On labeling of some graphs*

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ABSTRACT. A graph P_n^2, $n \geq 3$, is the graph obtained from a path P_n by adding edges that join all vertices u and v with $d(u,v) = 2$. A graph C_n^{+t}, $n \geq 3$ and $1 \leq t \leq n$, is formed by adding a single pendent edge to t vertices of a cycle of length n. A Web graph $W(2,n)$ is obtained by joining the pendent vertices of a Helm graph (i.e. a Wheel graph with a pendent edge at each cycle vertex) to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. In this paper, we find the gracefulness of P_n^2 for any n, of C_n^{+t} for $n \geq 3$ and $1 \leq t \leq n$, and of $W(2,n)$ for $n \geq 3$. Therefore, three conjectures about labeling graphs — Grace's, Koh's and Gallian's — are confirmed.

1 Introduction

Let $G = (V, E)$ be a simple graph and $g: V \to \{0, 1, \ldots, |E|\}$ be an injective map. Define an induced map $g^*: E \to \{1, 2, \ldots, |E|\}$ by setting $g^*(xy)$ equal to $|g(x) - g(y)|$ for $xy \in E$. If g^* maps E onto $\{1, 2, \ldots, |E|\}$, then

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g is said to be a *graceful labeling* (or *β-labeling*) of G, and the graph G is said to be *graceful*.

A graph P_n^2 is the graph obtained from a path P_n by adding edges that join all vertices u and v with $d(u, v) = 2$, where the minimum distance between two vertices u and v of the graph G, $d(u, v)$, is the length of the shortest path joining u and v. In 1983, T. Grace first posed the conjecture: all graphs P_n^2 are graceful for positive integer n. Later, D. Ropp, J.A. Gallian, J. Huang and S. Skiena verified this conjecture for $n \leq 32$ (see [2], [3], [4]) But, so far, the general question is still open. A. Gallian restated the conjecture in his recent survey [3]. In Section 2 of this paper we prove this conjecture completely.

For $n \geq 3$ and $1 \leq t \leq n$, let C_n^{+t} denote the class of graphs formed by adding a single pendent edge to t vertices of a cycle of length n. In 1986, Joseph A. Gallian conjectured that for every n and t the class C_n^{+t} contains a graceful graph. This conjecture was proved by Daniel Ropp [7]. Ropp and Gallian further conjectured that for all n and t, all members of C_n^{+t} are graceful. In Section 3 of this paper, we prove this conjecture.

K.M. Koh et al. [8] define a Web graph as one obtained by joining the pendent vertices of a Helm graph (i.e. a Wheel graph with a pendent edge at each cycle vertex) to form a cycle and then adding a single pendent edge to each vertex of this outer cycle. They ask whether such graphs are graceful. In Section 4 of this paper we will give a complete affirmative answer to this question. For $n \geq 3$ we denote by $W(2, n)$ the Web graph.

In what follows, the notations $(a, b \in Z)$:

$$[a, b] = \{x \in Z; a \leq x \leq b\} \text{ and}$$

$$[a, b]_k = \{x \in Z; a \leq x \leq b, x \equiv a \pmod{k}\} \text{ for } a \equiv b \pmod{k}$$

are used frequently.

2 The gracefulness of graph $\mathbf{P_n^2}$

Theorem 1. *All graphs P_n^2 are graceful for any positive integer $n \geq 3$.*

Construction: We denote the n vertices of graph P_n^2 by $x_1, x_2, \ldots, x_n$, successively. Its $2n-3$ edges are $x_1x_2, x_2x_3, \ldots, x_{n-1}x_n$ and $x_1x_3, x_2x_4, \ldots, x_{n-2}x_n$. For $3 \leq n \leq 16$, the labels of the n vertices in each P_n^2 are listed succesively as follows:

$P_3^2: 0, 1, 3;$

$P_4^2: 3, 0, 1, 5;$

$P_5^2: 6, 1, 0, 3, 7;$

$P_6^2: 9, 0, 1, 7, 5, 2;$

$P_7^2: 0, 11, 10, 2, 7, 9, 3;$

$P_8^2: 13, 1, 0, 11, 8, 2, 4, 9;$

$P_9^2: 15, 0, 6, 14, 1, 3, 13, 10, 9;$

$P_{10}^2: 15, 0, 1, 17, 14, 7, 2, 11, 13, 5;$

$P_{11}^2: 2, 18, 19, 0, 4, 14, 17, 5, 10, 16, 8;$

$P_{12}^2: 11, 21, 0, 7, 20, 1, 4, 19, 2, 14, 6, 15;$

$P_{13}^2: 12, 0, 23, 9, 1, 22, 6, 2, 21, 4, 3, 10, 13;$

$P_{14}^2: 13, 25, 0, 10, 24, 1, 7, 23, 2, 4, 22, 15, 14, 18;$

$P_{15}^2: 13, 0, 27, 11, 1, 26, 8, 2, 25, 5, 3, 24, 15, 16, 20;$

$P_{16}^2: 14, 29, 28, 0, 11, 27, 1, 20, 26, 2, 5, 25, 3, 16, 8, 4;$

For $n > 16$, we consider the following 6 cases according to the residue of $n \pmod 6$.

Case: n = 6m (m ≥ 3)

$$g(x_{3i+1}) = \begin{cases} 6m-1 & (i=0) \\ 4m+1-3i & (1 \le i \le m) \quad (*) \\ 4m-1+i & (m+1 \le i \le 2m-2) \\ 2m+2 & (i=2m-1) \end{cases}$$

$$g(x_{3i+2}) = \begin{cases} 12m-3-i & (0 \le i \le 2m-2) \\ 6m-2 & (i=2m-1) \end{cases}$$

$$g(x_{3i+3}) = \begin{cases} i & (0 \le i \le m-1) \\ 8m-4-3i & (m \le i \le 2m-3) \\ 10m-2 & (i=2m-2) \\ 6m+1 & (i=2m-1). \end{cases}$$

When $m \equiv 2 \pmod 3$, the row $(*)$ has to be replaced with

$$\begin{cases} 8m-4+3i & (1 \le i \le \frac{2m-1}{3}) \\ 4m+1-3i & (\frac{2m+2}{3} \le i \le m). \end{cases}$$

Case: $n = 6m + 1$ ($m \geq 3$)

$$g(x_{3i+1}) = \begin{cases} 6m & (i = 0) \\ 8m - 1 & (i = 1) \\ 4m + 1 - 3i & (2 \leq i \leq m) \quad (*) \\ 4m + i & (m + 1 \leq i \leq 2m - 2) \\ 2m & (i = 2m - 1) \\ 10m & (i = 2m) \end{cases}$$

$$g(x_{3i+2}) = \begin{cases} 12m - 1 - i & (0 \leq i \leq 2m - 2) \\ 6m + 2 & (i = 2m - 1) \end{cases}$$

$$g(x_{3i+3}) = \begin{cases} i & (0 \leq i \leq m - 1) \\ 8m - 3 - 3i & (m \leq i \leq 2m - 3) \\ 2m + 1 & (i = 2m - 2) \\ 10m - 3 & (i = 2m - 1). \end{cases}$$

When $m \equiv 0 \pmod 3$, the row $(*)$ will be replaced with

$$\begin{cases} 4m + 1 - 3i & (2 \leq i \leq m \text{ and } i \neq \frac{2m}{3}) \\ 10m - 2 & (i = \frac{2m}{3}). \end{cases}$$

When $m \equiv 1 \pmod 3$, the row $(*)$ will be replaced with

$$\begin{cases} 8m - 2 + 3i & (2 \leq i \leq \frac{2m-5}{3}) \\ 4m + 1 - 3i & (\frac{2m-2}{3} \leq i \leq m \text{ and } i \neq \frac{2m+1}{3}) \\ 10m - 1 & (i = \frac{2m+1}{3}). \end{cases}$$

Case: $n = 6m + 2$ ($m \geq 3$)

$$g(x_{3i+1}) = \begin{cases} 6m + 1 & (i = 0) \\ 4m + 5 - 3i & (1 \leq i \leq m + 1) \quad (*) \\ 4m - 1 + i & (m + 2 \leq i \leq 2m - 1) \\ 6m + 2 & (i = 2m) \end{cases}$$

$$g(x_{3i+2}) = \begin{cases} 12m + 1 - i & (0 \leq i \leq 2m - 1) \\ 6m + 5 & (i = 2m) \end{cases}$$

$$g(x_{3i+3}) = \begin{cases} i & (0 \leq i \leq m) \\ 8m - 3i & (m + 1 \leq i \leq 2m - 2) \\ 10m + 1 & (i = 2m - 1). \end{cases}$$

When $m \equiv 2 \pmod 3$, the row $(*)$ has to be replaced with

$$\begin{cases} 8m-4+3i & (1 \le i \le \frac{2m+2}{3}) \\ 4m+5-3i & (\frac{2m+5}{3} \le i \le m+1). \end{cases}$$

Case: $\mathbf{n = 6m+3}$ $\mathbf{(m \ge 3)}$

$$g(x_{3i+1}) = \begin{cases} 6m+2 & (i=0) \\ 12m+3-i & (1 \le i \le 2m-1) \\ 2m+1 & (i=2m) \end{cases}$$

$$g(x_{3i+2}) = \begin{cases} 0 & (i=0) \\ i+1 & (1 \le i \le m) \\ 8m+1-3i & (m+1 \le i \le 2m-2) \\ 10m+1 & (i=2m-1) \\ 10m & (i=2m) \end{cases}$$

$$g(x_{3i+3}) = \begin{cases} 12m+3 & (i=0) \\ 4m+3-3i & (1 \le i \le m) \quad (*) \\ 4m+1+i & (m+1 \le i \le 2m-1) \\ 6m+4 & (i=2m). \end{cases}$$

When $m \equiv 2 \pmod 3$, the row $(*)$ will be replaced with

$$\begin{cases} 8m+1+3i & (1 \le i \le \frac{2m-4}{3} \text{ and } i = \frac{2m+2}{3}) \\ 4m+3-3i & (\frac{2m+5}{3} \le i \le m \text{ and } i = \frac{2m-1}{3}). \end{cases}$$

Case: $\mathbf{n = 6m+4}$ $\mathbf{(m \ge 3)}$

$$g(x_{3i+1}) = \begin{cases} 6m+3 & (i=0) \\ 8m+1 & (i=1) \\ 4m+6-3i & (2 \le i \le m+1) \quad (*) \\ 4m+1+i & (m+2 \le i \le 2m) \\ 6m+5 & (i=2m+1) \end{cases}$$

$$g(x_{3i+2}) = \begin{cases} i & (0 \le i \le m+1) \\ 8m+5-3i & (m+2 \le i \le 2m-1) \\ 2m+4 & (i=2m) \end{cases}$$

$$g(x_{3i+3}) = \begin{cases} 12m+5-i & (0 \le i \le 2m-1) \\ 2m+5 & (i=2m). \end{cases}$$

When $m \not\equiv 0 \pmod 3$, the row (*) will be respectively replaced with

$$\begin{cases} 8m-1+3i & (2 \le i \le \lceil \frac{2m}{3} \rceil) \\ 4m+6-3i & (\lceil \frac{2m}{3} \rceil + 1 \le i \le m+1), \end{cases}$$

where the notation $\lceil x \rceil$ denotes the smallest integer not less than x.

Case: $\mathbf{n = 6m+5}$ $(\mathbf{m \ge 2})$

$$g(x_{3i+1}) = \begin{cases} 6m+4 & (i=0) \\ 12m+7-i & (1 \le i \le 2m+1) \end{cases}$$

$$g(x_{3i+2}) = \begin{cases} 0 & (i=0) \\ i+1 & (1 \le i \le m+1) \\ 8m+5-3i & (m+2 \le i \le 2m) \\ 6m+5 & (i=2m+1) \end{cases}$$

$$g(x_{3i+3}) = \begin{cases} 12m+7 & (i=0) \\ 4m+5-3i & (1 \le i \le m) \quad (*) \\ 4m+3+i & (m+1 \le i \le 2m). \end{cases}$$

When $m \equiv 0 \pmod 3$, the row (*) will be respectively replaced with

$$\begin{cases} 8m+3+3i & (1 \le i \le \frac{2m}{3}) \\ 4m+5-3i & (\frac{2m+3}{3} \le i \le m). \end{cases}$$

The fundamental idea to label the graph P_n^2 is as follows:

The edges of the graph P_n^2 can be partitioned into the following three paths:

(1) $x_1, x_2, x_4, x_5, x_7, x_8, x_{10}, x_{11}, \cdots$

(2) $x_2, x_3, x_5, x_6, x_8, x_9, x_{11}, x_{12}, \cdots$

(3) $x_1, x_3, x_4, x_6, x_7, x_9, x_{10}, x_{12}, \cdots$

Assigning appropriate labels to vertices x_{3i+1}, x_{3i+2} and x_{3i+3} respectively, we can obtain such edge-labels that the labels of most edges in every path as above-mentioned form sets of consecutive integers.

As an example, let us analyze the labels for P_{38}^2. The labels of vertices x_{3i+1} are:

$$\begin{matrix} x_1 & x_4 & x_7 & x_{10} & x_{13} & x_{16} & x_{19} & x_{22} & x_{25} & x_{28} & x_{31} & x_{34} & x_{37} \\ \underbrace{37 & 26 & 23 & 20 & 17 & 14 & 11 & 8}, & \underbrace{31 & 32 & 33 & 34}, & 38 \end{matrix}$$

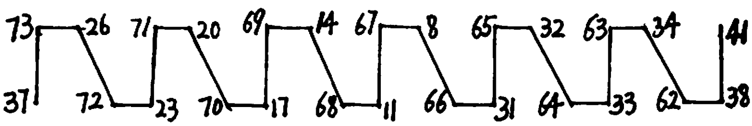
The labels of vertices x_{3i+2} are:

x_2	x_5	x_8	x_{11}	x_{14}	x_{17}	x_{20}	x_{23}	x_{26}	x_{29}	x_{32}	x_{35}	x_{38}
73	72	71	70	69	68	67	66	65	64	63	62	41

The labels of vertices x_{3i+3} are:

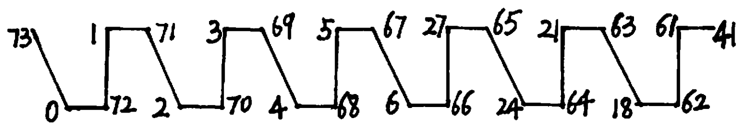
x_3	x_6	x_9	x_{12}	x_{15}	x_{18}	x_{21}	x_{24}	x_{27}	x_{30}	x_{33}	x_{36}
0	1	2	3	4	5	6	27	24	21	18	61

For path (1) consisting of vertices x_{3i+1} and x_{3i+2},



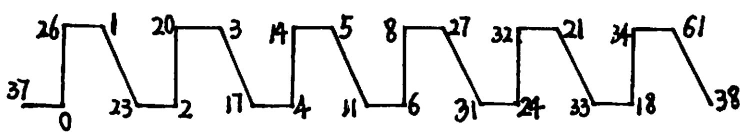
the edge-label are $\{36\} \cup [46, 59] \cup [28, 35] \cup \{24, 3\}$.

For path (2) consisting of vertices x_{3i+2} and x_{3i+3},



the edge-labels are $[60, 73] \cup [38, 45] \cup \{1, 20\}$.

For path (3) consisting of vertices x_{3i+3} and x_{3i+1},



the edge-labels are $\{37\} \cup [5, 25]_4 \cup [6, 26]_4 \cup \{2, 19, 4\} \cup [7, 15]_4 \cup$ $[8, 16]_4 \cup \{27, 23\}$.

For case $n = 6m$ the induced labels of the edges are as follows:

$$g^*(x_{3i+1}x_{3i+2})=\begin{cases}6m-2 & (i=0) & 6m-2 \\ 8m-4+2i & (1\le i\le m) & [8m-2,10m-4]_2 \quad (1) \\ 8m-2-2i & (m+1\le i\le 2m-2) & [4m+2,6m-4]_2 \\ 4m-4 & (i=2m-1) & 4m-4\end{cases}$$

$$g^*(x_{3i+2}x_{3i+3})=\begin{cases}12m-3-2i & (0\le i\le m-1) & [10m-1,12m-3]_2 \\ 4m+1+2i & (m\le i\le 2m-3) & [6m+1,8m-5]_2 \\ 1 & (i=2m-2) & 1 \\ 3 & (i=2m-1) & 3\end{cases}$$

$$g^*(x_{3i+3}x_{3i+4})=\begin{cases}4m-2-4i & (0\le i\le m-1) & [2,4m-2]_4 \quad (2) \\ 4i+4-4m & (m\le i\le 2m-3) & [4,4m-8]_4 \\ 8m-4 & (i=2m-2) & 8m-4\end{cases}$$

$$g^*(x_{3i+1}x_{3i+3})=\begin{cases}6m-1 & (i=0) & 6m-1 \\ 4m+1-4i & (1\le i\le m-1) & [5,4m-3]_4 \quad (3) \\ 4m-5 & (i=m) & 4m-5 \\ 4i+3-4m & (m+1\le i\le 2m-3) & [7,4m-9]_4 \\ 4m+1 & (i=2m-2) & 4m+1 \\ 4m-1 & (i=2m-1) & 4m-1\end{cases}$$

$$g^*(x_{3i+2}x_{3i+4})=\begin{cases}8m-1+2i & (0\le i\le m-1) & [8m-1,10m-3]_2 \quad (4) \\ 8m-3-2i & (m\le i\le 2m-3) & [4m+3,6m-3]_2 \\ 8m-3 & (i=2m-2) & 8m-3\end{cases}$$

$$g^*(x_{3i+3}x_{3i+5})=\begin{cases}12m-4-2i & (0\le i\le m-1) & [10m-2,12m-4]_2 \\ 4m+2i & (m\le i\le 2m-3) & [6m,8m-6]_2 \\ 4m & (i=2m-2) & 4m\end{cases}$$

When $m \equiv 2 \pmod 3$, by (*), the rows (1)-(4) will be respectively replaced with

$$(1)\begin{cases}4m+1-4i & (1\le i\le \frac{2m-1}{3}) & [\frac{4m+7}{3},4m-3]_4 \\ 8m-4+2i & (\frac{2m+2}{3}\le i\le m) & [\frac{28m-8}{3},10m-4]_2\end{cases}$$

$$(2)\begin{cases}8m-1+2i & (0\le i\le \frac{2m-4}{3}) & [8m-1,\frac{28m-11}{3}]_2 \\ 4m-2-4i & (\frac{2m-1}{3}\le i\le m-1) & [2,\frac{4m-2}{3}]_4\end{cases}$$

$$(3)\begin{cases}8m-4+2i & (1\le i\le \frac{2m-1}{3}) & [8m-2,\frac{28m-14}{3}]_2 \\ 4m+1-4i & (\frac{2m+2}{3}\le i\le m-1) & [5,\frac{4m-5}{3}]_4\end{cases}$$

$$(4)\begin{cases}4m-2-4i & (0\le i\le \frac{2m-4}{3}) & [\frac{4m+10}{3},4m-2]_4 \\ 8m-1+2i & (\frac{2m-1}{3}\le i\le m-1) & [\frac{28m-5}{3},10m-3]_2\end{cases}$$

It is not difficult to verify that the induced map g^* is a bijection from the set $\{x_i x_{i+1}; 1 \leq i \leq 6m-1\} \cup \{x_i x_{x+2}; 1 \leq i \leq 6m-2\}$ to the set $[1, 12m-3]$, whether $m \equiv 2 \pmod 3$ or not.

For other cases the proofs are similar.

3 The gracefulness of graphs $\mathbf{C_n^{+t}}$

Theorem 2. *The graph C_n^{+t} is graceful for any $n \geq 3$ and $1 \leq t \leq n$.*

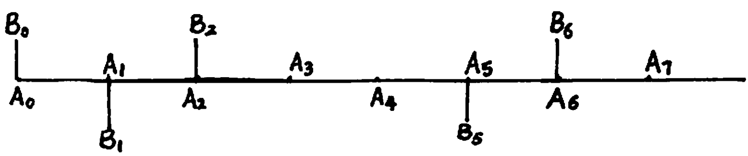
Construction: When $t = n$ the graph C_n^{+t} is just the Crown graph, which has been proved to be graceful [6]. Thus, our discussion will be only for $1 \leq t \leq n-1$.

All vertices on the cycle C_n are denoted by $A_0, \ldots, A_{n-1}$ successively such that there is a pendent edge adjoined to A_0 and there is no pendent edge adjoined to A_{n-1}. The pendent vertex of C_n^{+t}, which adjoins the cycle vertex A_i, is denoted by B_i. Note that the subscripts of all A_i run over interval $[0, n-1]$, but the subscripts of all B_i run only over a subset of $[0, n-1]$, with size t. All vertices in C_n^{+t}, except B_0 and A_{n-1}, are partitioned into two types as follows

M-type: A_{2k} and B_{2k+1};

N-type: A_{2k+1} and B_{2k}.

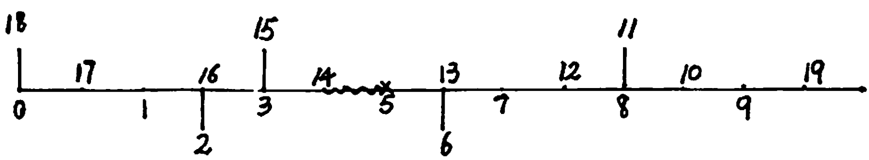
For each type, ordering vertices according the natural order of the subscripts, we get the M-sequence and the N-sequence, respectively. The first vertex in the M-sequence is A_0, and in the N-sequence the first vertex is A_1. For the edge $A_k A_{k+1}$, call A_k its *beginning* and A_{k+1} its *end*. For the edge $A_k B_k$, call A_k its *beginning* and B_k its *end*. For all edges except $A_0 B_0$, $A_{n-2} A_{n-1}$ and $A_{n-1} A_0$ we define the order "$\prec$"as follows: $A_{k-1} A_k \prec A_k A_{k+1}$ for $1 \leq k \leq n-2$, and $A_{k-1} A_k \prec A_k B_k \prec A_k A_{k+1}$ if there is a pendent edge $A_k B_k$ $(1 \leq k \leq n-2)$. The edge $A_0 A_1$ is called the first edge in the edge-sequence numbered. Suppose there are l pendent edges $A_{2k+1} B_{2k+1}$. Let $m = \lfloor \frac{n-1}{2} \rfloor + l$, where the notation $\lfloor x \rfloor$ denotes the greatest integer not greater than x. The end of the m-th edge in the edge-sequence is called *jump vertex*. For example, the following graph is a C_8^{+5}, where $n = 8$, $t = 5$, $l = 2$ and $m = 5$. Its M-sequence is $A_0, B_1, A_2, A_4, B_6, A_6$, N-sequence is A_1, B_2, A_3, A_5, B_6, and the other vertices are B_0 and A_7. The edge-sequence is $A_0 A_1$, $A_1 B_1$, $A_1 A_2$, $A_2 B_2$, $A_2 A_3$, $A_3 A_4$, $A_4 A_5$, $A_5 B_5$, $A_5 A_6, A_6 B_6$, and the other edges are $A_0 B_0$, $A_6 A_7$ and $A_7 A_0$. Thus the m-th edge is $A_2 A_3$ and the jump vertex is A_3.



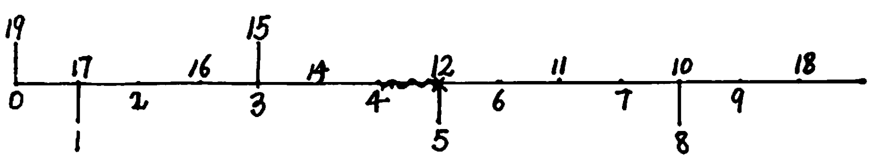
Now, let us give the graceful labeling f on all vertices A_k, and B_k as follows.

For the vertices in the M-sequence, the first vertex A_0 is labeled as $f(A_0) = 0$, and the labeling of the k-th vertex is 1 more than that of the $(k-1)$-th vertex except when the k-th vertex is the jump vertex (then it increases by 2). For the vertices in the N-sequence, the first vertex A_1 is labeled as $f(A_1) = n+t-2$, and the labeling of the k-th vertex is 1 less than that of the $(k-1)$-th vertex except when the k-th vertex is the jump vertex (then it decreases by 2). For vertices B_0 and A_{n-1}, define $f(B_0) = n+t-1$ (resp. $n+t$) and $f(A_{n-1}) = n+t$ (resp. $n+t-1$) when the jump vertex belongs to the M-sequence (resp. N-sequence).

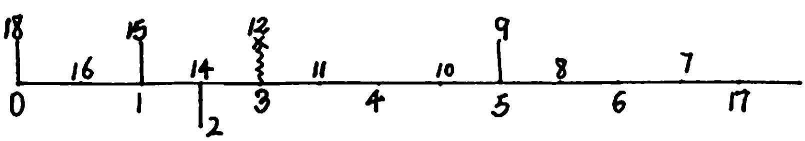
Examples. In each graph, the symbol "×" represents the jump vertex, and the symbol "~~" represents the m-th edge in the edges sequence.



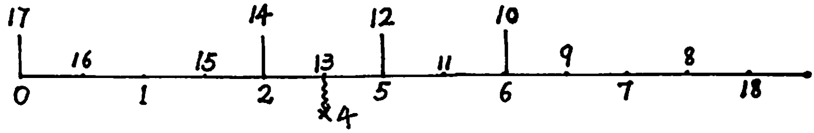
(n=14, t=5, l=2, m=8)



(n=14, t=5, l=3, m=9)

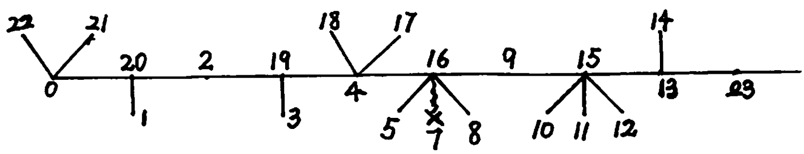


(n=13, t=5, l=1, m=7)



(n=13, t=5, l=1, m=7)

This labeling method can also be used to label the graph $C_n(r_1, \ldots, r_t)$, which is formed by adding $r_1 \ldots, r_t$ pendent edges to t vertices of C_n, respectively. For such a graph, denote by r_1 the number of pendent edges adjoined to A_0 and let $m = \lfloor \frac{n+1}{2} \rfloor + l - r_1$ (instead of $m = \lfloor \frac{n-1}{2} \rfloor + l$). And define $f(A_1) = n + (r_2 + \cdots + r_t) - 1$ (instead of $f(A_1) = n + t - 2$). We can still obtain a graceful labeling of $C_n(r_1 \ldots, r_t)$. As an example, we give a graceful labeling of a $C_{10}(2,1,1,2,3,3,1)$.



$(n = 10, l = 8, r_1 = 2, m = \lfloor \frac{10+1}{2} \rfloor + 8 - 2 = 11, n + \sum r_i = 23)$

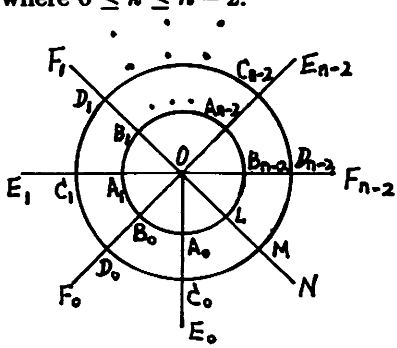
4 The gracefulness of Web graphs

Theorem 3. *Every Web graph $W(2,m)$ is graceful for $m \geq 3$.*

We will give graceful labelings of $W(2,m)$ for n odd and even, respectively.

Case: $\mathbf{m = 2n - 1}$

Denote $6n-2$ vertices of $W(2, 2n-1)$ by O (center), L, A_k, B_k (inner vertices), M, C_k, D_k (outer vertices), and N, E_k, F_k (pendent vertices) as in the following figure, where $0 \leq k \leq n-2$.



Define

$g(O) = O$, $g(L) = 10n - 5$, $g(M) = 5n - 3$, $g(N) = 8n - 3$, $g(E_0) = 2n$,

$g(A_k) = 9n + k - 4$, $g(B_k) = 2n - k - 2$, $g(C_k) = 2n + 3k - 1$,

$g(D_k) = 7n + k - 2$, where $0 \le k \le n - 2$,

$g(E_{2k}) = 3k + 1$, $1 \le k \le \lfloor \frac{n-2}{3} \rfloor$,

$$g(E_{2k-1}) = 3k - 1,\ 1 \le k \le \lfloor \tfrac{n}{3} \rfloor - \delta_n, \text{ where } \delta_n = \begin{cases} 1 & (n \equiv 0 \mod 6) \\ 0 & (\text{else}) \end{cases},$$

$g(E_{n-2k}) = 5n - 12k + 3$, $1 \le k \le \lfloor \frac{n+2}{3} \rfloor$,

$g(E_{n-2k-1}) = 5n - 12k - 2$, $1 \le k \le \lfloor \frac{n}{6} \rfloor$,

$g(F_k) = 7n - k - 5$, $0 \le k \le \lfloor \frac{n-5}{2} \rfloor$,

$$g(F_{n-2}) = \begin{cases} 5n - 1 & (n \equiv 0 \mod 6) \\ 5n - 5 & (\text{or else}) \end{cases},$$

$$g(F_{n-3}) = \begin{cases} 5n - 6 & (n \equiv 0 \mod 6) \\ 5n - 1 & (n \equiv 3 \mod 6), \\ 5n - 2 & (\text{or else}) \end{cases}$$

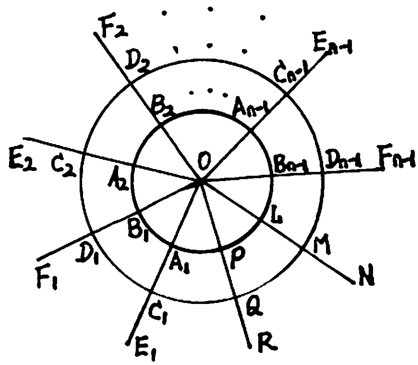
$$g(F_{n-3k-1}) = \begin{cases} 8n + 3k - 3 & (n \equiv 3 \mod 6) \\ 5n + 3k - 3 & (n \equiv 2 \mod 3) \\ 5n + 3k - 4 & (n \equiv 1 \mod 3) \\ 5n + 3k - 5 & (n \equiv 0 \mod 6) \end{cases} \quad 1 \le k \le \lfloor \tfrac{n+2}{6} \rfloor,$$

$$g(F_{n-3k-2}) = \begin{cases} 8n + 3k - 4 & (n \equiv 4,5 \mod 6) \\ 5n + 3k - 1 & (n \equiv 2 \mod 6) \\ 5n + 3k - 2 & (n \equiv 1 \mod 6) \\ 5n + 3k - 3 & (n \equiv 0 \mod 3) \end{cases} \quad 1 \le k \le \lfloor \tfrac{n}{6} \rfloor,$$

$$g(F_{n-3k-3}) = \begin{cases} 8n + 3k - 5 & (n \equiv 0,1,2 \mod 6) \\ 5n + 3k - 1 & (n \equiv 3 \mod 6) \\ 5n + 3k - 2 & (n \equiv 5 \mod 6) \\ 5n + 3k - 4 & (n \equiv 4 \mod 6) \end{cases} \quad 1 \le k \le \lfloor \tfrac{n-2}{6} \rfloor,$$

Case: m = 2n

Denote the $6n + 1$ vertices of $W(2, 2n)$ by O (center), L, P, A_k, B_k (inner vertices), M, Q, C_k, D_k (outer vertices), and N, R, E_k, F_k (pendent vertices) as in the following figure, where $1 \le k \le n - 1$ and $n > 3$.



Define

$g(O) = 0$, $g(L) = 10n$, $g(P) = 7n + 2$, $g(M) = 5n - 1$, $g(Q) = 5n + 2$,

$g(A_k) = 9n{+}k$, $g(B_k) = 2n{-}k{-}1$, $g(C_k) = 2n{+}3k{-}2$, $g(D_k) = 7n{+}k{+}2$, where $1 \le k \le n-1$,

$g(N) = 8n + 3$, $g(R) = 2n - 1$, $g(E_1) = 5n$,

$g(F_1) = 2n$, $g(E_{n-1}) = 5n - 6$, $g(F_{n-1}) = 5n + 1$,

$g(E_{2k}) = 3k - 1$ for $1 \le k \le \lfloor \frac{n-2}{3} \rfloor$,

$$g(E_{2k+1}) = 3k{+}1 \text{ for } 1 \le k \le \lfloor \tfrac{n-3}{3} \rfloor - \delta_n, \text{ where } \delta_n = \begin{cases} 1 & (n \equiv 0 \mod 6) \\ 0 & (\text{else}) \end{cases},$$

$$g(E_{n-2k}) = \begin{cases} 5n - 12k + 2 & (n \equiv 0,1,2,4 \mod 6) \\ 5n - 12k & (n \equiv 3,5 \mod 6) \end{cases} \quad 1 \le k \le \lfloor \tfrac{n+2}{6} \rfloor,$$

$$g(E_{n-2k-1}) = \begin{cases} 5n - 12k - 3 & (n \equiv 0,1,2,4 \mod 6) \\ 5n - 12k - 7 & (n \equiv 3,5 \mod 6) \end{cases} \quad 1 \le k \le \lfloor \tfrac{n}{6} \rfloor,$$

$g(F_k) = 7n - k + 1$, $2 \le k \le \lfloor \frac{n-2}{2} \rfloor$,

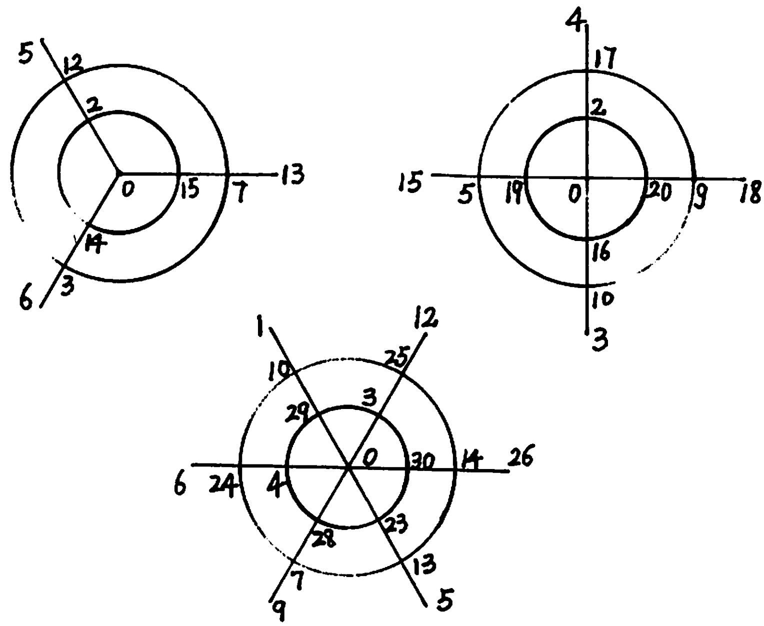
$$g(F_{n-2}) = \begin{cases} 5n + 3 & (n \equiv 0,1,4 \mod 6) \\ 8n + 2 & (n \equiv 3,5 \mod 6) \\ 8n + 6 & (n \equiv 2 \mod 6) \end{cases},$$

$$g(F_{n-3k}) = \begin{cases} 8n + 3k + 2 & (n \equiv 3,4 \mod 6) \\ 5n + 3k & (n \equiv 2 \mod 3) \\ 5n + 3k + 1 & (n \equiv 0 \mod 6) \\ 5n + 3k + 2 & (n \equiv 1 \mod 6) \end{cases} \quad 1 \le k \le \lfloor \tfrac{n+1}{6} \rfloor,$$

$$g(F_{n-3k-1}) = \begin{cases} 8n+3k+1 & (n \equiv 0,1,5 \mod 6) \\ 5n+3k & (n \equiv 3 \mod 6) \\ 5n+3k+1 & (n \equiv 4 \mod 6) \\ 5n+3k+2 & (n \equiv 2 \mod 6) \end{cases} \quad 1 \le k \le \lfloor \tfrac{n-1}{6} \rfloor,$$

$$g(F_{n-3k-2}) = \begin{cases} 8n+3k+6 & (n \equiv 2 \mod 6) \\ 5n+3k+1 & (n \equiv 5 \mod 6) \\ 5n+3k+2 & (n \equiv 0 \mod 3) \\ 5n+3k+3 & (n \equiv 1 \mod 3) \end{cases} \quad 1 \le k \le \lfloor \tfrac{n-3}{6} \rfloor.$$

For $m = 3, 4, 6$, we have the labels of $W(2,3)$, $W(2,4)$, $W(2,6)$ as follows:



Next, we give a proof only for case $m = 2n - 1$ as follows:

(1) The limits of the subscripts for the vertices E_i and F_i are appropriate. To see this, it is enough to show the following equalities:

i) If n is even, then

$$\left\lfloor\frac{n-2}{3}\right\rfloor+\left\lfloor\frac{n+2}{6}\right\rfloor=\frac{n-2}{2},$$
$$\left\lfloor\frac{n-3}{3}\right\rfloor+\left\lfloor\frac{n}{6}\right\rfloor=\begin{cases}\frac{n-2}{2} & (\text{if } n\equiv 0 \mod 6)\\ \frac{n-4}{2} & (\text{else})\end{cases}$$

ii) If n is odd, then

$$\left\lfloor\frac{n-3}{3}\right\rfloor+\left\lfloor\frac{n+2}{6}\right\rfloor=\frac{n-3}{2},$$
$$\left\lfloor\frac{n-2}{3}\right\rfloor+\left\lfloor\frac{n}{6}\right\rfloor=\frac{n-3}{2},$$
$$\left\lfloor\frac{n-2}{2}\right\rfloor+3\left\lfloor\frac{n+1}{6}\right\rfloor=n-1 \quad (n\equiv 0,5 \mod 6),$$
$$\left\lfloor\frac{n-2}{2}\right\rfloor+3\left\lfloor\frac{n-1}{6}\right\rfloor=n-2 \quad (n\equiv 1,2 \mod 6),$$
$$\left\lfloor\frac{n-2}{2}\right\rfloor+3\left\lfloor\frac{n-3}{6}\right\rfloor=n-3 \quad (n\equiv 3,4 \mod 6).$$

(2) All vertices receive distinct labels.

Below, the notation $A\cup B\cup C$ is vested with the extra meaning $A\cap B=A\cap C=B\cap C=\phi$. From the labeling of the vertices we have the following facts:

i) $g(O)\cup\{g(E_{2k})\}\cup\{g(E_{2k-1})\}\subseteq[0,n-1]$. In fact,

$$4<g(E_{2k})<3\left\lfloor\frac{n-2}{3}\right\rfloor+1\le n-1, g(E_{2k})\equiv 1\equiv 4 \pmod 3;$$
$$2<g(E_{2k-1})<3(\left\lfloor\frac{n}{3}\right\rfloor-\delta)-1\le n-1, g(E_{2k-1})\equiv 2 \pmod 3;$$

ii) $\{g(B_k)\}=[n,2n-2]$;

iii) $g(E_0)\cup\{g(E_{n-2k})\}\cup\{g(E_{n-2k-1})\}\cup\{g(C_k)\}\subseteq[2n-1,5n-7]$. Since

$$5n-9\ge g(E_{n-2k})\ge 5n-12\left\lfloor\frac{n+2}{6}\right\rfloor+3\ge 3n-1,$$
$$g(E_{n-2k})\equiv 5n-9\equiv 2n \pmod 3;$$
$$5n-14\ge g(E_{n-2k-1})\ge 5n-12\left\lfloor\frac{n}{6}\right\rfloor-2\ge 3n-2,$$
$$g(E_{n-2k-1})\equiv 5n-14\equiv 2n+1 \pmod 3;$$
$$5n-7\ge g(C_k)\ge 2n-1, g(C_k)\equiv 5n-7\equiv 2n+2 \pmod 3;$$

iv) $\{g(F_k)\} \subset [6.5n-2.5, 7n-5]$ since $7n - \lfloor\frac{n-5}{2}\rfloor - 5 \geq 6.5n - 2.5$;

v) $\{g(D_k)\} \cup g(N) = [7n-2, 8n-3]$;

vi) $\{g(A_k)\} \cup g(L) = [9n-4, 10n-5]$;

vii) $\{g(M), g(F_{n-2}), g(F_{n-3})\} \cup \{g(F_{n-3k-1})\} \cup \{g(F_{n-3k2})\} \cup \{g(F_{n-3k-3})\} \subseteq [5n-6, 5.5n-2] \cup [8n-2, 8.5n-4.5]$.

In fact, we have the table of values $g(F_i)$, where "$n \equiv$" means "$n \equiv$ (mod 6)".

$n \equiv$	$g(F_{n-2})$	$g(F_{n-3})$	$g(F_{n-3k-1})$	$g(F_{n-3k-2})$	$g(F_{n-3k-3})$
0	$5n-1$	$5n-6$	$[5n-2, 5.5n-5]_3$	$[5n, 5.5n-3]_3$	$[8n-2, 8.5n-8]_3$
1	$5n-5$	$5n-2$	$[5n-1, 5.5n-4.5]_3$	$[5n+1, 5.5n-2-5]_3$	$[8n-2, 8.5n-8.5]_3$
2	$5n-5$	$5n-2$	$[5n, 5.5n-4]_3$	$[5n+2.5, 5n-2]_3$	$[8n-2, 8.5n-6]_3$
3	$5n-5$	$5n-1$	$[8n, 8.5n-4.5]_3$	$[5n, 5.5n-4.5]_3$	$[5n+2, 5.5n-2.5]_3$
4	$5n-5$	$5n-2$	$[5n-1, 5.5n-3]_3$	$[8n-1, 8.5n-6]_3$	$[5n, 5.5n-5]_3$
5	$5n-5$	$5n-2$	$[5n, 5.5n-2.5]_3$	$[8n-1, 8.5n-6.5]_3$	$[5n+1, 5.5n-4.5]_3$

Therefore, from i)-vii), it is easy to see that all vertices in $W(2, 2n-1)$ receive distinct labels in $[0, 10n-5]$ when $n > 3$.

(3) The set $[1, 10n-5]$ is just filled with the induced labels of all edges in $W(2, 2n-1)$.

Below, for brevity, we will denote the edge X_kY_k by $(XY)_k$. The ranges of the induced labels for all edges are as follows:

i) $g^*(LA_0) \cup \{g^*(OB_k)\} = [n-1, 2n-2]$.

ii) $g^*(MC_0) \cup g^*(MD_{n-2}) \cup g^*(MN) = [3n-2, 3n]$.

iii) $\{g^*(CD)_k\} \cup \{g^*(C_{k+1}D_k)\} \cup g^*(LM) = [3n+2, 5n-1]$. This is because

$$3n+3 < g^*(CD) + k = 5n - 2k + 1 < 5n - 1 (0 < k < n-2),$$
$$g^*(CD)_k \equiv n+1 \pmod 2;$$
$$3n+2 < g^*(C_{k+1}D_k) = 5n - 2k - 4 < 5n - 4 (0 < k < n-3),$$
$$g^*(C_{k+1}D_k) \equiv n \pmod 2;$$
and $g^*(LM) = 5n-2$.

iv) $\{g^*(AB)_k\} \cup \{g^*(A_{k+1}B_k)\} \cup \{g^*(AC)_k\} \cup \{g^*(BD)_k\} = [5n, 9n-6]$.

This is because

$$7n-2 < g^*(AB)_k = 7n+2k-2 < 9n-6 (0 < k < n-2),$$
$$g^*(AB)_k \equiv n \pmod 2;$$
$$7n-1 < g^*(A_{k+1}B_k) = 7n+2k-1 < 9n-7 (0 < k < n-3),$$
$$g^*(A_{k+1}B_k) \equiv n-1 \pmod 2;$$
$$5n+1 < g^*(AC)_k = 7n-2k-3 < 7n-3 (0 < k < n-2),$$
$$g^*(AC)_k \equiv n+1 \pmod 2;$$
$$5n < g^*(BD)_k = 5n+2k < 7n-4 (0 < k < n-2),$$
$$g^*(BD)_k \equiv n \pmod 2.$$

v) $\{g^*(OA)_k\} \cup g^*(OL) \cup g^*(LB_{n-2}) = [9n-5, 10n-5]$.

vi) $\{g^*(CE)_k\} \cup \{g^*(DF)_k\} = [1, n-2] \cup [2n-1, 3n-3] \cup 3n+1$. In fact, we have the table of values $\{g^*(CE)_k\}$ and $\{g^*(DF)_k\}$:

$n \equiv \pmod 6$	0	1	2
$g^*(CE_0)$	1	1	1
$g^*(CE)_{n-2k-2}$	$[2, n-4]_6$	$[2, n-5]_6$	$[2, n-6]_6$
$g^*(CE)_{n-2k-1}$	$[4, n-2]_6$	$[4, n-3]_6$	$[4, n-4]_6$
$g^*(DF)_k$	$[3, n-3]_2$	$[3, n-2]_2$	$[3, n-3]_2$
$g^*(DF)_{n-3k-3}$	$[6, n-6]_6$	$[6, n-7]_6$	$[6, n-2]_6$
$g^*(DF)_{n-3k-1}$	$[2n+2, 3n-4]_6$	$[2n+2, 3n-5]_6$	$[2n+2, 3n-6]_6$
$g^*(DF)_{n-3k-2}$	$[2n-1, 3n-7]_6$	$[2n-1, 3n-8]_6$	$[2n-1, 3n-9]_6$
$g^*(CE)_{2k}$	$[2n+1, 3n-5]_3$	$[2n+1, 3n-6]_3$	$[2n+1, 3n-4]_3$
$g^*(CE)_{2k-1}$	$[2n, 3n-6]_3$	$[2n, 3n-4]_3$	$[2n, 3n-5]_3$
$g^*(DF)_{n-3}$	$3n+1$	$3n-3$	$3n-3$
$g^*(DF)_{n-2}$	$3n-3$	$3n+1$	$3n+1$

$n \equiv \pmod 6$	3	4	5
$g^*(CE_0)$	1	1	1
$g^*(CE)_{n-2k-2}$	$[2, n-7]_6$	$[2, n-2]_6$	$[2, n-3]_6$
$g^*(CE)_{n-2k-1}$	$[4, n-5]_6$	$[4, n-6]_6$	$[4, n-7]_6$
$g^*(DF)_k$	$[3, n-2]_2$	$[3, n-3]_2$	$[3, n-2]_2$
$g^*(DF)_{n-3k-3}$	$[2n-1, 3n-10]_6$	$[2n+2, 3n-8]_6$	$[2n+2, 3n-9]_6$
$g^*(DF)_{n-3k-1}$	$[6, n-3]_6$	$[2n-1, 3n-5]_6$	$[2n-1, 3n-6]_6$
$g^*(DF)_{n-3k-2}$	$[2n+2, 3n-7]_6$	$[6, n-4]_6$	$[6, n-5]_6$
$g^*(CE)_{2k}$	$[2n+1, 3n-5]_3$	$[2n+1, 3n-6]_3$	$[2n+1, 3n-4]_3$
$g^*(CE)_{2k-1}$	$[2n, 3n-3]_3$	$[2n, 3n-4]_3$	$[2n, 3n-5]_3$
$g^*(DF)_{n-3}$	$3n-4$	$3n-3$	$3n-4$
$g^*(DF)_{n-2}$	$3n+1$	$3n+1$	$3n+1$

Thus, by i)-vii), it is trivial to see that g^* maps $E(W(2, 2n-1))$ onto $[1, 10n-5]$.

One of the authors, Gui-hua Yang, has obtained graceful labeling for $W(3,n)$ and $W(4,n)$, which are extensions of the concept of a Web graph $W(2,n)$. In general, a graph $W(t,n)$ has t circles and n rays.

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