(27,13,6) Designs with an automorphism of order 3

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ABSTRACT. In this paper we construct all symmetric (27,13,6) designs with a fixed-point-free automorphism of order 3. There are 22 such designs.

1 Introduction

The only primes dividing the order of a non-trivial automorphism of a symmetric (27,13,6) design are 2, 3 or 13. Seven such designs were constructed by Tonchev [7] by assuming an automorphism of order 13. Those seven are the only (27,13,6) designs cited in [5]. This paper investigates the "order 3" problem, describing an exhaustive search which found twenty-two (27,13,6) designs with a fixed-point-free automorphism σ of order 3. One of these designs has an automorphism of order 13 and so this search reveals twenty-one designs not covered by [5]. These designs are given in Appendix I.

The search began by assuming an automorphism σ of order 3 which fixes no point (or block), finding all possible contracted incidence matrices corresponding with the action of σ, and finally using the software *GAP* to finish the problem with a computer search. The output of the search included sets of isomorphic designs; these were pared down by using a variety of techniques, including using *GAP* to construct isomorphisms between various designs.

2 The contracted incidence matrix

The semiregular automorphism, called σ throughout this paper, has 9 orbits of size 3 on points and similarly on blocks. Call the orbits on points $P_1 = \{p_1, p_2 = \sigma(p_1), p_3 = \sigma^2(p_1)\}$, $P_2 = \{p_4, p_5 = \sigma(p_4), p_6 = \sigma^2(p_4)\}, \ldots, P_9 = \{p_{25}, p_{26}, p_{27}\}$ and the nine orbits on blocks are $B_1 = \{b_1, b_2 = \sigma(b_1), b_3 = \sigma^2(b_1)\}$, $B_2 = \{b_4, b_5, b_6\}, \ldots, B_9 = \{b_{25}, b_{26}, b_{27}\}$. The automorphism σ is identified with the permutation

$$(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)$$
$$(16,17,18)(19,20,21)(22,23,24)(25,26,27)$$

on points and on blocks.

An incidence matrix A of blocks against points satisfies

$$AA^t = 7I + 6J, \tag{1}$$

$$AJ = JA = 13J. \tag{2}$$

Its rows and columns may be arranged according to the orbits of σ. Let A_{ij} be the 3×3 incidence matrix of the ith block class with the jth point class. View A as partitioned into the submatrices A_{ij}.

This partitioning of the 27×27 matrix A into blocks of size 3 is the key to our solution to this problem. The strategy is this: view A as a 9×9 matrix in various ways and find suitable solutions to equations (1) and (2) and piece these together to find (0,1) solutions to the equations.

More precisely, each matrix A_{ij} is a sum of powers of $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$; in particular, they are members of the ring $\mathbf{Z}(P)$ and we may view A as belonging to either $\mathrm{Mat}_{27}(\mathbf{Z})$ or $\mathrm{Mat}_9(\mathbf{Z}(P))$. The transpose map acts on the ring $Z(P)$ as an automorphism sending P to P^{-1}. We distinguish this map from the transpose map on the 27×27 matrix by calling the map $\tau\colon P \to P^{-1}$ the "conjugate" map. The transpose map on $\mathrm{Mat}_{27}(\mathbf{Z})$ has the following interpretation on $\mathrm{Mat}_9(\mathbf{Z}(P))$: A^t is obtained by transposing the 9×9 matrix and then conjugating (transposing) each (3×3) entry; that is, the transpose map on $\mathrm{Mat}_{27}(\mathbf{Z})$ is the conjugate transpose map on $\mathrm{Mat}_9(\mathbf{Z}(P))$.

The matrix P has 3 eigenvalues: 1, ω and ω^2 where $\omega = e^{2\pi i/3}$. These eigenvalues correspond to the right eigenvectors $\pi_0 = (1,1,1)^t$, $\pi_1 = (1,\omega,\omega^2)^t$ and $\pi_2 = (1,\omega^2,\omega)^t$, respectively, and these form an orthogonal basis for the column space $\mathbf{C}^3$. The maps $f_i\colon P^j \to \omega^{ij}$ $(i = 0,1,2)$ extend by linearity to $\mathbf{Z}$-module homomorphisms from $\mathbf{Z}(P)$ into the complex numbers $\mathbf{C}$; for any matrix X, $f_i(X)$ is merely the first coordinate of $X\pi_i$. The function f_0 maps a member of $\mathbf{Z}(P)$ to its row sum, and the maps f_1 and f_2 map $\mathbf{Z}(P)$ into the algebraic number field $\mathbf{Z}(\omega)$. The maps f_1 and f_2 have kernel equal to the multiples of $J_3 = I + P + P^2$ since $1 + \omega + \omega^2 = 0$. The "conjugate" map τ on $\mathbf{Z}(P)$ now corresponds to the conjugate map in $\mathbf{Z}(\omega)$, as $f_i(X^\tau)$ is the conjugate of $f_i(X)$.

We may extend the functions f_i to $\mathbf{Z}$-module homomorphisms from $\mathrm{Mat}_9(\mathbf{Z}(P))$ to $\mathrm{Mat}_9(\mathbf{C})$ by letting the functions act on each entry of the 9×9

matrix. Consider the equations (1) and (2) under the map f_0, that is, the "row sum" map. Since $f_0(J_{27}) = 3 \cdot J_9$ we have that

$$f_0(A) \cdot f_0(A)^t = 7I + 18J_9, \qquad (1_0)$$

$$f_0(A) \cdot J_9 = J_9 \cdot f_0(A) = 13J_9. \qquad (2_0)$$

Any solution to equations (1) and (2) must also solve equations (1_0) and (2_0).

The map f_1 sends P to ω and J to zero. So

$$f_1(A) \cdot f_1(A)^* = 7I_9 \qquad (1_1)$$

where * represents the conjugate transpose. (Equation (2) becomes, here, $f_1(A) \cdot 0 = 0 \cdot f_1(A) = 0$, which is useless.) Any solution to equations (1) and (2) must also solve (1_0), (2_0), and (1_1) since the functions f_i are homomorphisms.

If we find a solution to equations (1_0) and (2_0) and (1_1), will we have a design? The answer is yes. The set of solutions to equations (1) and (2), under the assumption that there is an automorphism of order three which fixes no point, is exactly the set of solutions to equations (1_0) and (2_0) and (1_1).

Theorem. *Let A be a 27×27 (0,1)-matrix with the property that it may be partitioned into blocks which form 3×3 circulants. A satisfies equations (1_0), (2_0) and (1_1) if and only if it is the incidence matrix of a (27,13,6) design.*

Proof: One direction is immediate from the fact that the functions f_i are **Z**-module homomorphisms. We need to show that if A satisfies equations (1_0), (2_0) and (1_1) then it is the incidence matrix of a design.

Replace each entry in $f_1(A)$ by its conjugate and thus create the matrix $f_2(A)$. Conjugation is a field automorphism which fixes the real numbers (and most particularly, the integers) and so the equation

$$f_2(A) \cdot f_2(A)^* = 7I_9 \qquad (1_2)$$

is satisfied.

A 27×27 matrix over the complex numbers is uniquely determined by its action on a basis of $\mathbf{C}^{27}$. We will choose a special basis for $\mathbf{C}^{27}$ as follows. Let e_j, $(j = 1, 2, 3, \ldots, 9)$ be the vector in $\mathbf{C}^9$ with a 1 in the jth entry and zeros elsewhere. Since the vectors π_0, π_1 and π_2 are an orthogonal basis for $\mathbf{C}^3$, the vectors $v_{jk} = e_j \otimes \pi_k$ form an orthogonal basis for $\mathbf{C}^{27}$. So it suffices to prove that the matrix A satisfies equations (1) and (2) when applied to the vectors v_{jk}, that is, it suffices to show that for all j, k,

$$(AA^t)v_{jk} = (7I + 6J)v_{jk}, \qquad (1')$$

$$(AJ)v_{jk} = (JA)v_{jk} = (13J)v_{jk}. \tag{2'}$$

(1′) and (2′) are clearly true for $k = 0$. The equation (2′) is trivial if $k = 1$ or 2.

If $k = 1$ then (1′) becomes $(AA^t)v_{j1} = (7I + 6J)v_{j1}$. Consider the lth partition of these 27×1 vectors, $l = 1, 2, \dots, 9$. Since $Jv_{j1} = 0$, the lth partition of $(7I + 6J)v_{j1}$ is $7\delta_{lj}\pi_1$ where δ is the Kronecker delta. Now $(A^t)v_{j1}$ is the vector $\sum_{i=1}^{9} f_1(A_{ij}^t)v_{i1}$ and the lth partition of $A(A^t)v_{j1}$ is the vector

$$\sum_{i=1}^{9} A_{li}(f_1(A_{ij}^t))v_{i1} = \sum_{i=1}^{9} f_1(A_{ij}^t)A_{li}v_{i1} = \sum_{i=1}^{9} \overline{f_1(A_{ji})} f(A_{li})\pi_1$$

Now $\sum_{i=1}^{9} \overline{f_1(A_{ji})} f(A_{li})\pi_1$ is the (l, j) entry of $f_1(A)f_1(A)^*$, which, by (1_1) is $7\delta_{lj}$. So (1′) is satisfied if $k = 1$.

If $k = 2$ then by a similar argument, (1′) is true. Therefore, A is the incidence matrix of a design.

This suggests the following attack. First, find all solutions to the equations (1_0) and (2_0), that is, find all possibilities for the matrix $f_0(A)$ up to row and column permutations. Then replace the entries 0, 1, 2, or 3 in $f_0(A)$ with an appropriate matrix ($1 \to I, P$, or $P^2, 2 \to J - I, J - P$, or $J - P^2$) so that equation (1_1) is also satisfied. This second step will presumably result in a large list of designs. The third step will then be to separate the list into isomorphism classes and retain one representative from each class.

3 Solving the equations (1_0) and (2_0)

Our first step is to find possibilities for the matrix $f_0(A)$, where A is a (0,1) matrix as in the theorem.

Set c_{ij} equal to the row sum of A_{ij}, that is, c_{ij} is the (i, j) entry of $f_0(A)$. Then, for a fixed i, $0 \le c_{ij} \le 3$ and

$$\sum_{j=1}^{3} c_{ij} = 13$$

$$\sum_{j=1}^{3} c_{ij}^2 = 7 + 18 = 25$$

$$\sum_{j=1}^{3} c_{ij}c_{ik} = 7\delta_{jk} + 18$$

Fix a row i and let $m_k = |\{c_{ij} : c_{ij} = k, 1 \le j \le 9\}|$. Then

$$\sum_{k=0}^{3} m_k = 9,\ \sum_{k=0}^{3} k m_k = 13 \text{ and } \sum_{k=0}^{3} k^2 m_k = 25.$$

The possible values for the set (m_0, m_1, m_2, m_3) are

$$(2,1,6,0), (1,4,3,1) \text{ and } (0,7,0,2).$$

The last can be ruled out by a parity argument: suppose the first row of the 9×9 matrix $f_0(A)$ is

$$1\,1\,1\,1\,1\,1\,1\,3\,3$$

and the second row is $x_1, x_2, x_3, \ldots, x_9$. Let $x = x_1 + x_2 + x_3 + \ldots x_7$ and let $y = x_8 + x_9$. Then $x + y = 13$ and $x + 3y = 18$, so $2y = 5$. But y is an integer and so this is impossible.

Therefore we may now assume the first row of $f_0(A)$ is one of

$$0\,0\,1\,2\,2\,2\,2\,2\,2 \qquad \text{(type 1)}$$

$$0\,1\,1\,1\,1\,2\,2\,2\,3 \qquad \text{(type 2)}$$

We now use inner-product and symmetry arguments to find the solutions to (1_0) and (2_0). The details are given in [1]. We suppose two cases. Case 1: there is *some* row of type 1; Case 2: *all* rows are of type 2.

Case 1 leads to three matrices $f_0(A)$ (up to equivalence):

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 0 & 1 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 3 & 2 & 1 & 1 & 1 & 1 \\ 2 & 0 & 3 & 1 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 0 & 0 & 2 & 2 & 2 \\ 2 & 2 & 1 & 2 & 0 & 3 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 2 & 1 & 0 & 1 & 3 \\ 2 & 2 & 1 & 1 & 2 & 1 & 1 & 3 & 0 \\ 2 & 2 & 1 & 1 & 2 & 1 & 3 & 0 & 1 \end{bmatrix}$$

This matrix has automorphism group of order 9, generated by (1,5,2)(3,4,6) (7,8,9). Call this **Matrix 1.**

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 1 & 3 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 3 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 & 2 & 0 & 2 & 3 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 & 3 & 0 & 1 & 1 & 1 \\ 2 & 1 & 2 & 3 & 0 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 & 1 & 0 & 2 & 3 \\ 2 & 2 & 1 & 1 & 1 & 1 & 2 & 3 & 0 \\ 2 & 2 & 1 & 1 & 1 & 1 & 3 & 0 & 2 \end{bmatrix}$$

This matrix, which we will call **Matrix 2**, has an automorphism group of order 18, generated by (1,2)(4,7)(5,8)(6,9),(7,8,9), and (4,5,6).

$$\begin{bmatrix}
1 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 1 & 3 & 1 & 1 & 1 & 2 & 2 & 2 \\
0 & 3 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\
2 & 1 & 2 & 3 & 1 & 1 & 0 & 1 & 2 \\
2 & 1 & 2 & 1 & 3 & 1 & 2 & 0 & 1 \\
2 & 1 & 2 & 1 & 1 & 3 & 1 & 2 & 0 \\
2 & 2 & 1 & 2 & 0 & 1 & 3 & 1 & 1 \\
2 & 2 & 1 & 1 & 2 & 0 & 1 & 3 & 1 \\
2 & 2 & 1 & 0 & 1 & 2 & 1 & 1 & 3
\end{bmatrix}$$

This is **Matrix 3**, which has an automorphism group of order 6 generated by (2,3)(4,9,6,8,5,7).

For case 2 we assume row 1 (and column 1) is 0 3 1 1 1 1 2 2 2 and that all rows (and columns) are type 2. We obtain the following two matrices.

$$\begin{bmatrix}
0 & 1 & 2 & 1 & 1 & 2 & 3 & 2 & 1 \\
1 & 2 & 0 & 1 & 2 & 1 & 2 & 1 & 3 \\
2 & 0 & 1 & 2 & 1 & 1 & 1 & 3 & 2 \\
1 & 1 & 2 & 3 & 1 & 2 & 1 & 0 & 2 \\
1 & 2 & 1 & 1 & 2 & 3 & 0 & 2 & 1 \\
2 & 1 & 1 & 2 & 3 & 1 & 2 & 1 & 0 \\
3 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 1 \\
2 & 1 & 3 & 0 & 2 & 1 & 1 & 1 & 2 \\
1 & 3 & 2 & 2 & 1 & 0 & 1 & 2 & 1
\end{bmatrix}$$

Call this **Matrix 4**. It has an automorphism group of order 3 generated by the permutation (1,2,3)(4,5,6)(7,8,9) acting on columns.

$$\begin{bmatrix}
0 & 3 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
3 & 1 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
1 & 0 & 3 & 1 & 1 & 1 & 2 & 2 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 & 0 & 3 & 1 \\
1 & 1 & 1 & 2 & 2 & 2 & 3 & 1 & 0 \\
1 & 1 & 1 & 2 & 2 & 2 & 1 & 0 & 3 \\
2 & 2 & 2 & 0 & 3 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 3 & 1 & 0 & 1 & 1 & 1 \\
2 & 2 & 2 & 1 & 0 & 3 & 1 & 1 & 1
\end{bmatrix}$$

We will call this **Matrix 5**. Matrix 5 has an automorphism group of order 81 generated by (4,5,6),(7,8,9) and (1,4,7)(2,5,8)(3,6,9).

We have five possible images for an incidence matrix under the map f_0, and now go on to the second stage of the project, which is solving equation (1_1).

4 Solving the equation (1_1)

Given one of the five matrices satisfying (1_0) and (2_0) we constructed two or three possible rows of a corresponding matrix which also satisfied the inner product conditions implied by (1_1). For example, Matrix 1 has rows 1, 2 and 5 equal to

$$\begin{array}{c} 001222222 \\ 220012222 \\ 122200222 \end{array}$$

after a permutation of columns. We may assume the rows and columns of the incidence matrix of the design have been permuted by the permutation $(3i+1, 3i+2, 3i+3)$, $i = 0, 1, \ldots, 8$, if needed, so that the the image of these three rows of A under f_1 is:

$$\begin{array}{ccccccccc} 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 1 & -\omega^a & -\omega^b & -\omega^c & -\omega^d \\ 1 & -\omega^e & -\omega^f & -\omega^g & 0 & 0 & -\omega^h & -\omega^i & -\omega^j \end{array}$$

where $\{a, b, \ldots, j\}$ are integers in $\{0, 1, 2\}$. The inner product of rows 1 and 2 is $-1+\omega^a+\omega^b+\omega^c+\omega^d$ which must be zero. The minimal polynomial for ω is x^2+x+1 and so the only way four positive terms and one negative could sum to zero is if three of the positive terms sum to zero and one positive term cancels the negative. In other words, this inner product could be zero only if $\{a, b, c, d\}$ is the multi-set $\{0, 0, 1, 2\}$

We may assume, after permuting the last three places, if needed, that row 2 of $f_1(A)$ is one of

$$\begin{array}{ccccccccc} -1 & -1 & 0 & 0 & 1 & -1 & -1 & -\omega^2 & -\omega, \\ -1 & -1 & 0 & 0 & 1 & -\omega & -1 & -1 & -\omega^2, \\ -1 & -1 & 0 & 0 & 1 & -\omega^2 & -1 & -1 & -\omega. \end{array}$$

The third case is the conjugate of the first and so to simplify our search we will conjugate the solution, if needed, and assume that the third case does not occur.

Matrix 1, Case 1.

The image of rows 1, 2 and 5 under f_1 are respectively:

$$\begin{array}{ccccccccc} 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 1 & -1 & -1 & -\omega^2 & -\omega \\ 1 & -\omega^e & -\omega^f & -\omega^g & 0 & 0 & -\omega^h & -\omega^i & -\omega^j \end{array}$$

The inner products of row 5 with rows 1 and 2 are

$$-\omega^f + \omega^g + \omega^h + \omega^i + \omega^j$$
$$-1 + \omega^e + \omega^h + \omega^{i-2} + \omega^{j-1}$$

These are required to be zero. The first expression forces $\{g-f, h-f, i-f, j-f\}$ to be the multi-set $\{0,0,1,2\}$ and the last expression forces $\{e, h, i-2, j-1\}$ to be the multi-set $\{0,0,1,2\}$. If $g=f$ then h, i, j are all different and this forces (e,h,i,j) to be $(0,0,1,2)$, $(0,1,2,0)$ or $(0,2,0,1)$. This gives nine possibilities for (e,f,g,h,i,j). If $g \neq f$ then two of h, i, and j are equal, and in fact equal to f and we find that $(e,f,g,h,i,j) = (2,0,2,0,0,1)$ or $(1,0,1,0,2,0)$.

Now we examine case 2 where the image under f_1 is:

$$\begin{array}{ccccccccc} 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 1 & -\omega & -1 & -1 & -\omega^2 \\ 1 & -\omega^e & -\omega^f & -\omega^g & 0 & 0 & -\omega^h & -\omega^i & -\omega^j \end{array}$$

Again we obtain a list of requirements on e, f, g, h, i, and j. They are

$$-\omega^f + \omega^g + \omega^h + \omega^i + \omega^j = 0$$
$$-1 + \omega^e + \omega^h + \omega^i + \omega^{j-2} = 0$$

and so $\{g-f, h-f, i-f, j-f\} = \{e,h,i,j-2\} = \{0,0,1,2\}$. If $g=f$ then h, i, j are distinct and we are forced to have $(e,h,i,j) = (2,0,1,2)$ or $(2,1,0,2)$. However, the seventh and eighth columns of the matrix agree on the first two rows and so we may permute those columns if necessary and assume the solution is $(e,h,i,j) = (2,0,1,2)$. The other case ($g \neq f$) leads to $(e,f,g,h,i,j) = (1,0,2,0,0,1)$, $(0,0,1,0,2,0)$, $(0,1,2,0,1,1)$, $(1,2,1,0,2,2)$, $(0,2,0,1,2,2)$, after permuting columns 7 and 8, if needed. This situation has 8 possibilities. (In addition there are 8 possibilities if the second row is the conjugate

$$-1 \quad -1 \quad 0 \quad 0 \quad 1 \quad -\omega^2 \quad -1 \quad -1 \quad -\omega$$

but we will not consider that possibility until the end of the search.)

So there are 19 possible starter sets of three rows arising from Matrix 1. At this point *GAP* was used to exhaustively search for the remaining rows. In the case of Matrix 1, no solutions were found.

Similar computations were made with each of the five matrices. Solutions for the first two or three rows were found and then an exhaustive search was made using *GAP*. (The starter rows for matrices 1 through 5 may be obtained by writing to the authors.)

5 Construction

Each design is stored as a 9×9 array, in which each of the entries is an ordered pair. The first entry of the ordered pair is the corresponding entry from the Matrix 1, 2, 3, 4 or 5. The second entry in the ordered pair indicates the power, or sum of powers of P being used. Specifically, $[0,0]$ is the 3 by 3 zero matrix, $[1,0]$ is $P^0 = I_3$, $[1,1]$ is P, $[1,2]$ is P^2, $[2,0]$ is $P+P^2$, $[2,1]$ is P^2+P^0, $[2,2]$ is $P+P^0$, and $[3,0]$ is J_3.

For example, the first of the 19 sets of starter rows for Matrix 1 is

$$\begin{array}{ccccccccc} 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & 0 & 1 & -1 & -1 & -\omega^2 & -\omega \\ 1 & -1 & -1 & -1 & 0 & 0 & -1 & -\omega & -\omega^2 \end{array}$$

which we encode as

[0,0]	[0,0]	[1,0]	[2,0]	[2,0]	[2,0]	[2,0]	[2,0]	[2,0]
[2,0]	[2,0]	[0,0]	[0,0]	[1,0]	[2,0]	[2,0]	[2,2]	[2,1]
[1,0]	[2,0]	[2,0]	[2,0]	[0,0]	[0,0]	[2,0]	[2,1]	[2,2]

in *GAP*.

The next step is to find solutions to equation (1_1) given one of the five solutions to (1_0) and (2_0). We will illustrate the method using Matrix 1. Any design arising from Matrix 1 has one of 19 possible first three rows in its array. Each of the rows 4 to 9 of the array have 3^7 possibilities since each row contains the entries $[0,0]$ and $[3,0]$. For each row, each of the 3^7 possibilities was checked using *GAP* to see if it had intersection size 6 with all of the first three rows of the array, and only those with the appropriate intersection size were retained. Further, given any two possibilities for the ith row (where i is fixed and $4 \le i \le 9$), which differ only in that all second entries of the ordered pairs in one of them are a fixed multiple of all second entries in the ordered pairs of the other, it is only necessary to consider one such row, since the multiple is simply a block permutation in the resulting design. An exhaustive search through the remaining rows yielded *no* (27,13,6) designs.

The same procedure was used on each of the matrices. Matrix 2 had 17 possible first two rows for its arrays. Matrix 3 also had 17 possible first two rows for its arrays. Matrix 4 had 12 possible first two rows and Matrix 5 had 27 possible first three rows. Obviously, for matrices 2, 3 and 4 one would also have to consider the 3^7 possible third rows of the array, and all possible rows 3 to 9 would be checked against the given first two rows.

Matrix 1 did not produce any designs; Matrix 2 gave 108 (possibly non-isomorphic) designs; Matrix 3 gave 99 (possibly distinct) designs; Matrix 4 gave 36 (possibly non-isomorphic) designs and Matrix 5 gave 24 (possibly non-isomorphic) designs.

6 Isomorphism Testing

Isomorphism testing consisted of a number of different stages and approaches. Initial isomorphism testing consisted of converting the arrays to designs with point set $P = \{1, 2, \ldots, 27\}$ and block set $\{b_1, b_2, \ldots, b_{27}\}$ and then calculating the intersection pattern for each design. (This idea is effectively used in [7].) We define the intersection pattern for a block of a design to be a 6-tuple whose ith entry corresponds to the number of pairs of blocks of the design having mutual intersection size $(i-1)$ with the given block. The intersection pattern for a point is a 6-tuple whose ith entry corresponds to the number of pairs of points of the design which appear together with the fixed point on $(i-1)$ blocks of the design. The intersection pattern for the design consists of the collection of intersection patterns over all blocks and points of the design. Two designs are non-isomorphic if either one has at least one block or point intersection pattern which is not present in the other design, or if the two designs have identical block and point intersection patterns that occur with different multiplicities.

For those designs having identical intersection patterns the next stage of isomorphism testing consisted of searching for isomorphisms on points amongst the group $\langle (3i+1, 3i+2, 3i+3) : i = 0, 1, \ldots, 8 \rangle$. Each block of the design was viewed as a set and the collection of blocks were also taken to be a set – thus alleviating the need to consider individual block permutations.

Using this group, the number of possibly non-isomorphic designs was reduced from 108 to 18 for Matrix 2, from 99 to 19 for Matrix 3, from 36 to 17 for Matrix 4 and from 24 to 8 for Matrix 5.

At this stage, using intersection patterns, it was possible to identify at least six isomorphism classes amongst the designs arising from Matrix 2. In fact, there are exactly six non-isomorphic designs, these being A, B, C, D, E and F (see Appendix I). There are seven designs from Matrix 3, these being J, $J2$, K, L, M, N and O. Matrix 4 yields a further six designs, namely, P, Q, R, S, T and U. Finally, Matrix 5 gives three designs, G, H and I (I being the design found by Tonchev in [7]). For the designs A to U, see Appendix I. Therefore the original collection of 267 designs found at the end of stage 2 has been reduced to 22 isomorphism classes. We make several remarks:

1. There are four pairs of non-isomorphic dual designs, namely, K and L, P and Q, R and S, and T and U.

2. Design I corresponds to $D(5\&25)$ found by Tonchev [7]; this is the only design having a transitive automorphism group.

3. Designs $J2$ and J are non-isomorphic even though they have the same intersection patterns. This pair of designs are the only non-isomorphic pair with the same intersection pattern.

7 Conclusion

There are 487 Hadamard matrices of order 28 (see [2], [3], [4] and [6]), and apparently there are 208310 non-isomorphic (27,13,6) designs [6]. It is remarked in [8] that a (27,13,6) design with an automorphism of order 3 fixing no points gives rise to a code over GF(7) as follows: a matrix $f_0(A)$ which satisfies equations (1_0) and (2_0) has the property that $X := 3J_9 - f_0(A)$ satisfies $XJ = 14J$, $XX^t = 7I + 21J$. X is the generator matrix for a (9,4) self-orthogonal code over GF(7).

References

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Appendix I

This appendix lists the 21 arrays for the new designs, together with design I. Each design is stored as a nine by nine matrix, in which each of the entries is an integer between 0 and 7. This integer is the decimal equivalent of the first row of each 3×3 matrix viewed as a binary number. For example, $J - P^2$, which has first row 110 is encoded as 6.

$$A := \begin{bmatrix} [014273623] \\ [173624203] \\ [430521173] \\ [453442730] \\ [722250513] \\ [341107523] \\ [542733440] \\ [207512253] \\ [333033034] \end{bmatrix};$$

$$B := \begin{bmatrix} [472345430] \\ [743254340] \\ [157134023] \\ [511743203] \\ [263401723] \\ [624310273] \\ [250272513] \\ [522027153] \\ [003333334] \end{bmatrix};$$

$$C := \begin{bmatrix} [011634723] \\ [105443273] \\ [234745430] \\ [647431610] \\ [342371023] \\ [436117203] \\ [722602513] \\ [275120153] \\ [330033334] \end{bmatrix};$$

$$D := \begin{bmatrix} [441273650] \\ [106731213] \\ [267052113] \\ [470661123] \\ [735644210] \\ [612140673] \\ [521126703] \\ [311217063] \\ [033303334] \end{bmatrix};$$

$$E := \begin{bmatrix} [474123560] \\ [743564120] \\ [450572213] \\ [165701223] \\ [237052123] \\ [542210573] \\ [611225703] \\ [322127053] \\ [003333334] \end{bmatrix};$$

$$F := \begin{bmatrix} [474123560] \\ [743564120] \\ [130671123] \\ [256702113] \\ [467061213] \\ [311120673] \\ [522116703] \\ [641217063] \\ [003333334] \end{bmatrix};$$

$$G := \begin{bmatrix} [264445037] \\ [605234471] \\ [452746430] \\ [427504646] \\ [434043751] \\ [546437201] \\ [044672546] \\ [676130112] \\ [740344343] \end{bmatrix};$$

$$H := \begin{bmatrix} [015426743] \\ [430231257] \\ [301611572] \\ [443743043] \\ [451137430] \\ [312571601] \\ [746025446] \\ [437450464] \\ [371302644] \end{bmatrix};$$

$$I := \begin{bmatrix} [470444333] \\ [704421365] \\ [047412563] \\ [414353170] \\ [441335701] \\ [422533017] \\ [333170121] \\ [365701112] \\ [356017211] \end{bmatrix};$$

$$J := \begin{bmatrix} [152572103] \\ [515227013] \\ [267120433] \\ [621702343] \\ [474543230] \\ [745434320] \\ [432015723] \\ [340251273] \\ [003333334] \end{bmatrix};$$

$$J2 := \begin{bmatrix} [452347160] \\ [316474520] \\ [167225103] \\ [612731013] \\ [711026263] \\ [170151623] \\ [204346713] \\ [023452173] \\ [333300334] \end{bmatrix};$$

$$K := \begin{bmatrix} [431620713] \\ [345402173] \\ [454743230] \\ [317461640] \\ [102372253] \\ [016127523] \\ [724515203] \\ [273251023] \\ [330033334] \end{bmatrix};$$

$$L := \begin{bmatrix} [432310723] \\ [346101273] \\ [234745160] \\ [647431520] \\ [202672153] \\ [025127513] \\ [711625203] \\ [175452023] \\ [330033334] \end{bmatrix}; \quad M := \begin{bmatrix} [125251073] \\ [247325103] \\ [571026223] \\ [230762413] \\ [344547230] \\ [235474340] \\ [012415733] \\ [702152343] \\ [333300334] \end{bmatrix}; \quad N := \begin{bmatrix} [712016543] \\ [271601343] \\ [127160643] \\ [304712233] \\ [430271433] \\ [043127133] \\ [356421470] \\ [444333740] \\ [333333004] \end{bmatrix};$$

$$O := \begin{bmatrix} [435614207] \\ [571202534] \\ [327150234] \\ [612725034] \\ [150271233] \\ [225027133] \\ [402512733] \\ [033333340] \\ [744433304] \end{bmatrix}; \quad P := \begin{bmatrix} [620234743] \\ [152705341] \\ [203446372] \\ [343137402] \\ [367210314] \\ [744344033] \\ [046652447] \\ [271045455] \\ [132674430] \end{bmatrix}; \quad Q := \begin{bmatrix} [612337021] \\ [250464473] \\ [023374612] \\ [304314547] \\ [274123606] \\ [456704254] \\ [733430444] \\ [447013453] \\ [312243750] \end{bmatrix};$$

$$R := \begin{bmatrix} [443710332] \\ [332012572] \\ [723424603] \\ [416454057] \\ [437463120] \\ [044433743] \\ [304647232] \\ [350574121] \\ [474306423] \end{bmatrix}; \quad S := \begin{bmatrix} [437443034] \\ [431230467] \\ [313574404] \\ [704445463] \\ [221654370] \\ [014437345] \\ [365021724] \\ [370613411] \\ [113701323] \end{bmatrix}; \quad T := \begin{bmatrix} [223232370] \\ [732140235] \\ [343374404] \\ [563047141] \\ [610346742] \\ [117602645] \\ [074134345] \\ [104266137] \\ [451716031] \end{bmatrix};$$

$$U := \begin{bmatrix} [173534044] \\ [462522703] \\ [616507124] \\ [113063227] \\ [647220664] \\ [201764163] \\ [344473340] \\ [750111463] \\ [061143574] \end{bmatrix};$$