Existence of Some (3,2,1)-HCOLS and (3,2,1)-ICOILS

Hantao Zhang*
Computer Science Department
The University of Iowa
Iowa City, IA 52242
hzhang@cs.uiowa.edu

Frank E. Bennett†
Department of Mathematics
Mount Saint Vincent University
Halifax, Nova Scotia B3M 2J6
frank.bennett@msvu.ca

ABSTRACT. A Latin square $(S, *)$ is said to be (3,2,1)-*conjugate-orthogonal* if $x * y = z * w$, $x *_{321} y = z *_{321} w$ imply $x = z$ and $y = w$, for all $x, y, z, w \in S$, where $x_3 *_{321} x_2 = x_1$ if and only if $x_1 * x_2 = x_3$. Such a Latin square is said to be *holey* ((3,2,1)-HCOLS for short) if it has disjoint and spanning holes corresponding to missing sub-Latin squares. Let (3,2,1)-HCOLS(h^n) denote a (3,2,1)-HCOLS of order hn with n holes of equal size h. We show that, for any $h \geq 1$, a (3,2,1)-HCOLS(h^n) exists if and only if $n \geq 4$, except $(n, h) = (6, 1)$ and except possibly $(n, h) = (6, 13)$. In addition, we show that a (3,2,1)-HCOLS with n holes of size 2 and one hole of size 3, exists if and only if $n \geq 4$, except for $n = 4$ and except possibly $n = 17$, 18, 19, 21, 22 and 23. Let (3,2,1)-ICOILS(v, k) denote an idempotent (3,2,1)-COLS of order v with a hole of size k. We provide 15 new (3,2,1)-ICOILS(v, k), where $k = 2$, 3 or 5.

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1 Introduction

Let $(S,*)$ be a quasigroup where the multiplication table of $*$ forms a Latin square indexed by S. The (i,j,k)-conjugate of $(S,*)$ is $(S,*_{ijk})$, where (i,j,k) is a permutation of (1,2,3) and $x_i *_{ijk} x_j = x_k$ if and only if $x_1 * x_2 = x_3$. Following the convention (see [2]), we call $(S,*)$ a Latin square. A Latin square is said to be (i,j,k)-*conjugate-orthogonal* ((i,j,k)-COLS for short) if $x*y = z*w$ and $x *_{ijk} y = z *_{ijk} w$ imply $x = z$ and $y = w$, where $x*y$ denotes the entry in the cell (x,y) of the square. We will use (i,j,k)-HCOLS$(h_1^{n_1} \ldots h_k^{n_k})$ to denote the type of holey (i,j,k)-COLS of order $\sum_{i=1}^{k} h_i n_i$, that have n_i holes of size h_i, $1 \le i \le k$, and all the holes are assumed to be mutually disjoint, and each of them corresponds to a missing sub-Latin square. It is well-known that there does not exist any (1,2,3)-HCOLS(h^n) for $n > 1$; a (1,3,2)-HCOLS(h^n) exists if and only if a (3,2,1)-HCOLS(h^n) exists; a (2,3,1)-HCOLS(h^n) exists if and only if a (3,1,2)-HCOLS(h^n) exists.

The existence of (2,1,3)-HCOLS(h^n) has been completely settled [2, 5]. In this paper, using a similar approach, we provide an alomost conclusive result to the existence of (3,2,1)-HCOLS. Note that an idempotent (3,2,1)-COLS of order v can be written as a (3,2,1)-HCOLS(1^v). An incomplete idempotent (3,2,1)-COLS of order v with a hole of size k, denoted by (3,2,1)-ICOILS(v,k), exists if and only if a (3,2,1)-HCOLS$(1^{v-k}k^1)$ exists.

The previous results concerning the existence of (3,2,1)-HCOLS(h^n) are summarized in the following theorem:

Theorem 1.1. ([1,2]) *There exists a (3,2,1)-HCOLS(h^n) if and only if $h \ge 1$ and $n \ge 4$, except $(n,h) = (6,1)$, and except possibly when $(n,h) = (12,1)$, when $n \in \{8,9,12\}$ and $h = 2$, and when $n = 6$ and $h \in \{5,7,13\}$.*

In this paper, we remove all but $(n,h) = (6,13)$ from the possible exceptions in the above theorem and thus obtain the following:

Theorem 1.2. *There exists a (3,2,1)-HCOLS(h^n) if and only if $h \ge 1$ and $n \ge 4$, except $(n,h) = (6,1)$ and possibly excepting $(n,h) = (6,13)$.*

In addition, we provide an almost conclusive result regarding the existence of (3,2,1)-HCOLS with n holes of size 2 and one hole of size 3:

Theorem 1.3. *There exists a (3,2,1)-HCOLS$(2^n 3^1)$ if and only if $n \ge 4$, except for $n = 4$ and except possibly $n = 17$, 18, 19, 21, 22 and 23.*

The previous result regarding the existence of (3,2,1)-ICOILS(v,k) is summarized in the following:

Theorem 1.4. ([2]) *For any integer $v \ge 1$, a (3,2,1)-ICOILS(v,k) exists if $v \ge (13/4)k+88$. For $2 \le k \le 6$, a (3,2,1)-ICOILS(v,k) exists if $v \ge 3k+1$*

except possibly when $(v,k) = (30,5)$ *and when*

$$\begin{aligned} k &= 2, \quad v \in \{16,17,19,20,21,23\}, \\ k &= 3, \quad v \in \{11,20,21,24,25,26,28,29,30\}. \end{aligned}$$

We are able to solve all the open cases when $2 \le k \le 5$ except $(v,k) = (11,3)$. That is, we have the following:

Theorem 1.5. *For* $2 \le k \le 6$, *a (3,2,1)-ICOILS*(v,k) *exists if and only if* $v \ge 3k+1$, *except possibly when* $(v,k) = (11,3)$.

The construction techniques that we used are conventional (such as the cyclic construction, the fill-in-holes construction and the group-divisible designs) and can be found in the survey paper [2]. The use of these techniques is similar to that of [5] where the existence of (2,1,3)-HCOLS(2^n3^1) is established.

The direct constructions reported in the paper were obtained by a computer program [6]. This program is a general theorem prover for propositional reasoning and has been used to solve various Latin square problems. The heavy use of computer techniques is crucial to our success.

2 (3,2,1)-HCOLS(h^n)

The new designs of (3,2,1)-HCOLS(h^n) were obtained by a starter-adder type construction, called the cyclic construction, which constructs a (3,2,1)-HCOLS of type h^nk^1 from its first row and first column using an Abelian group. In [2], this technique is described using the Abelian group Z_{hn} Below we present the construction using an arbitrary Abelian group of order hn.

The Cyclic Construction. Let $(G,+)$ be an Abelian group of order m and H a subgroup of order h. In general, we assume $G = \{0,1,\ldots,m-1\}$ and $H = \{i(m/h) \mid 0 \le i \le h\}$. Let $X = \{x_1,\ldots,x_k\} = \{m,\ldots,m+k-1\}$ and $S = G \cup X$. Suppose that L is a (3,2,1)-HCOLS(h^nk^1) based on S with a hole indexed by $X \times X$, and m/h holes indexed by $(g+H) \times (g+H)$, where $g+H$ runs over all cosets of H in G. We will denote by $(i * j)$ the entry in the cell (i,j) of L. The first row is given by the two vectors $\mathbf{e} = (0*0,\ldots,0*(m-1))$ and $\mathbf{f} = (0*m,\ldots,0*(m+k-1))$, and the last k elements of the first column are given by the vector $\mathbf{g} = (m*0,\ldots,(m+k-1)*0)$. For $a \in H$, we let $(0*a) = \emptyset$, which means that the cell $(0,a)$ is empty. The entire L is constructed from $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}$ as follows:

1. For $s \in G$ and $t \in G$,

$$s*(s+t) = \begin{cases} (0*t)+s & \text{if } (0*t) \in G \\ (0*t) & \text{otherwise.} \end{cases}$$

2. For $s \in G$, $t \in X$, $s * t = (0 * t) + s$.

3. For $s \in X$, $t \in G$, $s * t = (s * 0) + t$.

Note that + is the one in the Abelian group $(G, +)$.

There are obviously conditions that the vectors **e**, **f** and **g** must satisfy in order to produce a (3,2,1)-HCOLS($h^n k^1$) and they are given in the following lemma.

Lemma 2.1. *Let L be a holey Latin square generated by the cyclic construction using the Abelian group $(G, +)$. L is a (3,2,1)-HCOLS($h^n k^1$) if and only if* (i) *for any $x \notin H$, $0 * x \notin H$;* (ii) *for any $y \in G$, either $0 * y \in G$ or $y * 0 \in G$; and* (iii) *the following difference conditions hold:*

$$\begin{aligned} &\{(0 * x) + -(0 *_{321} x) \mid 0 * x \in G, x * 0 \in G, x \in S, x \notin H\} \\ \cup\ &\{(x * 0) + -(x *_{321} 0) \mid x \in X\} \\ =\ &G - H, \end{aligned}$$

where $-(x)$ is the inverse of x in the Abelian group $(G, +)$.

Lemma 2.2. *There exists a (3,2,1)-HCOLS(1^{12}), or equivalently, an idempotent (3,2,1)-COLS(12).*

Proof: Let **e** = (0 6 9 5 2 7 3 1 10 4 11 8), **f** = **g** = $\emptyset$. Using the Abelian group $Z_2 \times Z_2 \times Z_3$, we obtain a Latin square, shown in Figure 1, satisfying Lemma 2.1. □

In the following, when we use the cyclic construction to obtain a (3,2,1)-HCOLS($h^n k^1$), we always use the Abelian group Z_{hn}.

(a)

*	0	1	2	3	4	5	6	7	8	9	a	b
0	0	6	9	5	2	7	3	1	a	4	b	8
1	a	1	7	8	3	0	b	4	2	6	5	9
2	8	b	2	1	6	4	0	9	5	a	7	3
3	2	5	a	3	9	6	1	8	b	0	4	7
4	b	0	3	7	4	a	9	2	6	8	1	5
5	4	9	1	b	8	5	7	a	0	3	6	2
6	9	7	4	a	5	2	6	0	3	b	8	1
7	5	a	8	0	b	3	4	7	1	2	9	6
8	6	3	b	4	1	9	2	5	8	7	0	a
9	7	2	5	6	a	1	8	b	4	9	3	0
a	3	8	0	2	7	b	5	6	9	1	a	4
b	1	4	6	9	0	8	a	3	7	5	2	b

(b)

+	0	1	2	3	4	5	6	7	8	9	a	b
0	0	1	2	3	4	5	6	7	8	9	a	b
1	1	2	0	4	5	3	7	8	6	a	b	9
2	2	0	1	5	3	4	8	6	7	b	9	a
3	3	4	5	0	1	2	9	a	b	6	7	8
4	4	5	3	1	2	0	a	b	9	7	8	6
5	5	3	4	2	0	1	b	9	a	8	6	7
6	6	7	8	9	a	b	0	1	2	3	4	5
7	7	8	6	a	b	9	1	2	0	4	5	3
8	8	6	7	b	9	a	2	0	1	5	3	4
9	9	a	b	6	7	8	3	4	5	0	1	2
a	a	b	9	7	8	6	4	5	3	1	2	0
b	b	9	a	8	6	7	5	3	4	2	0	1

Figure 1.
(a) An idempotent (3,2,1)-COLS(12);
(b) The Abelian group used to obtain (a)

Lemma 2.3. *There exists a (3,2,1)-HCOLS(h^n) for* $(n,h) = (8,2), (9,2), (12,2), (6,5)$ *and* $(6,7)$.

Proof: It is sufficient to give the vectors **e**, **f** and **g**, as shown in Figure 2, which satisfy Lemma 2.1. □

type	e, f, g
2^8	$(\emptyset\ 11\ 10\ 4\ 6\ 2\ x_1\ \emptyset\ 12\ 1\ 13\ 9\ 3\ x_2), (5\ 8), (9\ 13)$
2^9	$(\emptyset\ 15\ 12\ 7\ 11\ 3\ 1\ 6\ 13\ \emptyset\ 16\ 5\ 14\ 10\ 4\ 8\ 17\ 2), \emptyset, \emptyset$
2^{12}	$(\emptyset\ 23\ 4\ 11\ 7\ 16\ 3\ 17\ 15\ 18\ 1\ 5\ \emptyset\ 6\ 9\ 21\ 20\ 13\ 19\ 8\ 10\ 2\ 14\ 22), \emptyset, \emptyset$
5^6	$(\emptyset\ 23\ 21\ x_4\ 22\ \emptyset\ 18\ 14\ x_1\ 12\ \emptyset\ x_3\ 16\ 9\ 3\ \emptyset\ 7\ 1\ 4\ 2\ \emptyset\ 13\ x_2\ 24\ x_5),$ $(6\ 8\ 11\ 17\ 19), (13\ 2\ 6\ 23\ 24)$
7^6	$(\emptyset\ 2\ 21\ 6\ 13\ \emptyset\ 17\ 19\ 22\ 31\ \emptyset\ 24\ 16\ 1\ 3\ \emptyset\ 18\ 33\ 4\ 26\ \emptyset\ 12\ 14\ 29\ x_4\ \emptyset\ 34\ x_5\ 11\ x_3$ $\emptyset\ x_6\ x_2\ x_7\ x_1), (7\ 8\ 9\ 23\ 27\ 28\ 32), (34\ 29\ 28\ 17\ 31\ 32\ 33)$
$2^5 3^1$	$(\emptyset\ 2\ 6\ x_2\ x_1\ \emptyset\ 3\ 9\ 4\ x_3), (1\ 7\ 8), (3\ 8\ 9)$
$2^6 3^1$	$(\emptyset\ x_1\ x_3\ 5\ 3 x_2\ \emptyset\ 8\ 1\ 4\ 2\ 9), (7\ 10\ 11), (3\ 8\ 9)$
$2^7 3^1$	$(\emptyset\ 5\ 8\ 4\ 13\ 2\ 9\ 10\ x_1\ 6\ x_2\ 3 x_3), (1\ 11\ 12), (8\ 12\ 13)$
$2^8 3^1$	$(\emptyset\ x_1\ 5\ 7\ 3\ x_2\ x_3\ 1\ \emptyset\ 11\ 6\ 4\ 10\ 2\ 9\ 12), (13\ 14\ 15), (1\ 6\ 7)$
$2^9 3^1$	$(\emptyset\ 2\ 5\ 14\ 16\ 11\ 10\ 15\ 13\ \emptyset\ x_1\ 7\ 4\ 8\ 3\ 17\ x_2\ x_3), (1\ 6\ 12), (15\ 16\ 17)$
$2^{10} 3^1$	$(\emptyset\ 8\ x_1\ 1\ 5\ 4\ 12\ 15\ 11\ 3\ \emptyset\ 13\ 7\ x_3\ x_2\ 6\ 9\ 2\ 14\ 16), (17\ 18\ 19), (4\ 9\ 12)$
$2^{11} 3^1$	$(\emptyset\ 13\ 16\ 7\ 10\ 21\ 8\ 14\ 1\ 4\ 19\ \emptyset\ 15\ 9\ 5\ 20\ 17\ 3\ 6\ x_1\ x_2\ x_3), (2\ 12\ 18), (19\ 20\ 21)$
$2^{12} 3^1$	$(\emptyset\ 6\ 17\ 11\ 10\ x_1\ x_3\ 9\ 3\ 18\ 2\ 1\ \emptyset\ 14\ 7\ 19\ 15\ 13\ 5\ 8\ x_2\ 4\ 16\ 20), (21\ 22\ 23),$ $(3\ 10\ 22)$
$2^{13} 3^1$	$(\emptyset\ 6\ 20\ 5\ 24\ 1\ 12\ 22\ 9\ x_1\ 19\ 23\ 15\ \emptyset\ 18\ 10\ 7\ 25\ 2\ 4\ 8\ 11\ 3\ 16\ x_2\ x_3),$ $(14\ 17\ 21), (23\ 24\ 25)$
$2^{14} 3^1$	$(\emptyset\ 13\ 6\ 20\ 7\ 23\ 19\ 3\ x_3\ 8\ x_2\ 9\ 5\ 15\ \emptyset\ 21\ 17\ 22\ 1\ x_1\ 11\ 2\ 16\ 10\ 12\ 4\ 18\ 24),$ $(25\ 26\ 27), (8\ 10\ 23)$
$2^{15} 3^1$	$(\emptyset\ 25\ 20\ 16\ 8\ 12\ 7\ 19\ 13\ 2\ 24\ 1\ 3\ 9\ 6\ \emptyset\ 14\ 26\ 21\ 18\ 22\ 10\ 17\ 4\ 11\ 5\ 23\ x_1\ x_2\ x_3),$ $(27\ 28\ 29), (6\ 16\ 8)$

Figure 2.
Vectors for some (3,2,1)-HCOLS(h^n) and (3,2,1)-HCOLS(2^n3^1).

Remark. Combining Lemmas 2.2 and 2.3 with Theorem 1.1, we have essentially proved Theorem 1.2.

3 (3,2,1)-HCOLS(2^n3^1)

Because a necessary condition for the existence of a (3,2,1)-COLS(h^nk^1) is that $n \geq 1 + 2(k/h)$, there is no (3,2,1)-HCOLS(2^n3^1) for $n \leq 3$. A (3,2,1)-HCOLS(2^43^1) does not exist by exhaustive computer search.

Lemma 3.1. *There exists a (3,2,1)-HCOLS(2^n3^1) for* $5 \leq n \leq 15$.

Proof: We give the vectors **e**, **f** and **g**, as shown in Figure 2, which satisfy Lemma 2.1. □

The use of recursive techniques is similar to that of [5].

Lemma 3.2. (Filling in holes)

(1) *If there exist a (3,2,1)-HCOLS($2^m h^1$) and a (3,2,1)-HCOLS($2^n u^1$) where $h = 2n + u$, then there exists a (3,2,1)-HCOLS($2^{m+n}u^1$).*

(2) *If there exists a (3,2,1)-HCOLS($(2m_1)^1(2m_2)^1 \cdots (2m_k)^1 h^1$) and a (3,2,1)-HCOLS($2^{m_i}u^1$) for $1 \le i \le k$, then there exists a (3,2,1)-HCOLS($2^n v^1$) where $n = m_1 + m_2 + \cdots + m_k$ and $v = h + u$.*

The weighting construction uses group-divisible designs [4, 5]. A *group-divisible design* (GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$, which satisfies the following properties:

1. $\mathcal{G}$ is a partition of X into subsets called *groups*.
2. $\mathcal{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point.
3. Every pair of points from distinct groups occurs in a unique block.

The following construction is used in [4]; see also [2, 5].

Lemma 3.3. (Weighting) *Let $(X, \mathcal{G}, \mathcal{B})$ be a GDD and let $w\colon X \to Z^+ \cup \{0\}$ be a weighing. Suppose that there exists a (3,2,1)-HCOLS of type $w(B)$ for every $B \in \mathcal{B}$. Then there exists a (3,2,1)-HCOLS of type $\{\sum_{x \in G} w(x)\colon G \in \mathcal{G}\}$.*

For most of our recursive constructions, we will make use of transversal designs. A *transversal design* TD(k, n) is a GDD with kn points, k groups of size n, and n^2 blocks of size k. It is well known that a TD(k, n) is equivalent to $k - 2$ MOLS of order n.

Lemma 3.4. ([3]) *There exists a TD$(5, m)$ if $m \ge 4$ and $m \ne 6$, 10 and a TD$(6, m)$ exists for all odd $m \ge 5$ and $m = 8$.*

Lemma 3.5. *Suppose a TD$(6, m)$ exists, $0 \le r \le m$, and $0 \le s \le m$. Then there exists a (3,2,1)-HCOLS of type $(2m)^4(2r)^1(2s)^1$.*

Proof: In a TD$(6, m)$ we give weight two to each point in the first four groups. In the fifth group, we give weight two to r points and weight zero to the other points. In the last group, we give weight two to s points and weight zero to the remaining points. We use the fact that there exist (3,2,1)-HCOLS of types 2^4, 2^5 and 2^6 to get the result from Lemma 3.3. □

Lemma 3.6. *There exists a (3,2,1)-HCOLS($2^n 3^1$) for $n = 20$, 25, 28, 30 and $32 \le n \le 90$.*

Proof: We first apply Lemmas 3.4 and 3.5 with $m \in \{5, 7, 8, 9, 11, 15\}$, and suitable choices of r and s as indicated in Table 1. Note that there exist (3,2,1)-HCOLS of types 2^m3^1, 2^r3^1 and 2^s3^1, so Table 1 guarantees the existence of (3,2,1)-HCOLS(2^n3^1), where $n = 4m + r + s$, by applying Lemma 3.2 (2). □

m	r	s	$n = 4m + r + s$
5	0,5	0,5	20,25,30
7	0,5 - 7	0,5 - 7	28,33 - 42
8	0	0	32
9	5 - 9	0,5 - 9	41 - 59
11	5 - 11	0,5 - 11	49 - 66
15	5 - 15	0,5 - 15	65 - 90

Table 1, Some (3,2,1)-HCOLS(2^n3^1)

Lemma 3.7. *There exists a (3,2,1)-HCOLS($2^{29}3^1$).*

Proof: From a TD(5,12) we delete two points from one group so as to form a $\{4,5\}$-GDD of type 12^410^1. Giving each point of this GDD weight one, we get a (3,2,1)-HCOLS(12^410^1). We then adjoin three infinite points to this design, using (3,2,1)-HCOLS of types 2^63^1 and 2^53^1 to get a (3,2,1)-HCOLS of type $2^{29}3^1$ by Lemma 3.2. □

Lemma 3.8. *There exists a (3,2,1)-HCOLS(2^n3^1) for $n = 16$ and 24.*

Proof: For $n = 16$, we start with a TD(5,8) and delete 5 points from one group to get a $\{4,5\}$-GDD of type 8^43^1. Giving each point of this GDD weight one, we get a (3,2,1)-HCOLS(8^43^1). Filling in the holes of size 8 with (3,2,1)-HCOLS(2^4), we get a (3,2,1)-HCOLS($2^{16}3^1$). Similarly, for $n = 24$, we start with a TD(5,12) and delete 9 points from one group so as to form a $\{4,5\}$-GDD of type 12^43^1. From this GDD, we get a (3,2,1)-HCOLS(12^43^1) and then a (3,2,1)-HCOLS($2^{24}3^1$), by filling in the holes of size 12 with a (3,2,1)-HCOLS(2^6). □

Lemma 3.9. *There exists a (3,2,1)-HCOLS(2^n3^1) for $n = 26$ and 27.*

Proof: We start with a TD(7,7) and adjoin an infinite point, say x, to the groups. From the resulting design, we delete one point different from x so as to form a $\{7,8\}$-GDD of type 6^77^1. For the case $n = 26$, we give each point in the groups of size 6 weight one. In the group of size 7, we give x weight zero, five other points weight two and the remaining point weight one. We need (3,2,1)-HCOLS of types 1^7 and 1^62^1, which exist from Theorem 1.4, to get a (3,2,1)-HCOLS(6^711^1). Using (3,2,1)-HCOLS(2^4), we can fill in the holes by adding two new points to get a (3,2,1)-HCOLS($2^{21}13^1$) from Lemma 3.2. We then fill in the hole of size 13 with a (3,2,1)-HCOLS(2^53^1) to get the (3,2,1)-HCOLS($2^{26}3^1$). Similarly, for the case $n = 27$, we use the $\{7,8\}$-GDD(6^77^1) again, except we give x weight one and the six remaining points of the group of size 7 each get weight two, so as to form a (3,2,1)-HCOLS($6^7 13^1$). Adding two new points and filling in the holes, we get a (3,2,1)-HCOLS($2^{21}15^1$). This produces a (3,2,1)-HCOLS($2^{27}3^1$) by further filling in the hole of size 15 with the type $2^6 3^1$. □

Lemma 3.10. *There exists a (3,2,1)-HCOLS($2^{31}3^1$).*

Proof: We start with a TD(5,7) and delete four points from one block so as to form a $\{4,5\}$-GDD of type $6^4 7^1$. We give each point of this GDD weight two to get a (3,2,1)-HCOLS($12^4 14^1$). We then introduce three new points and apply Lemma 3.2, using (3,2,1)-HCOLS of types $2^6 3^1$ and $2^7 3^1$ to get a (3,2,1)-HCOLS($2^{31}3^1$). □

Lemma 3.11. *If $m \geq 5$ and a TD($6, m$) exists, then there exists a (3,2,1)-HCOLS($2^{5m}u^1$) for $2 \leq u \leq 3m$.*

Proof: In a TD($6, m$) we give a weight of zero, two or three to each point in the last group so that the sum of the weights of all points in this group is u. In the first five groups, we give each point weight two. The input designs of types 2^m, 2^5, 2^6, $2^5 3^1$ all exist, and the result follows. □

Lemma 3.12. *For any integer $n \geq 80$, there exists a (3,2,1)-HCOLS($2^n 3^1$).*

Proof: If $n \geq 80$, then we may write $n = 5m + k$ where $m \geq 15$ is odd and $5 \leq k \leq 14$. Let $u = 2k + 3$. Then from Lemma 3.11 we know that a (3,2,1)-HCOLS($2^{5m}u^1$) exists where $2 \leq u \leq 31$. Since a (3,2,1)-HCOLS($2^k 3^1$) exists for $5 \leq k \leq 14$ by Lemma 3.1, we can fill in the hole of size u with (3,2,1)-HCOLS($2^k 3^1$) to get a (3,2,1)-HCOLS($2^{5m+k}3^1$) and hence the result. □

Remark. Combining Lemmas 3.1, 3.6 – 3.10 and 3.12, we have essentially established the result in Theorem 1.3.

4 (3,2,1)-ICOILS

Recall that (3,2,1)-ICOILS(v, k) denotes an idempotent (3,2,1)-COLS of order v with a hole of size k and is equivalent to a (3,2,1)-HCOLS($1^{v-k}k^1$). Using the cyclic construction, we are able to prove the following lemma.

Lemma 4.1. *There exists a (3,2,1)-ICOILS(v, k), where $(v, k) = (30, 5)$ and*

$$k = 2, \quad v \in \{16, 17, 19, 20, 21, 23\},$$
$$k = 3, \quad v \in \{20, 21, 24, 25, 26, 28, 29, 30\}.$$

We list below the vectors **e**, **f** and **g** for these cases, except the cases of (28,3) and (30,5) which are solved by the fill-in-hole construction. We assume $X = \{x_1, x_2\}$ for the cases of $(v, 2)$, $X = \{x_1, x_2, x_3\}$ for the cases of $(v, 3)$.

(v, n)	e	f	g
(16,2)	(0 4 9 12 8 x_1 3 13 10 5 11 2 x_2 7)	(1 6)	(12 13)
(17,2)	(0 6 14 10 x_2 11 1 8 x_1 12 4 13 5 9 3)	(2 7)	(13 14)
(19,2)	(0 14 16 11 8 7 12 2 x_2 3 15 4 13 5 x_1 1 6)	(9 10)	(15 16)
(20,2)	(0 14 17 6 11 13 15 8 10 5 x_1 4 x_2 7 2 1 3 9)	(12 16)	(16 17)
(21,2)	(0 16 18 8 5 12 x_1 13 1 11 14 2 6 3 17 10 x_2 9 7)	(4 15)	(17 18)
(23,2)	(0 18 20 13 5 x_2 19 9 12 14 x_1 6 15 4 8 1 3 2 11 7 10)	(16 17)	(19 20)
(20,3)	(0 12 14 16 7 10 13 x_1 9 2 1 x_2 x_3 15 6 4 3)	(5 8 11)	(14 15 16)
(21,3)	(0 5 14 17 11 x_3 15 13 x_2 10 12 6 x_1 3 7 2 1 9)	(4 8 16)	(15 16 17)
(24,3)	(0 16 18 20 x_3 8 12 11 17 19 2 13 5 14 4 x_2 3 1 x_1 10 6)	(7 9 15)	(18 19 20)
(25,3)	(0 17 19 21 18 10 9 13 1 x_3 14 2 20 15 x_1 16 3 x_2 7 4 8 11)	(5 6 12)	(19 20 21)
(26,3)	(0 18 20 22 9 13 8 x_1 12 15 11 21 19 4 6 1 x_3 5 7 x_2 10 14 2)	(3 16 17)	(20 21 22)
(29,3)	(0 4 15 9 22 19 14 12 17 21 3 7 6 8 24 x_1 23 18 20 10 5 25 11 13 x_2 x_3)	(1 2 16)	(23 24 25)
(30,3)	(0 18 15 6 24 19 16 8 20 x_2 1 26 14 5 3 11 21 23 12 x_1 2 25 17 x_3 4 9 7)	(10 13 22)	(24 25 26)

The case of (28,3) is solved by obtaining a (3,2,1)-HCOLS($5^5 3^1$) first and then using the fill-in-hole construction. That is, we fill the holes of size 5 in (3,2,1)-HCOLS($5^5 3^1$) with a (3,2,1)-COILS(5). Similarly, the case of (30,5) is solved by obtaining a (3,2,1)-HCOLS(5^6). The related vectors for (3,2,1)-HCOLS($5^5 3^1$) are:

$$\mathbf{e} = (\emptyset\ 4\ 13\ x_2\ 6\ \emptyset\ 3\ 8\ 17\ x_1\ \emptyset\ 2\ x_3\ 19\ 18\ \emptyset\ 12\ 11\ 1\ 7\ \emptyset\ 14\ 9\ 21\ 16)$$
$$\mathbf{f} = (22\ 23\ 24)$$
$$\mathbf{g} = (7\ 14\ 24)$$

Remark: Combining Lemma 5.1 with Theorem 1.4, we have established Theorem 1.5.

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References

[1] F. Bennett, L. Wu and L. Zhu, On the existence of COLS with equal-sized holes. *Ars. Combin.* **26B** (1988), 5–36.

[2] F. Bennett and L. Zhu, Conjugate-orthogonal Latin squares and related structures. J. Dinitz & D. Stinson (eds), *Contemporary Design Theory: A Collection of Surveys*, John Wiley & Sons, 1992. pp. 41–96.

[3] Th. Beth, D. Jungnickel and H. Lenz, *Design Theory*, Bibliographisches Institut, Zurich, 1985.

[4] J. H. Dinitz and D.R. Stinson, MOLS with holes. *Discrete Math.* **44** (1983), 145–154.

[5] D.R. Stinson and L. Zhu, On the existence of certain SOLS with holes. *J. of Combin. Math. and Combin. Compu.* **15** (1994), 33–45.

[6] H. Zhang and M. Stickel, Implementing the Davis-Putnam algorithm by tries. Technical Report, Dept. of Computer Science, The University of Iowa, 1994.