding to the vertices of the polygon and $E = \{(v_i, v_j) | v_i$ is visible to $v_j\}$. Note that two vertices v_i and v_j in P are visible if the closed line segment joining v_i and v_j does not intersect the exterior of the polygon. It may however touch the boundary. Consider a counter-clockwise traversal of the boundary chain $\{v_1, v_2, \ldots, v_n\}$. A vertex v_i is said to be a reflex vertex if the angle $v_{i-1}v_iv_{i+1} > 180^o$. Otherwise it is called a convex vertex. A chain $(x, r_1, r_2, \ldots, r_k, y)$ where $(r_1, r_2, \ldots, r_k)$ is a maximal chain of reflex vertices is called a *reflex chain*. Note that x and y are convex vertices. Similarly, a chain of vertices, $(x, c_1, c_2, \ldots, c_l, y)$ is called a *convex chain* if $c_1, \ldots, c_l$ is a maximal convex chain and x and y

are reflex vertices. A k-spiral is a simple polygon whose boundary chain has exactly k reflex chains and k convex chains such that the boundary chain forms an alternating sequence of convex and reflex chains. Let H be a Hamiltonian cycle in G, the visibility graph of a polygon. A cycle in G is said to be *ordered* with respect to H if v_i precedes v_j in the cycle implies that v_i precedes v_j in H [4]. A vertex v_p is a blocking vertex with respect to H for an invisible pair of vertices v_i and v_j, if no two vertices v_k in chain $(v_i, v_{i+1}, \ldots, v_{p-1})$ and v_m in chain $(v_{p+1}, v_{p+2}, \ldots, v_{j-1}, v_j)$ are adjacent in G.

Let P be a k-spiral polygon with reflex chains R_1, R_2, ..., R_k and convex chains C_1, C_2, ..., C_k labeled according to a counter-clockwise traversal of the boundary chain. Now, R_1, C_1, R_2, ..., R_k, C_k defines a Hamiltonian cycle H. Let the reflex vertices in chain R_i be labeled as $r_{i,1}$, $r_{i,2}$, ..., r_{i,n_i}, where n_i is the number of reflex vertices in the chain, R_i. For each R_i, extend the edges $(r_{i,j-1}, r_{i,j})$, $1 < j \leq n_j$ till they hit the boundary at $f_{i,j}$. Similarly, extend each edge $(r_{i,j+1}, r_{i,j})$ till it hits the boundary at $b_{i,j}$. Let B_i denote the set $\{b_{i,1}, b_{i,2}, \ldots, b_{i,n_i}\}$ and F_i denote the set $\{f_{i,1}, f_{i,2}, \ldots, f_{i,n_i}\}$ for chain R_i. We call $B_i \cup F_i$, the *extended vertices*. We define a **1-coil** as a 1-spiral with n_1 reflex vertices whose reflex chain possess the property that for any j, $1 \leq j < n_1 - 1$, the pair $(f_{1,j}, b_{1,j+2})$ is not mutually visible. Fig 1 gives an example.

Let $r_{1,1}$, $r_{1,2}$, $r_{1,3}$, ..., r_{1,n_1}, $c_{1,1}$, ..., c_{1,m_1} be the vertices of a 1-coil labelled in a counter-clockwise order. Let $B = \{b_{1,1}, b_{1,2}, \ldots, b_{1,n_1}\}$ and $F = \{f_{1,1}, f_{1,2}, \ldots, f_{1,n_1}\}$ be the extended vertices. Let $C(x, y)$ denote the counter-clockwise boundary chain segment between two vertices x and y (x and y inclusive). The extended edge $(c_{1,m_1}, f_{1,1})$ together with $C(f_{1,1}, c_{1,m_1})$ encloses a convex region, say CR_1. Similarly, the extended edge $(b_{1,1}, f_{1,2})$ along with the boundary segment $C(f_{1,2}, b_{1,1})$ forms region CR_2. Thus the set of extended edges $(b_{1,i-1}, f_{1,i})$, $1 \leq i < n_1$ and the chain segments $C(f_{1,i}, b_{1,i-1})$ defines n_1 distinct convex regions. The last reflex vertex n_1 and $C(f_{1,n_1}, r_{1,n_1})$ defines the region CR_{n_1+1}. For each i, $1 \leq i < n_1$, CR_i and CR_{i+1} has a non-empty intersection bounded by the line $(r_{1,i}, f_{1,i})$, $C(f_{1,i}, b_{1,i})$ and the line $(b_{1,i}, r_{1,i})$, say I_i. Corresponding to each reflex vertex $r_{1,i}$, there is an I_i. There are $n_1 + 1$ regions and therefore n_1 such intersection areas. A guard posted anywhere in I_i can see the regions CR_i and CR_{i+1}. He can also see the guard located at I_{i+1}.

Example 1 *Fig 1 is an example of a 1-coil with 5 reflex vertices.*



Here,

$$
\begin{aligned}
C_1 &= \{r_{1,5}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}, c_{1,5}, c_{1,6}, c_{1,7}, r_{1,1}\} \\
R_1 &= \{c_{1,7}, r_{1,1}, r_{1,2}, r_{1,3}, r_{1,4}, r_{1,5}, c_{1,1}\} \\
CR_1 &= \{r_{1,1}, c_{1,7}, b_{1,1}, c_{1,6}, f_{1,1}, r_{1,1}\} \\
CR_2 &= \{b_{1,1}, c_{1,6}, f_{1,1}, b_{1,2}, c_{1,5}, f_{1,2}, r_{1,2}, r_{1,1}, b_{1,1}\} \\
CR_3 &= \{b_{1,2}, c_{1,5}, f_{1,2}, b_{1,3}, c_{1,4}, f_{1,3}, r_{1,3}, r_{1,2}, b_{1,2}\} \\
CR_4 &= \{b_{1,3}, c_{1,4}, f_{1,3}, b_{1,4}, c_{1,3}, f_{1,4}, r_{1,4}, r_{1,3}, b_{1,3}\} \\
CR_5 &= \{b_{1,4}, c_{1,3}, f_{1,4}, b_{1,5}, c_{1,2}, f_{1,5}, r_{1,5}, r_{1,4}, b_{1,4}\} \\
CR_6 &= \{c_{1,1}, r_{1,5}, b_{1,5}, c_{1,2}, f_{1,5}, c_{1,1}\} \\
B_1 &= \{b_{1,1}, b_{1,2}, b_{1,3}, b_{1,4}, b_{1,5}\} \\
F_1 &= \{f_{1,1}, f_{1,2}, f_{1,3}, f_{1,4}, f_{1,5}\} \\
I_1 &= \{b_{1,1}, c_{1,6}, f_{1,1}, r_{1,1}, b_{1,1}\} \\
I_2 &= \{b_{1,2}, c_{1,5}, f_{1,2}, r_{1,2}, b_{1,2}\} \\
I_3 &= \{b_{1,3}, c_{1,4}, f_{1,3}, r_{1,3}, b_{1,3}\} \\
I_4 &= \{b_{1,4}, c_{1,3}, f_{1,4}, r_{1,4}, b_{1,4}\} \\
I_5 &= \{b_{1,5}, c_{1,2}, f_{1,5}, r_{1,5}, b_{1,5}\}
\end{aligned}
$$

Let x be a vertex in the given k-spiral. Let $RV_i(x) = \{r_{i,l}, r_{i,m}, \ldots, r_{i,q}\}$ denote the set of vertices in reflex chain R_i which are visible to x. We call $r_{i,k} \in RV_i(x)$, **the left peripheral** of x wrt R_i if $r_{i,j}$ is not visible to x for all $j < k$ and $r_{i,k}$ is visible to x. Similarly, $r_{i,l} \in RV_i$ will be termed the **right peripheral** of x if $r_{i,j}$ is not visible to x for all $j > l$ and $r_{i,l}$ is visible to x. We will use the term **spiral–degree** to denote the maximum number of reflex chains in a spiral polygon.

Let p and q be two non-adjacent boundary vertices. If p and q are visible to each other, then we can draw a line segment between p and q that would be completely contained in the polygon. We call such a line segment, an **internal diagonal**. An internal diagonal drawn between two vertices on reflex chains R_i and R_j where $i \neq j$ is called a **cross diagonal**. Such a cross diagonal is said to be incident on R_i and R_j. A set of cross diagonals is said to **span a set of reflex chains** if at least one cross diagonal is incident to each chain.

A reflex chain R_i is said to be **E–visible** to another reflex chain R_j if r_{i,n_i}, an end vertex of R_i is visible to $r_{j,1}$, an end vertex of R_j or if $r_{i,1}$ the other end vertex of R_i is visible to r_{j,n_j}, an end vertex of R_j. If R_i and R_{i+1} are E–visible then the convex chain C_i is said to have **end-to-end visibility**. It follows that the part of the polygon, bounded by the arc segment $(r_{i,n_i}, c_{i,1}, c_{i,2}, \ldots, r_{i+1,1})$ and the internal diagonal $(r_{i,n_i}, r_{i+1,1})$ forms a convex region.

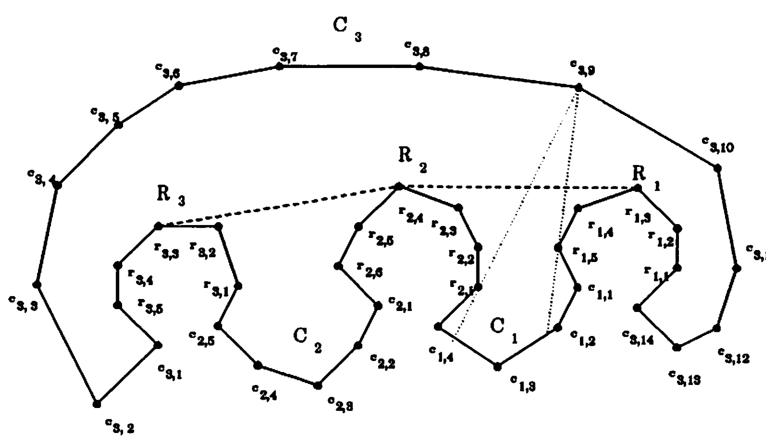


Fig 2

Example 2 *Fig 2 is a spiral polygon of spiral-degree 3. Here,*

$$
\begin{aligned}
C_1 &= \{r_{1,5},\ c_{1,1},\ c_{1,2},\ c_{1,3},\ c_{1,4},\ r_{2,1}\}; \\
C_2 &= \{r_{2,6},\ c_{2,1},\ c_{2,2},\ c_{2,3},\ c_{2,4},\ c_{2,5},\ r_{3,1}\}; \\
C_3 &= \{r_{3,5},\ c_{3,1},\ c_{3,2},\ c_{3,3},\ c_{3,4},\ c_{3,5},\ c_{3,6},\ c_{3,7}, c_{3,8},\ c_{3,9},\ c_{3,10}, \\
&\quad c_{3,11},\ c_{3,12},\ c_{3,13},\ c_{3,14},\ r_{1,1}\}; \\
R_1 &= \{c_{3,14},\ r_{1,1},\ r_{1,2},\ r_{1,3},\ r_{1,4},\ r_{1,5},\ c_{1,1}\}; \\
R_2 &= \{c_{1,4},\ r_{2,1},\ r_{2,2},\ r_{2,3},\ r_{2,4},\ r_{2,5},\ r_{2,6},\ c_{2,1}\}; \\
R_3 &= \{c_{2,5},\ r_{3,1},\ r_{3,2},\ r_{3,3},\ r_{3,4},\ r_{3,5},\ c_{3,1}\}.
\end{aligned}
$$

The line segments $(r_{1,3},\ r_{2,4})$ *and* $(r_{2,4},\ r_{3,3})$ *are cross diagonals which span all three reflex chains. Consider vertex* $c_{3,9}$. $RV_2(c_{3,9}) = \{r_{2,1},\ r_{2,2},\ r_{2,3},\ r_{2,4}\}$. *The left peripheral of* $c_{3,9}$ *w.r.t* R_2 *is* $r_{2,1}$. *The right peripheral of* $c_{3,9}$ *w.r.t.* R_2 *is* $r_{2,4}$. *Similarly,* $r_{1,2}$, *and* $r_{1,5}$ *are respectively, the left and right peripherals of* $c_{3,9}$ *w.r.t.* R_1. *In this example,* R_1 *is E–visible to* R_2 *since* $r_{1,5}$ *is visible to* $r_{2,1}$. R_2 *is E–visible to* R_3 *since the last reflex vertex in chain* R_2, $r_{2,6}$ *is visible to* $r_{3,1}$, *the first reflex vertex in chain* R_3. *But* R_3 *is not E–visible to* R_1 *or vice-versa.*

3 Upper bound on the number of M.C. Guards

The original art gallery problem needs at most $\lceil n/3 \rceil$ vertex guards. Chvátal's famous art gallery theorem states that $\lceil n/3 \rceil$ guards are occasionally necessary and always sufficient to cover a polygon of n vertices. For mobile guards, O'Rourke has proved that any polygon P of $n \geq 4$ edges can be covered by $\lfloor n/4 \rfloor$ geometric diagonal or line guards. We do not know of any such bounds on the number of guards for the MCG problem. In this section, we show that the minimum number of guards required for the MCG problem is at most N_k, where $N_k = n_1 + n_2 + \ldots + n_k$.

Lemma 1 *Let* P *be a 1-coil with* n_1 *reflex vertices and intersection areas* I_i, $1 \leq i < n_1$ *as defined in section 2. Given an* I_i *and* I_j, *a point in* I_i *is visible to some point in* I_j *iff* $|i-j| <= 1$.

Proof: ($\Rightarrow$) Let p and q be two points in I_i and I_j respectively, where $|i-j| \leq 1$. We will prove that p and q are mutually visible. If $|i-j| = 0$, then p and q both belong to the same convex region and hence are mutually visible. Let $|i-j| = 1$. With no loss of generality, we will choose p to be at $b_{1,i}$ and q to be at $f_{1,j}$. Now $b_{1,i}$ and $f_{1,j}$ are the extreme ends of the line segment passing through $r_{1,i}$ and $r_{1,j}$. Hence they are mutually visible. Also they are the farthest two points when I_i and I_j are considered. Therefore p located anywhere in I_i and q located anywhere in I_j are mutually visible.

($\Leftarrow$) Let p and q be two mutually visible points in I_i and I_j respectively. We will show that $|i-j| \leq 1$. Proof is by contradiction. Assume $|i-j| > 1$. Let $j = i+2$. Consider p to be at f_i and q to be at b_{i+2}, the nearest points as far as I_i and I_j are concerned. By definition of 1-coil, p and q are not visible. This contradicts the assumption and hence the result.

Corollary 1 *A 1-coil with n_1 reflex vertices needs at least n_1 guards for a minimum co-operative guard placement.*

Proof: Any optimal algorithm for minimum co-operative guard placement has to take advantage of the intersection areas in placing the guards so that each guard can completely cover two convex regions and be visible to the next guard. By lemma 1, a single guard can completely cover no more than two convex regions, CR_i and CR_j such that $CR_i \cap CR_j \neq \phi$. Assume that we can achieve a minimum guard placement with $n_1 - 1$ guards. that means one of the intersection areas, say I_i has no guard placed in it. There are two cases to consider.

Case 1: Suppose $1 < i < n_1$. Now none of the guards in I_1 to I_{i-1} can see any of the guards in areas I_{i+1} to I_{n_1}. So the guards visibility graph is disconnected. Hence the guard placement does not comprise a feasible solution to the minimum co-operative guard problem.

Case 2: $i = 1$ or $i = n_1$. With no loss of generality, assume that $i = n_1$. Now the vertex c_{1,m_1} and the region enclosing c_{1,m_1} which is beyond the line segment (r_{1,n_1}, f_{1,n_1}) is not visible to any guard. This again implies that the guard placement does not comprise a feasible solution to the minimum co-operative guard problem.

The above discussion suggests a simple, but optimal algorithm for minimum co-operative guard-placement in a 1-coil. Place a guard at each $f_{1,i}$, $1 \leq i \leq n_1$. This algorithm needs exactly n_1 guards. The optimality is proved by corollary 1.

Now, consider a k-spiral P_k whose reflex chains, ordered according to a counter-clockwise traversal of the boundary chain, are respectively $R_1, R_2, R_3, \ldots R_k$. We call R_i a **neighbor** of R_j if at least one reflex vertex of R_i is visible to a reflex vertex of R_j. P_k will be defined as a *k-coil* if it meets the following criteria.

1. The extended vertices $f_{i,j}$ and $b_{i,j+2}$ are not visible for each j, $1 \leq j < n_i$ and for all i, $1 \leq i < k$, where n_i denotes the number of reflex vertices in chain R_i.

2. A reflex chain R_i has at most two neighbors.

3. If R_i and R_j are neighbors, then there exists exactly one reflex vertex in R_i which is visible to exactly one reflex vertex in R_j.

In Fig. 3.a, we show an example of a k-coil, where $k = 5$. However, the polygon in Fig. 3.b does not qualify to be a k-coil since it violates conditions 2 and 3.

Condition 1 above and the counter-clockwise labeling of the alternating sequence of reflex and convex chains in the boundary chain defines the following **chain of neighbors**. The **neighbor-chain** of a k-spiral is $R_1, R_k, \ldots, R_i, R_{k-i+1}, \ldots, R_{\lceil k/2 \rceil}$. Given two neighbors R_i and R_j, reflex vertex r_{i,n_i} (r_{j,n_j}) is visible to $r_{j,1}$ ($r_{i,1}$) if R_i (R_j) appears before R_j (R_i) in the chain of neighbors.

Lemma 2 *A k-coil needs exactly N_k guards, where $N_k = n_1 + n_2 + \ldots + n_k$, for a minimum co-operative guard placement problem.*

Proof: Consider the chain of neighbors, $R_1, R_k, R_2, \ldots$. The polygon bounded by $c_{k,m_k}, r_{1,1}, r_{1,2}, \ldots, r_{1,n_1}, f_{k,n_k}, b_{k,n_k}, r_{k,n_k}$ and the counter-clockwise boundary segment $C(r_{k,n_k}, c_{k,m_k})$ forms a 1-coil. Similarly $c_{1,1}, f_{k,n_k}, r_{k,n_k}, r_{k,n_k-1}, r_{k,n_k-2}, \ldots, r_{k,1}, b_{2,1}, f_{2,1}, r_{2,1}$ and the counter-clockwise boundary segment $C(r_{2,1}, f_{k,n_k})$ forms a 1-coil. Thus, each reflex chain has a 1-coil associated with it. The optimal algorithm discussed above can be applied for the k-coil. By condition 1 and corollary 2, each such coil needs n_i guards. Thus the k-coil needs $N_k = \sum_{i=1}^{k} n_i$ guards.

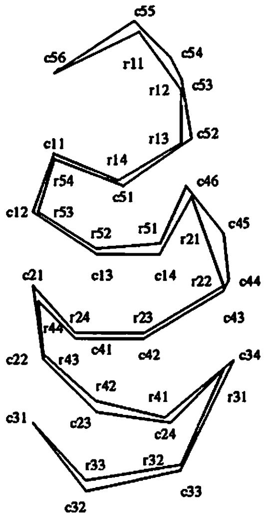


Fig 3.a

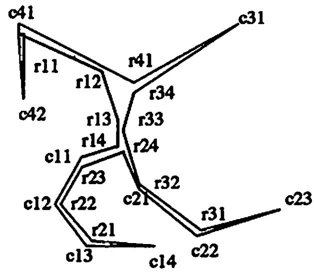


Fig. 3.b

Theorem 1 *Let P be a k-spiral polygon with N_k reflex vertices, where $N_k = n_1 + n_2 + \ldots + n_k$, and n_i is the number of vertices in R_i. P needs at most N_k guards for the minimum cooperative placement of guards, for $N_k \geq 1$.*

Proof: It is quite obvious that by placing a guard at each of the reflex vertices, the entire polygon can be covered and at the same time the guards visibility graph forms a chain. Thus the sufficiency of N_k guards for a k-spiral is established. Now consider the worst case scenario of a k-coil with N_k reflex vertices. By lemma 2, it needs N_K guards. Thus for the special case where a k-spiral becomes a k-coil, N_k guards are necessary for a minimum co-operative guard placement.

4 Classification of 2–spirals

A better understanding of the possible structures of 2–spirals is essential for the optimal placement of guards. To do the classification, we need the following definitions. Consider a cross diagonal (a, b), where $a = r_{i,l}$ is in chain R_i and $b = r_{j,k}$ is in chain R_j.

The line obtained by extending this diagonal to both directions ba and ab divides the plane into two half-planes. If the two boundary edges incident on a lies in the same half-plane w.r.t. (a, b) and both edges incident on b lies in the other half–plane then (a, b) is called a *cross tangent*. If the edges incident on a as well as b lie in the same half–plane w.r.t. (a, b) then (a, b) is called a *common tangent*.

Lemma 3 *Let (a,b) be a cross tangent, where $a = r_{i,l}$ is in reflex chain R_i and $b = r_{j,k}$ is in reflex chain R_j. Then a is the left(right) peripheral of b w.r.t. R_i and b is the left(right) peripheral of a w.r.t. R_j.*

Proof: We will prove the result for the case of left peripheral. The proof for the right peripheral is similar. Proof is by contradiction. Assume that (a, b) is a cross tangent. In the case of the left peripheral, there are two cases to consider:

Case 1: Assume that a is the left peripheral of b w.r.t. R_i but b is not the left peripheral of a w.r.t. R_j.

Then there exists some vertex $r_{j,m}$ whose index is smaller than that of $b = r_{j,k}$ but is visible to a. Let $m = k - 1$. This implies that edge $(r_{j,k-1}, r_{j,k})$ is in one half–plane and edge $(r_{j,k}, r_{j,k+1})$ is in the other half–plane with respect to (a, b), which implies that (a, b) is not a cross tangent, a contradiction and hence the result.

Case 2: b is the left peripheral of a but a is not the left peripheral of b

The proof for this case is similar to that of Case 1 and hence omitted.

Lemma 4 *Let (a,b) be a common tangent, where $a = r_{i,l}$ is in reflex chain R_i and $b = r_{j,k}$ is in reflex chain R_j. Then a is the left(right) peripheral of b w.r.t. R_i and b is the right(left) peripheral of a w.r.t. R_j.*

proof: Suppose a is the left peripheral of b but b is not the right peripheral of a. Then at least the vertex $r_{j,k+1}$ should be visible to a. This would then imply that the edges $(r_{j,k}, r_{j,k+1})$ and $(r_{j,k-1}, r_{j,k})$ are in different half–planes w.r.t. (a, b), which contradicts the assumption that (a, b) is a common tangent. The proof for the following cases are quite similar to this and hence omitted.

Case 1: a is the right peripheral of b but b is not the left peripheral of a

Case 2: a is not the left peripheral of b but b is the right peripheral of a

Case 3: a is not the right peripheral of b but b is the left peripheral of a

Two spirals can be classified into the following types based on the number of common tangents and cross tangents (Fig 4 gives examples of each type).

Type A: those with two common tangents and two cross tangents

Type B: those with one common tangent and two cross tangents

Type C: those with one common tangent and one cross tangent

Type D: those with one common tangent and zero cross tangents

Type E: those with zero common tangents and two cross tangents

Type F: those with zero common tangents and one cross tangent

Type G: those with zero common tangent and zero cross tangents

Next, we discuss the algorithm for minimum co-operative guard placement in each of these types. Since this algorithm is related to earlier work on minimum co-operative guards by Bern-Cherng *et al*, we first discuss their algorithm. We claim that the algorithm presented by Bern-Cherng *et al*[1] is incomplete since it does not completely cover all possible cases of 2–spirals. Their algorithm covers Type A 2-spirals. We present an algorithm that can handle all possible types of 2-spirals such as types A through G. Then we extend it for 3-spirals in a forth coming paper.

4.1 Discussion of 2–spiral algorithm

First, we discuss the algorithm proposed by Bern-Cherng *et al* for minimum co-operative guard placement in 1–spirals. Given a 1-spiral P and a point p within P, their algorithm places the minimum number of guards inside P such that a guard is stationed at

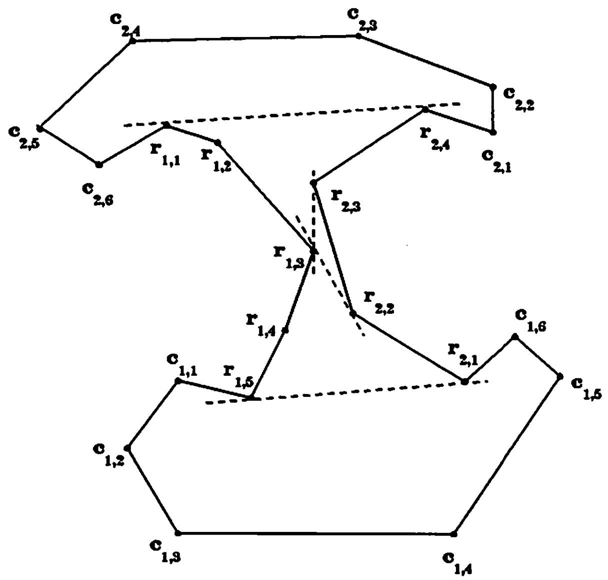


Fig 4.a Type A 2-spiral (2 common tangents & 2 cross tangents)

p. Let the vertices of the polygon, labeled according to a counter-clockwise traversal of the boundary chain be r_1, r_2, r_3, ... , r_i, c_1, c_2, ... , c_j where $(c_j, r_1, r_2, \ldots, r_i, c_1)$ constitute the reflex chain R and $(r_i, c_1, c_2, \ldots, c_j, r_1)$ constitute the convex chain. Extend the edge (c_j, r_1) till it hits the convex boundary chain at l_1. Similarly, extend the edge (c_1, r_i) till it hits the convex boundary at e_1. From l_1, draw a tangent to R in the clockwise direction. Let l_2 be the point of intersection of this tangent with the convex chain. From l_2, draw a tangent to R to hit the convex boundary chain at l_3. Proceed this way till the tangent (l_{max-1}, l_{max}) hits the convex region bounded by the line segment (c_1, e_1) and the counter-clockwise chain segment $(c_1, c_2, \ldots, e_1)$. Similarly, draw tangent from e_1 to R in the clockwise direction. Let e_2 be the point of intersection of this tangent with the convex chain. Repeat this process till tangent (e_{max-1}, e_{max}) hits the convex region bounded by (c_j, l_1) and the boundary segment $(c_j, c_{j-1}, \ldots, l_1)$. Let L_i denote the convex region enclosed by (l_i, l_{i-1}) and the convex boundary segment (l_{i-1}, l_i). Let E_i denote the convex region enclosed by (e_i, e_{i-1})

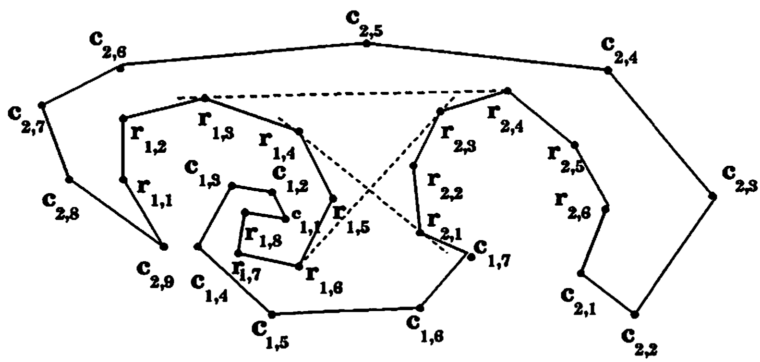


Fig 4. b Type B 2-spiral (1 common tangent & 2 cross tangents)

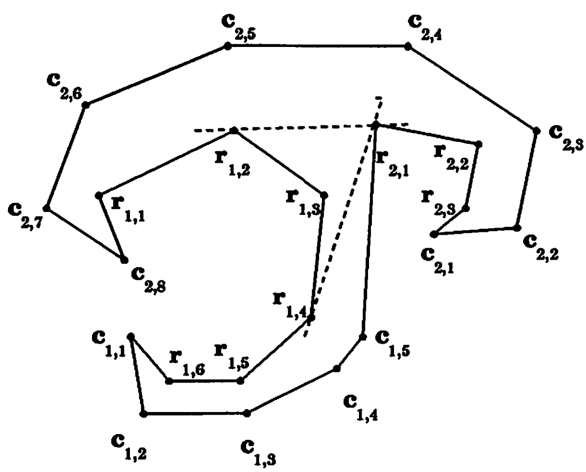


Fig 4.c Type C 2-spiral (one common tangent & one cross tangent)

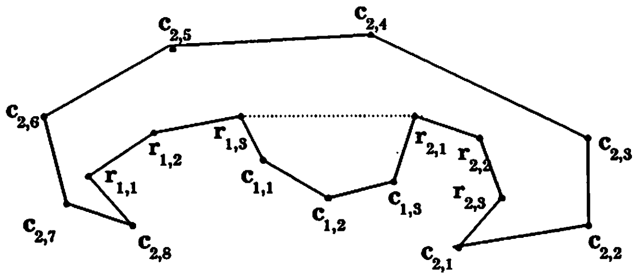


Fig 4. d Type D 2-spiral (1 common tangent & 0 cross tangents)

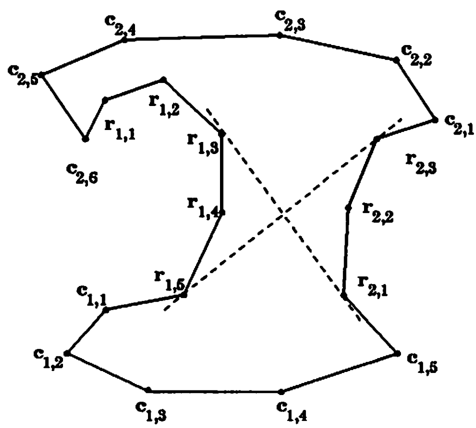


Fig 4. e Type E 2-spiral (0 common tangent & 2 cross tangents)

and the convex boundary segment (e_{i-1}, e_i). Place a guard at the given point p. If p lies in the non-empty intersection of regions L_k and $E_{max-k+1}$ for some k then place guards at each of the remaining $max - 1$ such regions. Otherwise, the algorithm places the guards at $l_1, l_2, \ldots, l_k$.

In the description of their algorithm for 2-spirals, they assume the existence of all four tangents. However, there exists 2–spirals that do not have all four tangents as can be seen in Fig 4 for Types B through G. Bern–Cherng's algorithm is therefore based on an incorrect assumption and does not work for general 2-spirals. We present a detailed algorithm for minimum co-operative guard placement in a general 2-spiral that covers all cases.

First, we define some terms that are used in the algorithm. We call the process of drawing tangents (l_0, l_1), (l_1, l_2), (l_2, l_3), ..., (l_{max-1}, l_{max}) the *forward-sweep*. The term *reverse-sweep* denotes the process of drawing tangents (e_0, e_1), (e_1, e_2), ..., (e_{max-1}, e_{max}). We use the term **bold chain** defined in [1] to denote the convex boundary segments $(l_i, e_{max-i+1})$ for $0 < i \leq max$. The common tangents in a 2-spiral are denoted as CT_1 and CT_2 and the cross tangents are denoted as XT_1 and XT_2. Let ct'_i and ct''_i denote the reflex vertices where common tangent CT_i touches R_1 and R_2 respectively for $1 \leq i \leq 2$. Similarly, let xt'_i and xt''_i denote the points where XT_i touches R_1 and R_2 respectively. We use the term *pred*(x) to denote the vertex which appears immediately before vertex x in the boundary chain ordered according to the counter-clockwise traversal and *succ*(x) to denote the vertex which appears immediately after vertex x in the boundary chain. If vertex x appears earlier than y in the counterclockwise ordering scheme then we denote it as $x <_L y$. For a 2-spiral with at least one common tangent CT_1, P_1 denotes the 1-spiral formed by ct'_1, $pred(ct'_1), \ldots, r_1, pred(r_1), \ldots, c_{2,1}, r_{2,n_2}, r_{2,n_2-1}, \ldots, succ(ct''_1)$, ct''_1, ct'_1. The sub-polygon enclosed by ct'_1, $succ(ct'_1), \ldots, r_{1,n_1}$, $c_{1,1}, succ(c_{1,1}), \ldots, pred(r_{2,1}), r_{2,1}, \ldots, pred(ct''_1), ct''_1$ is denoted as $\overline{P_1}$. The points of intersection of XT_i with the boundary chains of P_1 and $\overline{P_1}$ are denoted as α'_i and α''_i respectively. The points of intersection of CT_i with the convex boundary chains of P_1 and $\overline{P_1}$ are denoted as β'_i and β''_i respectively.

$\overline{P_1}$ is further split into a 1 or 2-spiral P_2 and a 1-spiral P_3. It may be noted that, for Type F, P_2 degenerates into an empty polygon. Bern-Cherng's 1-spiral algorithm is used in placing the guards optimally in P_1 and P_3 such that the combined solution of the two 1-spirals yields a minimum co-operative guard placement for the 2-spiral. In order to obtain the optimal solution, we may introduce *fake cross tangents* if there are fewer than two cross tangents in the 2-spiral (as in the case of Type C and Type G). For purposes of placing guards, fake cross tangents are treated as real cross tangents. In particular, we use notations α'_i and α''_i to denote the points of intersection of fake cross tangent XT_i with the convex boundary chains of P_1 and $\overline{P_1}$ respectively. The construction of fake tangents is described in algorithm two–split. We call the parts of the convex boundary chains between α'_1 and α'_2 and that between α''_1 and α''_2 the ***visible regions*** VR_1 and VR_2.

Algorithm two–split
Case polygon of

Type A: (two common tangents and two cross tangents)

$$P_1 \leftarrow ct'_1,\ pred(ct'_1),\ \ldots,\ r_{1,1},\ pred(r_{1,1}),\ \ldots,\ c_{2,1},\ r_{2,n_2},\ r_{2,n_2-1},\ \ldots,\ succ(ct''_1),\ ct''_1,\ ct'_1$$

$$P_2 \leftarrow ct'_1,\ succ(ct'_1),\ \ldots,\ ct'_2,\ ct''_2,\ succ(ct''_2),\ \ldots,\ pred(ct''_1),\ ct''_1,\ ct'_1$$

$$P_3 \leftarrow ct'_2,\ succ(ct'_2),\ \ldots,\ r_{1,n_1},\ c_{1,1},\ c_{1,2},\ \ldots,\ pred(r_{2,1}),\ r_{2,1},\ succ(r_{2,1}),\ \ldots,\ pred(ct''_2),\ ct''_2,\ ct'_2$$

Type B: (one common tangent and two cross tangents)

$$P_1 \leftarrow ct'_1,\ pred(ct'_1),\ \ldots,\ r_{1,1},\ pred(r_{1,1}),\ \ldots,\ c_{2,1},\ r_{2,n_2},\ r_{2,n_2-1},\ \ldots,\ succ(ct''_1),\ ct''_1,\ ct'_1$$

$$P_2 \leftarrow ct'_1,\ succ(ct'_1),\ \ldots,\ pred(xt'_2),\ xt'_2,\ xt''_2,\ succ(xt''_2),\ \ldots,\ pred(ct''_1),\ ct''_1,\ ct'_1$$

$$P_3 \leftarrow xt'_2,\ succ(xt'_2),\ \ldots,\ r_{1,n_1},\ c_{1,1},\ c_{1,2},\ \ldots,\ pred(r_{2,1}),\ r_{2,1},\ \ldots,\ xt''_1,\ xt'_2$$

Type C: (One common tangent and one cross tangent)

$$P_1 \leftarrow ct'_1,\ pred(ct'_1),\ \ldots,\ r_{1,1},\ pred(r_{1,1}),\ \ldots,\ c_{2,1},\ r_{2,n_2},\ r_{2,n_2-1},\ \ldots,\ succ(ct''_1),\ ct''_1,\ ct'_1$$

In this case there is only one cross tangent, say XT_1. We construct a fake cross tangent as follows. With no loss of generality, assume that xt'_1 is in $\overline{P_1}$ and xt''_1 is in P_1. Let y denote the vertex adjacent to xt'_1 which is not visible from P_1. To construct a fake cross tangent, extend the edge (y, xt'_1) till it hits the boundary chain at α''_2. From α''_2, draw a line segment such that it is tangential to the left peripheral of α''_2 w.r.t. R_1. Let α'_2 denote the point of intersection of this line with the convex boundary of P_1. Then the line joining α''_2 and α'_2 define the fake cross tangent.

If $y = succ(xt'_1)$ then

$P_2 \leftarrow ct'_1,\ succ(ct'_1),\ \ldots,\ xt'_1,\ \alpha''_2,\ \ldots,\ xt''_1,\ succ(xt''_1),\ \ldots,\ ct''_1,\ ct'_1$

$P_3 \leftarrow \alpha''_2,\ xt'_1,\ y,\ succ(y),\ succ(succ(y)),\ \ldots,\ \alpha''_2$

else

$P_2 \leftarrow ct'_1,\ succ(ct'_1),\ \ldots,\ \alpha''_2,\ xt'_1,\ succ(xt'_1)\ \ldots,\ pred(ct''_1),\ ct''_1,\ ct'_1$

$P_3 \leftarrow xt'_1,\ \alpha''_2,\ succ(\alpha''_2),\ \ldots,\ r_{2,1},\ \ldots,\ xt'_1$

Type D: (One common tangent and zero cross tangents)

In this case, no further splitting or fake tangents is needed.

$$P_1 \leftarrow ct'_1,\ pred(ct'_1),\ \ldots,\ r_{1,1},\ pred(r_{1,1}),\ \ldots,\ c_{2,1},\ r_{2,n_2},\ r_{2,n_2-1},\ \ldots,\ succ(ct''_1),\ ct''_1,\ ct'_1$$

Type E: (Zero common tangent and two cross tangents)

With no loss of generality assume that $xt'_1 <_L xt'_2$ and $xt''_1 <_L xt''_2$

Let x_1 = left peripheral of xt''_2 with respect to. R_1
y_1 = right peripheral of x_1 with respect to. R_2
x_2 = right peripheral of xt''_1 with respect to. R_1
y_2 = left peripheral of x_2 with respect to. R_2

$$P_1 \leftarrow x_1,\ pred(x_1),\ pred(pred(x_1)),\ \ldots,\ r_{1,1},\ pred(r_{1,1}),\ \ldots,\ c_{2,1},\ r_{2,n_2},\ \ldots,\ y_1,\ x_1$$

$$P_2 \leftarrow x_1,\ succ(x_1),\ \ldots,\ r_{1,n_1},\ c_{1,1},\ \ldots,\ pred(r_{2,1}),\ r_{2,1},$$

$$\ldots,\ y_1,\ x_1$$

$$P_3 \leftarrow x_2,\ succ(x_2),\ \ldots,\ r_{1,n_1},\ c_{1,1},\ \ldots,\ pred(r_{2,1}),\ y_2,\ x_2$$

Type F: (One cross tangent and no common tangents)

In this case, the cross tangent splits the 2-spiral into two 1-spirals P_1 and P_3 (P_2 is empty).

$$P_1 \leftarrow xt_1',\ pred(xt_1'),\ \ldots,\ r_{1,1},\ pred(r_{1,1}),\ \ldots,\ c_{2,1},\ r_{2,n_2},\ \ldots,\ xt_1''$$

$$P_3 \leftarrow xt_1',\ succ(xt_1'),\ \ldots,\ r_{1,n_1},\ c_{1,1},\ \ldots,\ pred(r_{2,1}),\ r_{2,1},\ \ldots,\ xt_1''$$

Type G: (No common tangent and no cross tangent)

Locate a pair $r_{1,s}, r_{2,l}$ such that $r_{1,s}$ is the left peripheral of $r_{2,l}$ w.r.t. R_1 and $r_{2,l}$ is the right peripheral of $r_{1,s}$ w.r.t. R_2. Similarly, locate another pair $r_{1,t}, r_{2,m}$ such that $r_{1,t}$ is the right peripheral of $r_{2,m}$ w.r.t. R_1 and $r_{2,m}$ is the left peripheral of $r_{1,t}$ w.r.t. R_2.

$$P_1 \leftarrow r_{1,s},\ r_{1,s-1},\ \ldots,\ r_{1,1},\ pred(r_{1,1}),\ \ldots,\ c_{2,2},\ c_{2,1},\ \ldots,\ r_{2,l},\ r_{1,s}$$

$$P_2 \leftarrow r_{1,s},\ succ(r_{1,s}),\ \ldots,\ pred(r_{1,t}),\ r_{1,t},\ r_{2,m},\ succ(r_{2,m}),\ \ldots,\ pred(r_{2,l}),\ r_{2,l}$$

$$P_3 \leftarrow r_{1,t},\ succ(r_{1,t}),\ \ldots,\ pred(r_{2,m}),\ r_{2,m},\ r_{1,t}$$

Let A_1, A_2, B_1 and B_2 denote respectively, the directed line segments obtained by extending the edges $(pred(r_{1,s}),\ r_{1,s})$, $(succ(r_{1,t}),\ r_{1,t})$, $(succ(r_{2,l}),\ r_{2,l})$ and $(pred(r_{2,m}),\ r_{2,m})$.

If A_1 and A_2 intersect then XT_1 is constructed by drawing the line segment joining $r_{1,s}$ and the left peripheral of $r_{1,s}$ w.r.t. R_2 and extending it till it hits the convex boundary at α_1''. Similarly XT_2 is constructed by extending the line segment joining $r_{1,t}$ and the right peripheral of $r_{1,t}$ wrt. R_2

extended to meet the convex boundary at α_2'. Note that α_1' $= r_{1,s}$ and $\alpha_2'' = r_{1,t}$.

IF A_1 and A_2 do not intersect then XT_1 is constructed by drawing a line segment joining $r_{2,l}$ with its right peripheral w.r.t. R_1 and extended, if needed to meet the convex boundary at α_1''. Similarly XT_2 is constructed by joining $r_{2,m}$ with its left peripheral w.r.t. R_1 and extended to meet at α_2'. In this case $\alpha_1' = r_{2,l}$ and $\alpha_2'' = r_{2,m}$.

The fake cross tangent XT_1 is defined by α_1' and α_1'' and XT_2 is defined by α_2' and α_2''.

Next we give the algorithm for placing the guards in the 1-spirals and combining the solutions. For Type A, we use the algorithm given in [1].

Algorithm Combine:

Construct bold chains in P_1 and P_3.

Case polygon of

Type A: There are four cases to consider.

Case 1: There exists two bold chains BS_1 in VR_1 and BS_3 in VR_2. Choose point x from BS_1 and y from BS_3 such that x and y are mutually visible.

Case 2: There exists one bold chain that is either BS_1 in VR_1 or BS_3 in VR_2.

Let x denote a point in $BS_1(BS_3)$ which is in the visible region. Choose y in $BS_3(BS_1)$ such that x and y are mutually visible.

Case 3: No bold chain is present in VR_1 and VR_2 but there exists a point z in P_2 which is visible to some point x in BS_1 and some point y in BS_3. Place a guard at z and choose x and y.

Case 4: No bold chain in VR_1 and VR_2 and no point z in P_2 which is visible to both BS_1 and BS_3.

Choose some x in VR_1 and some y in VR_2.

Apply Bern–Cherng *et al* 's 1–spiral algorithm for P_1 and P_3 with guards placed at x and y respectively.

Type B, C, E, G: Apply the same technique as in type A.

Type D: Construct two lines by extending the edges $(c_{1,1}, r_{1,n_1})$ and $(pred(r_{2,1}), r_{2,1}))$ till they intersect the boundary chain C_2 at x and y respectively. Now there are three cases to consider.

Case 1: The lines intersect at z within P_1.

Case 2: The lines intersect at z' within $\overline{P_1}$. Place a guard at z' and choose a point z on the convex boundary chain in P_1 which is visible to z'.

Case 3: The lines do not intersect within the 2-spiral. Here there are two subcases:

1. The convex boundary segment xy has non-empty intersection with some bold chain in P_1.
   Choose z to be a point in this intersection.
2. xy does not have a non-empty intersection with any bold sub-chain in P_1.

Choose z to be x.

Apply Bern-Cherng's 1-spiral algorithm on P_1 with z as the special node.

Type F: Let XT_1 denote the only one cross tangent. Apply Bern–Cherng's 1-spiral algorithm on P_1 with α_1' as the special node and on P_3 with α_1'' as the special node.

So, given a 2-spiral, we apply algorithm two–split and then algorithm combine to place the minimum co-operative guards.

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