A note on mod $(2p + 1)$-orientable graphs

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ABSTRACT. Fix a positive integer k. A *mod k-orientation* of a graph G is an assignment of edge directions to $E(G)$ so that at each vertex $v \in V(G)$, the number of edges directed in is congruent to the number of edges directed out modulo k. The main purpose of this note is to correct an error in [JCMCC, 9 (1991), 201–207] by showing that a connected graph G has a mod $(2p+1)$-orientation for any $p \geq 1$ if and only if G is eulerion.

Graphs in this note are finite and loopless. See [1] for undefined terms. Let G be a graph and let $O(G)$ denote the set of odd vertices of G. A graph G is *even* if $O(G) = \emptyset$, and is *eulerian* if G is both even and connected.

A *mod k-orientation* of G is an orientation of $E(G)$ such that at every vertex v, the number of the edges directed into v is congruent to the number of edges directed out from v modulo k. The graph K_1 is regarded as having a mod k-orientation for any $k \geq 1$.

Proposition 1. *Let G be a graph. Each of the following holds.*

(i) *In general, G has a mod k-orientation if and only if each component of G has a mod k-orientation.*

(ii) *If $k > 0$ is an even integer, then G has a mod k-orientation if and only if G is even.*

(iii) *If $k > 0$ is an odd integer, and if G has a mod k-orientation, then for any $v \in V(G)$, if the the degree $d(v)$ of v is less than k, then $d(v)$ is even.*

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Proof: The proof is left to the readers. □

Theorem 2. *Let G be a graph. The following are equivalent:*

(i) *G is even.*

(ii) *G has a mod $(2p+1)$-orientation for all positive integers p.*

(iii) *G has a mod $(2p+1)$-orientation for infinitely many positive integers p.*

Proof: It suffices to prove the case when G is connected. Note that (ii) trivially implles (iii).

Assume (i). Then G has an Euler tour. Orient the edges in $E(G)$ along that Euler tour. In the resulting orientation, at every vertex, the number of edges directed into the vertex is equal to the number of edges directed out from the vertex, and so this orientation is a mod $(2p+1)$-orientation for any integer $p \geq 1$, and so (ii) holds.

Now assume (iii). Then G has a mod $(2p+1)$-orientation for infinitely many integers $p \geq 1$. If G has a vertex of odd degree k, then by (iii) of Proposition 1, G cannot have a mod $(2p+1)$-orientation for any $2p+1 > k$, a contradiction. Hence G must be an eulerian raph, and so (i) holds. □

In [2], it is claimed that if a 2-edge-connected graph G does not have a subdivision of K_4, then G has a mod $(2p+1)$-orientation for any $p \geq 1$. This claim is incorrect, and T' eorem 2 corrects this error.

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, American Elsevier, New York (1976).

[2] H-.J. Lai and H.Y. Lai, Cycle covers in graphs without subdivisions of K_4, *J. Combin. Math. and Combin. Computing*, **9** (1991), 201–207.