On Veronese's Decomposition Theorem and the Geometry of outer automorphisms of S_6

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ABSTRACT. In this paper the Desargues Configuration in $P^2(k)$, where k is a $char(k) \neq 2$ field, is characterized combinatorially en route to define Desargues Block Designs and associate them with certain families of dihedral subgroups of S_6 through the use of the outer automorphisms of S_6.

This ideas let us understand more of the geometry associated to this group morphisms and, as a by-product, generalize and show a more conceptual approach to classical (i.e. for $k = R$) Veronese's Decomposition Theorem about Pascal's straight lines configuration (60_3).

1 Introduction

Lets recall first that six cyclically ordered points in $P^2(R)$, the projective plane over the real field R, define a hexagon in that plane. As it is well known, Pascal [5] found (ca. 1640) the necessary and sufficient condition for those points to lie in a conic. Namely, that the intersection points of pairs of opposite sides of this hexagon should belong to one and the same straight line, which is called the Pascal line associated to the mentioned hexagon (see Fig. 1).

With this theorem, Pascal started, may be without knowing so, the construction of the *Hexagrammum Misticum (Mystic Hexagon)* Configuration.

Next we sketch some steps of the construction of this configuration.

After G. Salmon [6]: "M. Steiner was the first who directed the attention of geometers to the complete figure obtained by joining in every possible

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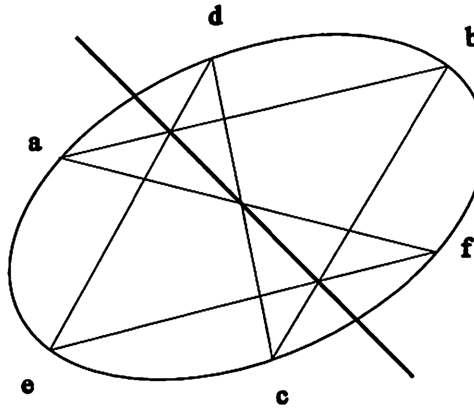

Figure 1: Inscribed hexagon (*abcdef*) and it's Pascal line

way six points on a conic. ...".

So, those six points, cyclically ordered in all possible ways determine 60 ($= \frac{6!}{6\times 2}$) hexagons and, consequently, 60 Pascal lines. And from then on, the figure gradually developed through more than 150 years and from the work of J. Steiner, T.P. Kirkman, A. Cayley, G. Salmon, G. Veronese, L. Cremona, and some other mathematicians.

Steiner (see [10]) proved at the beginning of the XIX century that the 60 Pascal lines are concurrent by triples in 20 points since then named Steiner points after him.

Kirkman's [3] main contribution to this construction, by the middle of that same century, was an extension of Steiner's results, observing that Pascal lines meet also by triples over 60 points, which are different from the Steiner points, called more recently Kirkman points, and so building up a (60_3)-type configuration (notation on this geometric objects comes in the next section).

Afterwards Veronese ([11], 1877) proved that this (60_3)-configuration properly splits into six Desargues Configurations $(10_3)_1$'s. This is Veronese's Decomposition Theorem which we refer to in the title of this paper. The proof given by Veronese goes much along Kirkman's work; i.e., relying on a clever choice of the straight lines involved and on a heavy use of Desargues two triangle Theorem quoted as Theorem 2 in section 3 below. In this paper we give S_6-based criteria for those choices, beginning from the very construction of the Pascal lines and up to Veronese's result.

Also in section 3 there will be established a combinatorial property to characterize Desargues Configuration $(10_3)_1$ (up to projective equivalence) in $P^2(k)$. Then this property will be used to define Desargues Block Designs (DBD), actually a class of 1-designs, which will be needed in the fourth and last section of this paper.

In section 4 we identify a particular block design structure in a particular family of dihedral D_6 subgroups of S_6 (which is chosen with the aid of allotropic embeddings of S_5 in S_6 through the outer automorphisms of S_6) and then prove that the Pascal lines in the main configuration associated to this family constitute also a DBD and so carry a $(10_3)_1$-configuration structure, recovering in this way Veronese's result but in an invariant way and also in a more general context. This is the main purpose in this paper, which appears summarized as the claim in this same section. We also give a "semi-algorithmic" procedure to reconstruct the Veronese components (just called *figures* by G. Veronese himself) starting with any one of the constituent lines of that component.

Generalizations of this theorem to some types of balanced incomplete block designs will be studied in a forthcoming paper [9] and also, in a separate communication, a natural extension of the relations presented here, mainly through the study of the links between the subgraph lattices of K_6 and $L(K_6)$ and the subgroups lattices of S_6 and M_{12}, to understand better the "English" and "German" configurations, the first "Grand" configuration (80_4), and Veronese's infinite sequences of configurations [7], which are some of the associated configurations to the Mystic Hexagon Configuration.

2 Configurations in $P^2(k)$

We will start defining projective plane configurations and some of the classical terminology for k a field with $char(k) \neq 2$ and $|k| \gg 0$. The restriction on the field characteristic is to avoid Fano's configuration and also to ensure the existence of enough and effectively distinct points on each straight line of $P^2(k)$.

Definition 1 *A configuration in projective plane $P^2(k)$ is a system of v points and b straight lines arranged in such a way that*

(i) through each point of the system there pass a fixed number r of lines of the system, and

(ii) each line of the system contains exactly π points of the system.

Something that one notes in a Projective Geometry course, immediately after this definition, is that the quantities v, b, r, and π are not independent, but relate to each other through the following fundamental equation:

$$v\pi = br \tag{1}$$

Furthermore, one notes that this property, being purely combinatorial, applies also to other types of structures, like certain Block Designs, which we will consider in following sections of this paper. Actually, v, r, and π are the usual parameters for a regular design, where we have changed the k in the usual notation (see e.g. Wallis[12] pp. 5 and 6) for π to be able to reserve that symbol k for the base field as is common in projective and algebraic geometry. Also, in sight of (1), we will refer to the $1-(v,\pi,r)$ design defined by such a configuration.

It is usual to assign the Coxeter symbol

$$\begin{pmatrix} v & \pi \\ r & b \end{pmatrix} \tag{2}$$

to such a configuration and, if the number of points equals the number of straight lines in the system, i.e. $v = b$, we have also, by (1), that $r = \pi$, and so a symbol (v_π) would be enough to refer to it. These may be called also Symmetric Configurations, after the symmetric block designs.

Recall that the general symbol (2) does not characterize, in general, a specific configuration; namely, it may refer to a whole scheme of configurations or even to an empty one. Two very interesting examples are the next (see Figure 2) configurations of type (10_3) which are not even projectively related. Furthermore, even if there was a coherent way to fill up an

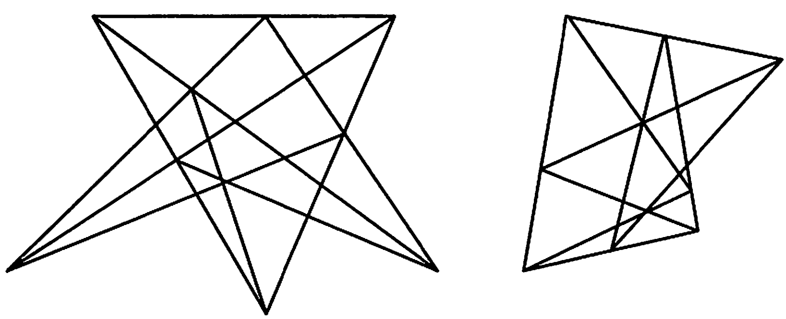

Figure 2: Two Non-Desarguesian (10_3)-type configurations

"incidence table" for a symbol (2), there is no guaranty that there would exist even a single one corresponding configuration. This is the very interesting k-realizability problem for configurations which we will not address in this paper mainly because the configurations that will be considered in this paper are already k-realizable.

Because it is so important in projective and algebraic geometry as well, Desargues Configuration is distinguished among the other (10_3)-type configurations and usually labeled with the symbol $(10_3)_1$.

3 Desargues Block Designs

In this section, it will be recalled a particular construction of Desargues Configuration which will be helpful to characterize combinatorially that configuration. Then we define Desargues Block Designs (as a special type of t-designs) also in a suitable way to understand combinatorially Veronese's Decomposition Theorem.

To start with, we quote Desargues' two triangle Theorem which "builds up" Desargues configuration:

Theorem 1 *If two triangles are in perspective, then the intersections of pairs of corresponding edges are aligned.*

So, we start considering the six vertices and the edges of two triangles in perspective, plus the center of perspective and the straight lines joining corresponding vertices under the considered perspective as a system of seven points and nine straight lines in $P^2(k)$.

Observe that, by now, those elements do not satisfy the required incidence conditions (i.e. (i) and (ii) in Definition 1) to constitute a configuration as already defined.

So, the quoted Desargues' Theorem provides us with the necessary elements (i.e. three more points and an extra line) satisfying the necessary incidence and concurrence conditions which, together with the former ones, constitute a (10_3)-configuration: (see Fig. 3).

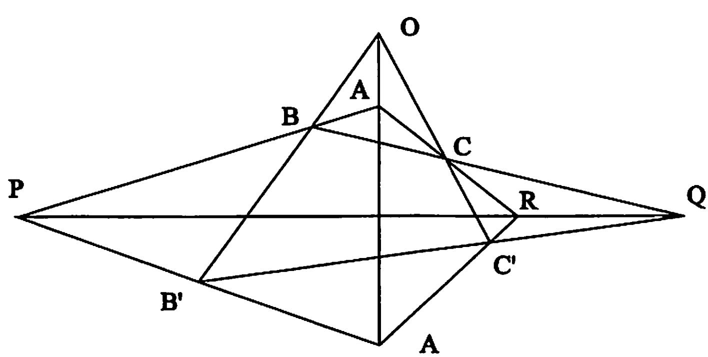

Figure 3: Desargues two triangle configuration

3.1 Combinatorial Property of $(10_3)_1$

The following well known property of Desargues Configuration it is used almost verbatim in classical geometry to emphasize the homogeneity of the configuration in the sense that no point on it plays a distinguished role in the configuration, for instance, as the center of perspective.

Property 1 If any point of the configuration is removed, along with the three straight lines of this configuration containing it, the remaining three points determine a straight line which also belongs to the configuration.

For example, in Figure 3, the points A, C, and R are the only non-neighbor points of B', which also happen to be in one of the straight lines of this configuration. Also note that both configurations in Figure 2 do not have this property, *for at least one of its points* in each case.

The next theorem proves that Property 1 also characterizes combinatorially the Desargues Configuration. To emphasize the only dependence of the following proof of the theorem on the "natural" block design structure of the involved configuration $\mathcal{D}$, let's define a block of the design to be the complete set of points of $\mathcal{D}$ belonging to a line of $\mathcal{D}$.

Theorem 2 *A* (10_3) *type configuration with the Property 1 above is a* $(10_3)_1$*-configuration.*

Let $\mathcal{D}$ be a configuration as in the hypothesis. Let N be any one of the points of $\mathcal{D}$ and let l_S, l_T, and l_U be the blocks to which N belongs; say

$$l_S = \{N, P, P'\},\ l_T = \{N, Q, Q'\}, \text{ and } l_U = \{N, R, R'\}.$$

Then, we have to prove that the three points $PQ \cdot P'Q'$, $QR \cdot Q'R'$, and $RP \cdot R'P'$ constitute the block l_N determined by the element N through the mentioned property.

Let's say that $l_N = \{S, T, U\}$ and also that U belongs to the blocks l_A and l_B, besides l_N.

First of all, we have that, at most one of the elements of each couple of points $\{P, P'\}$, $\{Q, Q'\}$, and $\{R, R'\}$ belongs to l_A (resp. l_B). So, let's suppose that P, $Q \in l_A$. Then P' and Q' should belong to l_B: if $R \in l_B$, then R', N, and P' (or Q') would belong to the block l_U , which contradicts Property 1. Also we have that $R' \notin l_B$, and so we may suppose further that

$$l_A (= l_{R'}) = \{U, P, Q\} \text{ and } l_B (= l_R) = \{U, P', Q'\}$$

(see Figure 4).

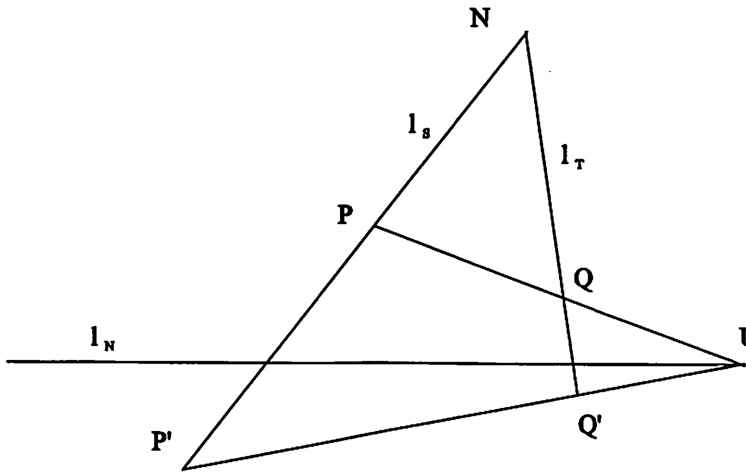

Figure 4:

Likewise, being S an element of the block $\{S, Q, R\}$ implies that also the block $\{S, Q', R'\}$ belongs to $\mathcal{D}$ and then we have that

$$l_{P'} = \{S, Q, R\} \text{ and } l_P = \{S, Q', R'\}.$$

Now that we have the triangles ΔPQR and $\Delta P'Q'R'$ in perspective from the point N, we just have to identify the rest of a Desargues Configuration to end the proof.

So, if the block $\{P, R', T\}$ would belong to $\mathcal{D}$ (see Fig. 5), then property 1 applied to point P would imply that Q', R, and S constitute a block of $\mathcal{D}$. So, the only possibility for the not yet identified blocks in D are:

$$l_{Q'} = \{P, R, T\} \text{ and } l_Q = \{P', R', T\}.$$

The following lemma will be used in the next section in the search of the Pascal lines that constitute, together with a given one, a Desargues configuration, accordingly to Veronese's result.

Lemma 1 *For a* $(10_3)_1$ *configuration in* $P^2(k)$,

$$Aut\left((10_3)_1\right) \cong S_5. \tag{3}$$

Just note that the incidence graph of such a configuration is $L(K_5)$, the edge graph of a complete graph on five vertices, for which it is well known that $Aut(L(K_5)) \cong S_5$.

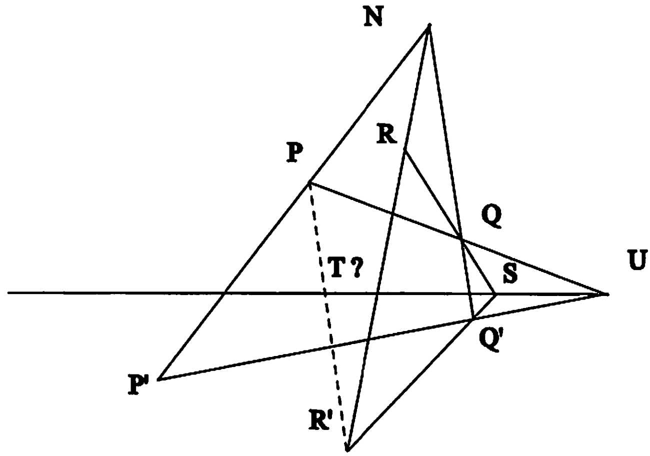

Figure 5:

Definition 2 *A Desargues Block Design (DBD) $\mathcal{D}$ is a (10_3)-type symmetric design such that*

For each point P_i, its non-neighbor points; i.e., the point set

$$\mathcal{D} - (l_{1_i} \cup l_{2_i} \cup l_{3_i}) \tag{4}$$

constitute a block l_{P_i} of $\mathcal{D}$.

The points in the set $l_{1_i} \cup l_{2_i} \cup l_{3_i}$ plus those same blocks constitute a block design $S_{P_i}D$ that may be called the Star of P_i in $\mathcal{D}$ and also the point set (4) with the only block l_{P_i} defined by Property 1 above will be $R_{P_i}\mathcal{D}$, the residual design to P_i in $\mathcal{D}$.

In general, for a Block Design D, we define $S_P D$, the Star of a point P of D, as the set of neighbor points of P in D with blocks: $\{l \text{ block of } D \mid P \in l\}$ and $R_P D$, the residual block design of D respect to P, as the set of non-neighbor points of P in D with blocks: $\{l \text{ block of } D \mid Q \in l, \text{ for a } Q \in R_P D\}$. This is the treatment followed in San Agustín [9] to study triangular double association schemes.

Warning: Those definitions are not the usual ones in Design Theory! (compare with [12] for instance).

4 Relationship with the outer automorphisms of S_6

We will use the following notation:

1. $(P_1P_2P_3P_4P_5P_6) :=$ hexagon obtained by joining the $P^2(k)$ points P_1, P_2, ..., and P_6 in that cyclic order.

2. $\overline{12...6} :=$ Pascal line for the inscribed (in a conic) hexagon

$$(P_1P_2P_3P_4P_5P_6).$$

3. Also we will use cyclic notation for elements in S_n.

Recall that any $G(\cong S_5) \prec S_6$ belongs to one and only one of the two following types:

1. $Stab(n)$, for $n \in \{1,2,3,4,5,6\}$, and

2. $\omega(Stab(n))$, for an outer automorphism of S_6:

$$\omega : S_6 \longmapsto S_6. \tag{5}$$

Also recall that two outer automorphisms of S_6 differ (under operation in $Aut(S_6)$) by an inner automorphism, which do not change the group type of G as described above and that classes (1) and (2) above are stable under S_6 inner automorphism action.

Proposition 1 *For each $6-cycle$ $\alpha \in S_6$, there is a unique $S_5 \prec S_6$ such that $\alpha \in S_5$.*

If ω is any outer automorphism of S_6, $\omega(\alpha)$ is the product of a 3-cycle, say (a,b,c), and a transposition, say (d,e), disjoint to each other, which involve therefore only five of the elements of the set $\{1,2,3,4,5,6\}$; i.e. the subgroup $Stab(n)$, for the only element $n \in \{1,2,3,4,5,6\} - \{a,b,c,d,e\}$ is the only subgroup of S_6 isomorphic to S_5 and containing $\omega(\alpha)$.

Then, the S_5 we are looking for is the pull back under ω of that stabilizer subgroup of S_6, $Stab(n)$.

Finally, by the bijectivity of ω shows the uniqueness of this group:

Suppose that there is another such subgroup T of S_6 containing α. Again, it must be of type 2 and so $\omega(T) = Stab(m)$, for $m \in \{1,2,3,4,5,6\}$.

For what it was said before, $m = n$ and so $T = \omega^{-1}(Stab(n))$.

By the same argument, this construction does not depend on the outer automorphism ω employed in it.

For a $6 - cycle$ α in S_6, we will call S' this special S_5.

Let's consider now $N_{S'}(\alpha)$, the normalizer of a $6 - cycle$ α in S', which is isomorphic to D_6, the dihedral group of 12 elements. It is easy to see that $N_{S_6}(N_{S'}(\alpha)) = N_{S'}(\alpha)$ and so the definition of this group, call it simply $N(\alpha)$, does not depend on S'.

So, for six cyclically ordered points a, b, c, d, e, and f in a conic in $P^2(k)$, we may associate to the Pascal line $\overline{abcdef}$ of the hexagon $(abcdef)$ the normalizer group $N(\alpha)$ corresponding to the $6-cycle$ $\alpha = (a,b,c,d,e,f)$:

$$\overline{abcdef} \longmapsto N(\alpha) \tag{6}$$

This map is defined from the set of 60 Pascal lines determined by those six points (as explained at paragraph no.1 of this paper) to the lattice of subgroups of S_6.

Now, from the above comments and for a fixed $6 - cycle$ α, we naturally consider the 10 conjugate subgroups of $N(\alpha)$ in the S' detected by Proposition 1. Call them $N_1 := N(\alpha)$, N_2, ...N_{10}. Let l_1, l_2, ...l_{10} be the pullbacks of N_1, N_2, ...N_{10}, respectively, under the map (6).

Claim *The set of Pascal lines l_1, l_2,...l_{10} above constitute a Desargues configuration.*

Despite the proof of the following Proposition is a straightforward one, we will give it because the counting involved in it strongly relates Veronese's decomposition theorem and Desargues Block Designs, which is the tool we use to prove the just stated claim, our ultimate goal in this paper.

Proposition 2 *There are exactly 10 triples of this dihedral subgroups of S' that intersect exactly over a Z_2.*

Again, this fact is very easy to see by using an outer automorphism ω of S_6:

Knowing that also $\omega(N(\alpha)) \cong D_6$, $\omega(\alpha)$ is of the form $(a,b,c)(d,e)$ and so $N(\omega(\alpha))$ may be generated by $\omega(\alpha)$ and (b,c) for instance. Then the dihedral subgroups $N' := \langle (d,b,c)(a,e),(b,c)\rangle$ and $N'' := \langle (e,b,c)(a,d),(b,c)\rangle$ of $\omega(S')$ intersect along the subgroup ($\cong Z_2$) generated by (b,c).

Considering the choice of (b,c) just made (see the item no. 2 in the list below), these are the only triples with that property: being $\omega(\alpha)^3$ a transposition, if it belongs to $N(\beta)$, another such dihedral subgroup of S', $\omega(\alpha)^3 \neq \beta^3$ (or $N(\alpha) = N(\beta)$). But in such a case, the transpositions product $\omega(\alpha)^3\beta^3 \in N(\beta)$, enlarging then the intersection.

Now, to count them, note that

1. There are 20 order six elements in S', being 10 of them the inverses of the other ones.

2. Any one of the 3 order two *generators*[1] of $\omega(N(\alpha))$ will be one of the mentioned permutations .

3. Each order six element appears in 3 of these triples (this might be checked even by inspection).

So we have $\frac{3\times 10}{3} = 10$ triples as mentioned. □

Remark Without using the outer automorphisms (and also without loss of generality) we may take $N(\alpha) = \langle (a,b,c,d,e,f), (a,f)(b,e)(c,d) \rangle$. Then, the other two of those dihedral subgroups mentioned at the Proposition 2 are

$$N(\alpha') = \langle (a,c,b,e,d,f), (a,f)(b,e)(c,d) \rangle \text{ and}$$

$$N(\alpha'') = \langle (a,b,e,f,d,c), (a,f)(b,e)(c,d) \rangle,$$

which are precisely the conjugates of $N(\alpha)$ by, for example, $(c,d)(e,f)$ and $(cedf)$.

Finally, the corresponding Pascal lines $\overline{abcdef}$, $\overline{acbdef}$, and $\overline{abefdc}$ (projective geometry exercise, see Kirkman [3] or Salmon [6]) are concurrent in a point. This is the typical Kirkman point as mentioned at the introduction.

Also note that we may associate, for a fixed outer automorphism ω and by the proof of Proposition 2, to each Pascal line l a $3-cycle$, say (a,b,c), and also to a Kirkman point K a transposition (l,m) in such a way that

$$K \in l \Longleftrightarrow \{l,m\} \subset \{a,b,c\} \tag{7}$$

On the other hand we observe by the last remark that three Pascal lines are concurrent if[2]

(i) A set of three alternate edges (i.e., non adjacent by pairs) of one of the corresponding hexagons also belong, and also as an alternate set of edges, to another one of them, and

(ii)The remaining edges in both hexagons constitute, in like manner, the edges of the last of the three considered hexagons.

[1] $\omega(N(\alpha))$ may be generated as well with $\omega(\alpha) = (a,b,c)(d,e)$ and any one of the two transpositions (a,b) and (c,a).

The only other transposition in $\omega(N(\alpha))$ *not* to be considered is $(\omega(\alpha))^3$, because it does not generate, together with $\omega(\alpha)$, the group $\omega(N(\alpha))$.

[2] There are another triads of concurrent Pascal lines, which determine the so called Steiner points of the Mystic Hexagon Configuration [6]. We won't discuss them in this paper.

So, in this case, the edges of the three hexagons determine just 9 different straight lines (instead of the 18 expected for the general case) in $P^2(k)$, which, together with the three Pascal's first considered, meet by triples in 16 points, which, in turn, belong by four's to this 12 straight lines and so constitute a $\begin{pmatrix} 16 & 3 \\ 4 & 12 \end{pmatrix}$ type configuration.

Now, given a Pascal line $\overline{abcdef}$, we use the last observation to give a simple construction of the other Pascal lines meeting by pairs on it:

1. First we represent *diagrammatically* the hexagon $(abcdef)$ as in Figure 6.
2. Then we substitute the following scheme of diagonals (there are three possibilities) for triples of alternate edges on the hexagon $(abcdef)$ in such a way that the new hexagons formed this manner exhaust all six vertices P_1, P_2, ..., and P_6 (there are also two possibilities)[3]:

In a moderner language we are treating each of this "H" shaped diagonal schemes and the two triples of alternate sides as 1-factors of the K_6 graph determined by the points P_1, P_2, ..., and P_6 which constitute, by couples, the three hexagons (including the original one) who's Pascal lines are meant to concur, according to Kirkman's result.

Explicitly, this construction produces the following sets of straight Pascal lines:

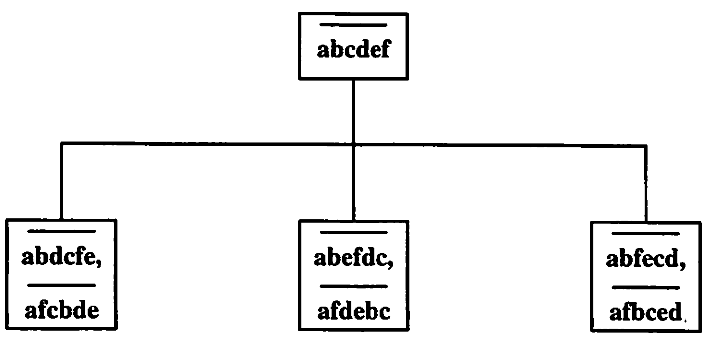

Figure 7:

[3] This construction is essentially due to Steiner, who used a similar scheme to grasp the triples of Pascal lines concurrent over the now called Steiner points, using only main diagonals in the schematic representation of Figure 6 for that purpose. Then Kirkman modified that scheme of diagonals to get the one we just described. Nor Steiner, nor Kirkman posed their schemes in any explicit form. They and all other authors, in any source available to us, described at most typographically their methods (which proved to be enough for them).

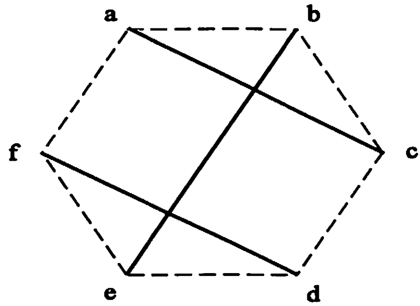

Figure 6:

Applying this process once more to this "second generation" Pascal lines set we get only 3 new lines among all the "third generation" lines. The rest of them are just repetitions of the "older"ones.

There are no new elements at the "fourth generation".

The reason of this is just that the given construction corresponds, via the map (6), to conjugation of $\alpha = (a, b, c, d, e, f)$ by $(c, d)(e, f)$ and $(becd)$ as explained in the preceding remark, respectively, *and none of those elements belongs to* $N(\alpha)$. So $N(\alpha) \neq N'$ and $N(\alpha) \neq N''$.

Finally, to prove the claim, the counting process used to prove Proposition 2 gives us (item by item) that:

1. Each Pascal line contains three Kirkman points.

2. There are 10 Pascal lines in this set.

3. Through each Kirkman point there pass three Pascal lines.

   And, by the very statement of Proposition 2,

4. There are 10 Kirkman points.

That is, putting together both the Kirkman points and the Pascal lines so far considered, we have a (10_3) type configuration in $P^2(k)$.

To see that this is a Desargues configuration (using Proposition 1) we rely again in an outer automorphism ω of S_6:

We have that the residual points to the Kirkman point associated to the transposition (a, b) in our configuration are precisely the ones asscciated to the transpositions (l, m) where $\{a, b\} \cap \{l, m\} = \emptyset$, the empty set. That is,

$$\{l, m\} \in \{c, d, e\}$$

and so, regarding (7), those residual points belong to the Pascal line associated to the $3 - cycle$ (c, d, e) and, by Property 1, this residual set of points and line satisfies Definition 2, proving our claim.

Acknowledgments

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The pictures of Figure 2 were inspired by the ones in [1].

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