Connecting the Permutation Representations of a Group

Johannes Siemons
School of Mathematics
UEA Norwich
Norwich NR4 7TJ
United Kingdom
email: j.siemons@uea.ac.uk

ABSTRACT. Suppose that a finite group G acts on two sets X, Y and that FX, FY are the natural permutation modules for a field F. We examine conditions which imply that FX can be embedded in FY in other words that (∗): There is an injective G-map $FX \to FY$. For primitive groups we show that (∗) holds if the stabilizer of a point in Y has a 'maximally overlapping' orbit on X. For groups of rank three we show that (∗) holds unless a specific divisibility condition on the eigenvalues of an orbital matrix of G is satisfied. Both results are obtained by constructing suitable incidence geometries.

1 Introduction

Many finite groups of Lie type have natural permutation representations which are doubly transitive or at least have low rank, and the same can be said about sporadic simple groups [1,3]. We are interested in permutation representations of such groups other than these natural ones and in particular in the following question: When is it possible to make inference about permutational properties of such actions in terms of the natural representation? Here is an example of what we have in mind.

Theorem. *Suppose that G acts on two finite sets X and Y where G is doubly transitive on X. Then precisely one of the following holds:*

(i) *For all $y \in Y$ the stabilizer G_y is transitive on X.*

(ii) *For every subgroup $H \leq G$ there are at least as many H-orbits on Y as there are H-orbits on X.*

This result is from [4] where we have discussed this problem also for infinite groups. There are similar questions: Can we relate the permutation ranks of subgroups? Can we compare the cycle structures of group elements in the two representations? Some of these have been answered and a pattern has emerged. The question about cycle structures, for instance, has been investigated in [2]. Important for the theorem above is that a non-trivial G_y-orbit on X implies the existence of an injective G-map $FX \to FY$. (FX and FY denote the natural permutation modules over $\mathbb{C}$). In other words, if (i) fails, then FX is isomorphic to a G-submodule of FY, and it is this property which allows us to answer some of the other questions mentioned above.

The purpose of this note therefore is to investigate what kinds of interconnections between (G, X) and (G, Y) are needed to guarantee an embedding of FX in FY. This question is best approached from the view point of incidence geometry, we explain this in section 2 below.

In section 3 we investigate primitive groups in which point stabilizers have orbits on the other set with maximal 'overlap' – see the first alternative in the following

Theorem A. *Suppose that G acts on two finite sets X and Y where G is primitive on X. Then one of the following holds:*

(i) *For any $y, y^* \in Y$ let Γ and Γ^* be unions of orbits on X of G_y and G_{y^*} respectively, with $|\Gamma| = |\Gamma^*|$ and $\Gamma^* = \Gamma^g$ if $y^{*g} = y$. Then $\Gamma = \Gamma^*$ or $|\Gamma \cap \Gamma^*| < |\Gamma| - 1$.*

(ii) *If F is a field whose characteristic is 0 or sufficiently large then there is an injective G-map ϕ: $FX \to FY$.*

The overlapping property is least stringent when G_y has sufficiently many short orbits on X and as a result such groups will generally allow an embedding of the permutation modules. After Lemma 3.2 we comment how the overlapping condition can be translated into a condition on the G_x-orbits on Y. The second result is about groups of small rank.

Theorem B. *Suppose that G acts on two finite sets X and Y where G has rank 3 on X. Then one of the following holds:*

(i) *For $x \in X$ and any non-trivial union $\emptyset \neq \Delta \neq Y$ of G_x-orbits on Y the set $\{|\Delta \cap \Delta^g| \mid g \in G, \Delta \neq \Delta^g\}$ consists of two values $\lambda_1 \neq \lambda_2$ where $(\lambda_1 - \lambda_2)$ divides $(\lambda_1 - |\Delta|)$ and $(\lambda_2 - |\Delta|)/(\lambda_1 - \lambda_2)$ is a non-maximal eigenvalue of an orbital matrix of G on X.*

(ii) *If F is a field whose characteristic is 0 or sufficiently large then there is an injective G-map ϕ: $FX \to FY$.*

This is proved in section 4 and we show that similar considerations apply to groups of rank up to 5.

2 Incidence Maps

For the remainder we assume G is a group acting on two finite sets X and Y where neither action needs to be faithful. If F denotes a field then FX and FY are the vector spaces with X and Y as bases. So these are the natural permutation modules for G over F. We are interested in G-maps $\phi\colon FX \to FY$ which are injective. As the sets are finite every $\phi\colon FX \to FY$ has a transpose $\phi^*: FY \to FX$ with respect to the standard inner products. Clearly, ϕ^* is a G-map if and only if ϕ is a G-map and as ranks are equal, ϕ^* is surjective if and only if ϕ is injective.

Now suppose that $I \subseteq X \times Y$ is some relation. We regard I as an incidence relation and define *incidence maps* $\phi_I\colon FX \to FY$ and $\phi_I^*\colon FY \to FX$ by $\phi_I(x) = \sum_{(x,y)\in I} y$ and $\phi_I^*(y) = \sum_{(x,y)\in I} x$. The *incidence matrix* $M(I)$ of I is the matrix of ϕ_I with respect to X and Y. Its rows are indexed by X, its columns by Y so that the (x, y)-entry is 1 if $(x, y) \in I$ and 0 otherwise. It is clear that ϕ_I^* is the transpose of ϕ_I and both maps have the same rank.

Furthermore, ϕ_I and ϕ_I^* are G-maps if and only if G preserves the relation. Therefore one approach to finding injective G-maps $\phi\colon FX \to FY$ is to examine the G-relations on (X, Y) and proving that at least one of the corresponding incidence maps have maximal rank.

To make this idea quite clear let I be any relation on $X \times Y$. For $x \in X$ and $y \in Y$ denote $xI := \{y \mid (x, y) \in I\}$ and $Iy := \{x \mid (x, y) \in I\}$. The following is easy to establish.

Lemma 2.1. *Let G act on X and Y and suppose that I is some relation on $X \times Y$. If I is G-invariant, then xI is a union of G_x-orbits on Y and Iy is a union of G_y-orbits on X. Conversely, if Δ is a union of G_x-orbits on Y, then there is a G-invariant relation I such that $xI = \Delta$ and I is unique if (G, X) is transitive. Similarly, if Γ is a union of G_y-orbits on X, then there is a G-invariant relation I with $Iy = \Gamma$ which is unique if (G, Y) is transitive.*

We denote a relation with $xI = \Delta$ or $Iy = \Gamma$ by I_Δ and ${}_\Gamma I$ respectively. Note that I_Δ and ${}_\Gamma I$ are unique if G acts transitively on the sets. In fact, the lemma implies the following: If G acts transitively on both sets and if $x \in X$ and $y \in Y$, then $I_\Delta = {}_\Gamma I$ defines a one-to-one correspondence between unions of G_x-orbits on Y and unions of G_y-orbits on X. Therefore conditions on G_x-orbits on Y can be translated into conditions on G_y-orbits on X and vice versa. We will make use of this later.

To give an example consider again the doubly transitive group from the theorem in the introduction. So G acts doubly transitively on X and arbitrarily on Y. Regard the elements of Y as *blocks* and if y is a block let $\Gamma \subset X$ be some non-trivial union of G_y-orbits. Regarding X as *points*, the definition $I = {}_\Gamma I$ declares that Γ is the set of points incident with y. Duality then says that for any $x \in X$ there is some union Δ of G_x-orbits on blocks for which $I_\Delta = I$. Obviously, Δ is the pencil of blocks through x. But notice, as G is doubly transitive on X, there is a constant number of blocks through any pair of distinct points. So the relation we have just defined is a 2-design, and the conclusion of the theorem is the well-known lemma of Block.

Finiteness is important for these arguments. In infinite designs Block's Lemma does not hold and in [4] we have given counter examples. We also show that some form of Block's Lemma does remain valid if block sizes are finite and if FX is an almost irreducible G-module. For the full details see the paper.

3 Primitivity

Let X, Y be sets and I an incidence relation on $X \times Y$. Elements of X are called *points* and elements of Y *blocks*. If y is a block, then Iy is the set of points incident with y and so it makes sense to call $|Iy| =: |y|$ the *size* of y. Similarly, xI is the collection of all blocks through x, and $|x| := |xI|$ is the *degree* of x. Double counting shows that $\sum_{x\in X} |x| = \sum_{y\in Y} |y|$.

For the remainder assume that $|x|$ and $|y|$ are constant, in particular $|x|.|X| = |y|.|Y|$. We are interested in the situation when $|y|$ is small, or equivalently, when $|x|$ is small with regards to $|Y|/|X|$. For $|y| = 1$ the map $\phi_I\colon FX \to FY$ is injective trivially, regardless of the field. For $|y| = 2$ we can think of I as an undirected graph, possibly with repeated edges, but without loops. The following is well-known:

Lemma 3.1. *Let I be a graph with vertex set X and edge set Y. Suppose that the number of connected components is c and that b of them are bipartite. Then the rank of $\phi_I\colon FX \to FY$ is $|X|-c$ if F has characteristic 2 and $|X| - b$ otherwise.*

Proof: It is sufficient to consider connected graphs. So let $\partial\colon FY \to FX$ be the transpose of ϕ_I and fix a vertex v^*. If v is some other vertex then there is a path $\{v^*, v_1\} = e_i, \{v_1, v_2\} = e_2, \ldots, \{v_{s-1}, v\} = e_s$ and $\partial(e_1 - e_2 + \cdots + (-1)^{s-1}e_s) = v^* + (-1)^{s-1}v$. Hence ∂ has rank $\geq |X|-1$. If $X = X_1 \cup X_2$ is a bipartition, then $\phi_I(\sum_{v\in X_1} v) = \sum_{y\in Y} y = \phi_I(\sum_{v\in X_2} v)$ and this shows that $\mathrm{rank}(\phi_I) = \mathrm{rank}(\partial) = |X| - 1$.

If the graph is not bipartite then v^* is on some odd cycle. Hence we can take $v = v^*$ and $2 \leq s-1$ above so that $\partial(e_1 - e_2 + \cdots + (-1)^{s-1}e_s) = 2v^*$. If the characteristic of F is $\neq 2$ then ∂ is a surjection and ϕ has rank $|X|$. If F has characteristic 2, then $\phi_I(\sum_{v\in X} v) = 2\sum_{y\in Y} y = 0$ from which it follows that ϕ_I has rank $|X|-1$. □

This proof is given because some of the argument can be extended to incidence structures with $|y|>2$ if they admit a point-primitive automorphism group. Notice, if a graph admits such a group then it is necessarily connected and non-bipartite, since components, or parts of a bipartition, are blocks of imprimitivity.

Lemma 3.2. *Let G act primitively on X and arbitrarily on Y. Further, let I be a relation on $X \times Y$ which is preserved by G. Then (i) implies (ii):*

(i) *There are y, y^* in Y with $|y|-1=|Iy \cap Iy^*|=|y^*|-1$.*

(ii) *$\phi_I\colon FX \to FY$ has rank at least $|X|-1$, and equal to $|X|$ if $|Iy| \neq 0$ in F.*

Proof: Let v be the point in $Iy \setminus Iy^*$ and let v^* be the point in $Iy^* \setminus Iy$. If $\partial\colon FY \to FX$ is the transpose of $\phi_I\colon FX \to FY$ then $\partial(y-y^*)=v-v^*$. Now we define a relation on X by $x \sim x^*$ if and only if $x-x^* \in \partial(FY)$. It is easy to see that this is an equivalence relation. As G is primitive on X we conclude that $\sim$ is the 'all' relation. Therefore, if x, x^* are any two points in X, then $x-x^*$ belongs to $\partial(FY)$.

Hence ∂ and ϕ_I have rank at least $|X|-1$. Now fix some $y \in Y$ and let $Iy=\{v, v_1, v_2, \ldots, v_{k-1}\}$. Then there are elements $w_1, w_2, \ldots, w_{k-1}$ in FY for which $\partial(w_i)=v-v_i$. From this we see that $\partial(y+w_1+w_2+\ldots+w_{k-1})=kv$ and so ∂ is surjective unless $k=0$ in F. □

Remarks: 1). It may be worth to write down the dual of the overlap condition (i), at least when block sizes are constant:

$(i)^\perp$ There exist points $x_0, x_1, \ldots, x_k \in X$ for which $\bigcap_{i=10,\ldots,k-1} x_i I$ and $\bigcap_{i=1,\ldots,k} x_i I$ are non-empty, where k is the maximal integer for which there exist points $x_1^*, x_2^*, \ldots, x_k^*$ with $\bigcap_{i=1,\ldots,k} x_i^* I$ non-empty.

To see the duality take y in $\bigcap_{i=1,\ldots,k-1} x_i I$ and y^* in $\bigcap_{i=1,\ldots,k} x_i I$. Then $|y|=k=|y^*|$ by definition and $|Iy \cap Iy^*|=k-1$.

2). For $|y|=1$ the overlap condition is trivial, and it also holds automatically when $|y|=2$. For if $|Iy \cap Iy^*|$ is always either 0 or 2, then Iy is a block of imprimitivity for the G-action on X. In terms of condition $(i)^\perp$ we see therefore that if $|y|$ is constant and $|x|<3|Y|/|X|$ then condition (ii) holds. In other words, (ii) holds whenever the degree of a point is less than $3|Y|/|X|$. In this form the graph nature of condition (i) is not quite so immediate.

3). When $k=3$, we can think of I as a triple system, and for condition (i) to fail any two (distinct) points must be on at most one block. So I is either a Steiner triple system (any two points are on a unique block) and

(ii) holds nevertheless by Fisher's inequality. Or otherwise I is a partial Steiner triple system (some pairs of points fail to be on a common block) and here conclusion (ii) is quite open.

The proof of theorem A is now immediate. If (i) fails then there is an incidence relation I, defined consistently by ${}_{\Gamma}I = {}_{\Gamma^*}I$ for which $\Gamma = Iy$ and $\Gamma^* = Iy^*$ satisfy the overlap condition of Lemma 3.2.

4 Groups of small rank

Now we suppose that G has rank at most 5 in the action on X. Throughout we fix the field F and suppose that its characteristic is 0 or sufficiently large. Again we are interested in whether injective G-maps $FX \to FY$ exist. The first result is well known and deals with the question we have raised at the beginning of the introduction. We give a proof because it illustrates the arguments later.

Lemma 4.1. (2-Designs) *Let G act doubly transitively on the set X and arbitrarily on Y, and let F be a field with characteristic* 0 *or sufficiently large. Then the following are equivalent:*

(i) *For $x \in X$ there is a set $\Delta \subset Y$ fixed setwise by G_x but not by G.*

(i)$^{\perp}$ *For some $y \in Y$ there is a set $\Gamma \subset X$ fixed setwise by G_y but not by G.*

(ii) *There is a relation I on $X \times Y$, $I = I_\Delta$ or $I = I_\Gamma$, for which $\phi_I \colon FX \to FY$ is injective.*

Proof: Conditions (i) and $(i)^{\perp}$ are dual to each other, following the discussion after lemma 2.1. Suppose therefore that (i) holds and define a relation by $I = I_\Delta$. Let $M = M(I)$ be its incidence matrix and put $N := M.M^T$ where M^T denotes the transpose of M. Clearly, if $g \in G$ is written as a permutation matrix, then $gNg^{-1} = N$ as I is G-invariant. Further, as G is doubly transitive, all off-diagonal entries are equal to λ, say, and all diagonal entries are equal to $r = |\Delta|$. Now compute the determinant $d = (r-\lambda)^{n-1}(r + (|X|-1)\lambda)$ of N and note that $d \neq 0$ iff $r \neq \lambda$. But λ is the cardinality of $\Delta \cap \Delta^g$ for any g with $v \neq v^g$. Therefore $d \neq 0$ and $\phi_I \colon FX \to FY$ has rank $|X|$. Conversely, if Δ is a G-orbit, then $\phi_I(v) = \phi_I(v^g)$ for all $g \in G$, and so ϕ_I is not injective. □

Remarks: 1). Statements (i) and $(i)^{\perp}$ can of course be replaced by saying that I is G-invariant and non-trivial. The simple but significant observation is that then I is a 2-design on X, and this has motivated the analysis of incidence maps in a more general setting. The lemma gives also a simple verification of the theorem at the beginning of the introduction. For alternative (ii) in the theorem is just Block's Lemma which holds in any finite 2-design.

2). If we replace condition (ii) by a weakening, namely that there exists an injection ϕ: $FX \rightarrow FY$ (for instance when F are the complex numbers) then it is unclear if this already implies the existence of an injective incidence map. This might be an interesting problem.

Now suppose that G acts on X as a group of rank m. Then there are m G-orbits D (the diagonal orbit), $0_1, 0_2, \ldots, 0_{m-1}$ on $X \times X$. We represent each by an $|X| \times |X|$ matrix whose (x, x^*)-entry is 1 if (x, x^*) belongs to the orbit, and zero otherwise. These are the orbital matrices of the action and they span an m-dimensional algebra $\mathbb{A}$ which is called the *centralizer algebra* of the action. $\mathbb{A}$ contains all matrices C for which $g^{-1}Cg = C$ whenever $g \in G$ is written as a permutation matrix. As we have seen in the proof of Lemma 4.1, if I is a G-invariant relation with matrix M, then $M.M^T$ belongs to $\mathbb{A}$. Our strategy is to determine the rank of M by knowing enough about $\mathbb{A}$.

From centralizer theory (see chapter V of Wielandt's book [5]) it is known that $m = \sum n_i^2$ when the permutation character $\pi = \sum n_i \chi_i$ is written as a sum of irreducible characters χ_i and furthermore that A is commutative if and only if all multiplicities are at most 1. This is the case automatically if $m \leq 5$. It is sufficient to rule out $5 = 1 + 2^2$, that is $\pi = 1 + 2\chi$: Note that G must contain a fixed point free element g for which $0 = \pi(g) = 1 + 2\chi(g)$, contradicting the fact that $\chi(g)$ is an algebraic integer.

We first deal with the case of rank 3 groups where one can say a little more:

Lemma 4.2. (Rank 3 Groups): *Let G act as a rank three group on and arbitrarily on Y. Suppose that $\emptyset \neq I \neq X \times Y$ is a G-invariant relation and that F is a field of characteristic 0 or sufficiently large. Then one of the following holds*

(i) $\{|xI \cap x^*I| \mid x, x^* \in X$ *and* $xI \neq x^*I\} = \{\lambda_1, \lambda_2\}$ *with* $\lambda_1 \neq \lambda_2$ *where* $(\lambda_1 - \lambda_2)$ *divides* $(\lambda_1 - |x|)$ *and* $(\lambda_2 - |x|)/(\lambda_1 - \lambda_2)$ *is a non-maximal eigenvalue for one of the orbital matrices of* (G, X).

(ii) ϕ_I: $FX \rightarrow FY$ *is an injective G-map.*

Proof: As G has rank three on X we have three orbital matrices Id, A and $J - Id - A$ (where J is the all 1-matrix) and $\mathbb{A} = \langle Id, A, J\rangle$ has dimension 3. Let $M = M(I)$ be the matrix of I and $N := M.M^T$. Clearly, N belongs to $\mathbb{A}$ and we are going to show that N is non-singular unless (i) holds.

So if $N = rId + \lambda_1 A + \lambda_2(J - Id - A)$, then $r = |X|$ is the number of blocks through a point and the other two coefficients count how many blocks pass through a pair (x, x^*). This number is $|xI \cap x^*I|$ and it only depends on the orbital to which (x, x^*) belongs. We can assume $\lambda_1 \neq \lambda_2$ for otherwise I is a non-trivial 2-design and Lemma 4.1 applies.

If G is primitive on X, then Sims' theorem implies that A, viewed as the adjacency matrix of a graph, is connected. So $\mathbb{A} = \langle Id, A \rangle$ as J is a polynomial in A, and A has precisely three eigenvalues $\alpha_0 > \alpha_1 > \alpha_2$ where α_0, the row sum of A, has multiplicity 1. If G is imprimitive on X then A represents disconnected copies of complete graphs. In this case, however, the other orbital is connected and so $J - Id - A$ has three eigenvalues which satisfy the above condition. So, after interchanging 0_1 and 0_2 if necessary, we can assume that A has three eigenvalues $\alpha_0 > \alpha_1 > \alpha_2$ where α_0 has multiplicity 1.

As $\mathbb{A}$ is commutative, the eigenvalues $\nu_0 > \nu_1$, ν_2 of $N = rId + \lambda_1 A + \lambda_2(J - Id - A)$ are easily computed. Clearly $\nu_0 = r + \lambda_1\alpha_0 + \lambda_2(|X| - 1 - \alpha_0) > 0$ is the row sum of N and the remaining values are $\nu_i = r + \lambda_1\alpha_1 + \lambda_2(-1 - \alpha_i)$. So if N is singular then $\alpha_i = \lambda_2 - r/(\lambda_1 - \lambda_2)$, for $i = 1$ or $i = 2$, and $(\lambda_1 - \lambda_2)$ divides $\lambda_2 - r$ as α_i is an algebraic integer. □

Remark. The Lemma is best possible in the sense that (i) occasionally does occur. For instance, there are generalized quadrangles which have rank 3 automorphism groups for which the incidence maps are singular and where the point module can not be embedded in the line module. The smallest one has A_5 as automorphism group, and the characters on X and Y are genuinely different.

The proof of Theorem B follows directly from the Lemma. We are going to extend it to groups of rank 4 and 5. The result is not as concise simply because there are many more possibilities.

Lemma 4.3. (Groups of Rank 4 or 5): *Let G act on X as a rank m group with $4 \leq m \leq 5$, and arbitrarily on Y. Suppose that $\emptyset \neq I \neq X \times Y$ is a G-invariant relation and that F is a field of characteristic 0 or sufficiently large. Then ϕ_I: $FX \to FY$ is an injective G-map unless the following holds:*

(∗) $\{|xI \cap x^*I| \mid x, x^* \in X, xI \neq x^*I\} = \{\lambda_1, \ldots, \lambda_{m-1}\}$ *and* $0 = (|x| - \lambda_3) + (\lambda_1 - \lambda_3)\alpha + (\lambda_2 - \lambda_3)\beta$ *if* $m = 4$, *or* $0 = (|x| - \lambda_4) + (\lambda_1 - \lambda_4)\alpha + (\lambda_2 - \lambda_4)\beta + (\lambda_3 - \lambda_4)\gamma$ *if* $m = 5$ *where* α, β, γ *are eigenvalues of distinct orbital matrices for the G-action on X.*

Proof: Again the centralizer algebra is generated by orbital matrices, $\mathbb{A} = \langle Id, A, B, J - Id - A - B \rangle$ or $\mathbb{A} = \langle Id, A, B, C, J - Id - A - B - C \rangle$ depending on whether G has rank 4 or 5 on X. Denote the eigenvalues of A by $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$, those of B by $\beta_0, \beta_1, \ldots, \beta_{m-1}$ etc.

Let $M = M(I)$ be the incidence matrix of I. Then as before $N := M.M^T$ belongs to $\mathbb{A}$. Expressing N by the generators gives $N = rId + \lambda_1 A +$

$\lambda_2 B + \lambda_3(J - Id - A - B)$ or $N = rId + \lambda_1 A + \lambda_2 B + \lambda_3 C + \lambda_4(J - Id - A - B - C)$. We have $r = |x|$ and $\lambda_i = |xI \cap x^* I|$ counts the number of blocks through a pair (x, x^*). As $\mathbb{A}$ is commutative the eigenvalues of N are sums of eigenvalues of the orbital matrices. In particular, the non-maximal eigenvalues are $\nu_i = r + \lambda_1\alpha_i + \lambda_2\beta_i + \lambda_3(-1 - \alpha_i - \beta_i)$ and $\nu_i = r + \lambda_1\alpha_i + \lambda_2\beta_i + \lambda_3\gamma_i + \lambda_4(-1 - \alpha_i - \beta_i - \gamma_i)$ respectively. So if N is singular, then $0 = (r - \lambda_3) + (\lambda_1 - \lambda_3)\alpha)i + (\lambda_2 - \lambda_3)\beta_i$ or $0 = (r - \lambda_4) + (\lambda_1 - \lambda_4)\alpha_i + (\lambda_2 - \lambda_4)\beta_i + (\lambda_3 - \lambda_4)\gamma_i$ for some i and eigenvalues α_i, β_i, γ_i of distinct orbital matrices. This proves the lemma. □

References

[1] P.J. Cameron, Finite permutation groups and finite simple groups, *Bull. London Math Soc.* **13** (1981), 1–22.

[2] R.J. Chapman and B.S. Webb, A note on a Lemma of Brauer, *Discrete Mathematics* **138** (1995), 187–191.

[3] C.W. Curtis, W.M. Kantor and G. Seitz, The 2-transitive permutation representations of the finite Chevalley groups, *Transactions American Math Soc.* **218** (1976), 1–57.

[4] D.M. Evans and I.J. Siemons, On the number of orbits of a group in two permutation representations, *Archiv der Mathematik*, **60** (1993), 420–424.

[5] H. Wielandt, *Finite permutation groups*, Academic Press, 1964.