# Computing Geodetic Bases of Chordal and Split Graphs

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**Abstract**

The *geodetic cover* of a graph $G = (V, E)$ is a set $C \subseteq V$ such that any vertex not in $C$ is on some shortest path between two vertices of $C$. A minimum geodetic cover is called a *geodetic basis*, and the size of a geodetic basis is called the *geodetic number*. Recently Harary, Loukakis, and Tsouros announced that finding the geodetic number of a graph is NP-Complete. In this paper, we prove a stronger result, namely that the problem remains NP-Complete even when restricted to chordal graphs. We also show that the problem of computing the geodetic number for split graphs is solvable in polynomial time.

## 1 Introduction

Let $G = (V, E)$ be a simple, undirected, connected graph. If $X \subseteq V$, then an *X-geodesic* is a shortest path between two vertices of $X$. The *geodetic closure* of $X$, denoted $(X)$, is the set of all vertices which lie on some $X$-geodesic. Note that $X \subseteq (X)$. A set $X$ is a *geodetic cover* of $G$ if $(X) = V$. An alternate definition is that X is a geodetic cover if any vertex in $V - X$ lies on some $X$-geodesic. A mininum cardinality geodetic cover is called a *geodetic basis*. The size of a geodetic basis is called the *geodetic number* of $G$, and is denoted $g(G)$.

As an example, consider the graph in Figure 1. The geodetic closure of the set $\{1,3,5\}$ is

$$(\{1,3,5\}) = \{1,2,3,4,5,8\}.$$

The set $\{3,6,9\}$ is a geodetic basis of the graph. The geodetic number of the graph is $|\{3,6,9\}| = 3$.

Given a vertex $v \in V$, the *(closed) neighborhood* of $v$ is the set of $v$ and all vertices adjacent to $v$. A vertex is *simplicial* if its neighborhood induces a

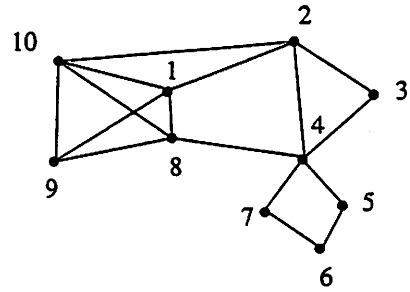


Figure 1: The set $\{3, 6, 9\}$ is a geodetic basis for this graph.

clique. In Figure 1, vertices 3 and 9 are simplicial. The following theorem will be very useful in later discussion.

**Theorem 1.1** *Let $S$ be the set of simplicial vertices of a graph, and let $C$ be any geodetic cover. Then $S \subseteq C$.*

**Proof:** Assume vertex $s$ is simplicial and that there exists a geodetic cover $C$ such that $s \notin C$. Then there must be a $C$-geodesic which includes $s$ but does not have $s$ as an endpoint. Let $\pi$ be such a geodesic, and let $x$ and $y$ the two vertices of $\pi$ which are adjacent to $s$ (refer to Figure 2). Since $s$ is simplicial,

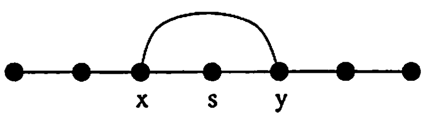


Figure 2: A path $\pi$ containing simplicial vertex $s$.

$x$ and $y$ are adjacent, and it follows that $\pi$ is not a shortest path between its endpoints, a contradiction. □

A graph is *chordal* if all cycles of length greater than three in the graph have a chord. Chordal graphs are a well known class of perfect graphs [6], and many problems in computational graph theory which are NP-Complete for general graphs are known to be solvable in polynomial time for chordal graphs. Such problems include the coloring, clique, independent set, and clique covering problems.

Chordal graphs have structural properties that make them interesting from the standpoint of computing the geodetic basis. In particular, they are the intersection graphs of subtrees of a tree [3, 5, 10]. Let $\mu(G)$ be the set of maximal cliques of a chordal graph $G$. A tree $T$ can be constructed so that each vertex of $T$ corresponds to a member of $\mu(G)$, and two vertices of $T$ are adjacent if and only if their corresponding maximal cliques in $\mu(G)$ have an edge in common. The tree $T$ is called the *clique tree* of $G$ (See Figure 3). Every

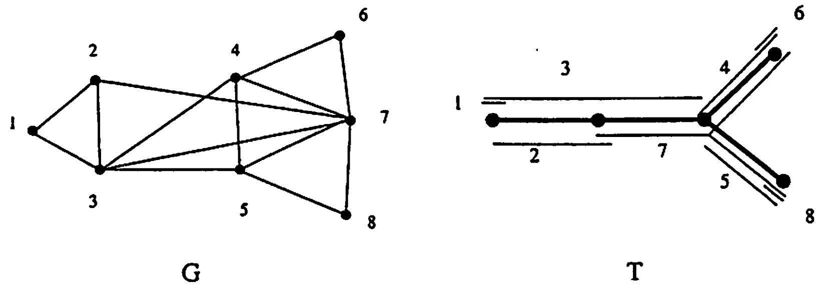


Figure 3: A chordal graph and its corresponding clique tree.

vertex of $G$ corresponds to a subtree of $T$, and two vertices of $G$ are adjacent if and only if the corresponding subtrees of $T$ overlap.

Since it is clear that the geodetic basis of a tree is simply the leaves of the tree [7], the clique-tree structure in chordal graphs suggests that their geodetic bases should also be easily computed. This suggestion is reinforced by the following theorem from [9].

**Theorem 1.2** *Let $G$ be a chordal graph, and $T$ the clique tree of $G$. If $\ell$ is a leaf of $T$, then the clique of $G$ corresponding to $\ell$ contains at least one simplicial vertex.*

For example, in Figure 3, the vertices 1, 6, and 8 are simplicial vertices of the chordal graph $G$. Equivalently, they are in exactly one maximal clique of $G$. Further, the maximal cliques which they occupy correspond to the leaves of the clique-tree $T$.

A *split graph* is a chordal graph whose complement is also chordal. Equivalently, a split graph is a graph whose vertex set can be partitioned into a clique and an independent set [6]. An example of a split graph is shown in Figure 4.

In Section 2 we prove that, in spite of the above observations about the structure of chordal graphs, computing geodetic bases of chordal graphs is an NP-Complete problem. However, in Section 3, we develop and analyze an efficient algorithm for finding geodetic bases of split graphs in polynomial time.

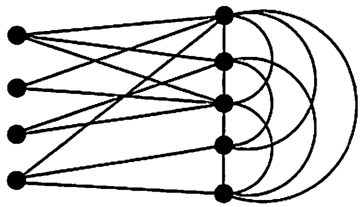

Figure 4: A split graph, with the independent set on the left, and the clique on the right.

## 2 Geodetic Bases of Chordal Graphs

To prove the NP-Completeness of finding geodetic bases for chordal graphs, we must first express the problem as a decision problem, the *geodetic cover problem for chordal graphs:* "Given a chordal graph $G = (V, E)$ and an integer $k$, with $1 \leq k \leq |V|$, does $G$ have a vertex set $X \subseteq V$ such that $|X| = k$ and every vertex in $V - X$ is on some $X$-geodesic?"

We mention in passing that this decision problem is solvable in polynomial time by a non-deterministic Turing machine; once a set of vertices is "guessed," one need only calculate their geodetic closure in polynomial time, then compare the closure to the vertex set $V$ of the graph.

All that remains in the proof of NP-Completeness is to show that a known NP-Complete problem has a polynomial transformation to the geodetic cover problem for chordal graphs. We choose the *dominating set problem for chordal graphs:* "Given a chordal graph $G = (V, E)$ and an integer $k$, with $1 \leq k \leq |V|$, is there a vertex set $X \subseteq V$ such that every vertex in $V - X$ is adjacent to some member of $X$?" This problem is known to be NP-Complete [2]. We show how this problem is polynomially transformable to the geodesic cover problem for chordal graphs.

Let $G = (V, E)$ be a non-trivial chordal graph, with $V = \{a_1, a_2, \ldots, a_p\}$, and let $k$ be an integer, with $1 \leq k \leq p$. We construct a supergraph of $G$ as follows. Let $G' = (V', E')$, with $V' = V \cup \{b_1, b_2, \ldots, b_p\} \cup \{c\}$, and $E' = E \cup \{a_1b_1, a_2b_2, \ldots, a_pb_p\} \cup \{a_1c, a_2c, \ldots, a_pc\} \cup \{b_1c, b_2c, \ldots, b_pc\}$. An example of this construction is shown in Figure 5. Clearly, the construction can be performed in polynomial time.

Note that $G$ is the subgraph of $G'$ induced by the set $\{a_1, a_2, \ldots a_p\}$. Also, $c$ is adjacent to every vertex in $G'$ and each $b_i$ is adjacent to only $a_i$ and c, for all $i$.

First we show that $G'$ is a chordal graph. If it is not, it will have a chordless $N$-cycle $v_1v_2\ldots v_Nv_1$, with $N > 3$. If every vertex of this cycle is in the set $V$, then the subgraph $G$ will not be chordal, a contradiction. Therefore the cycle

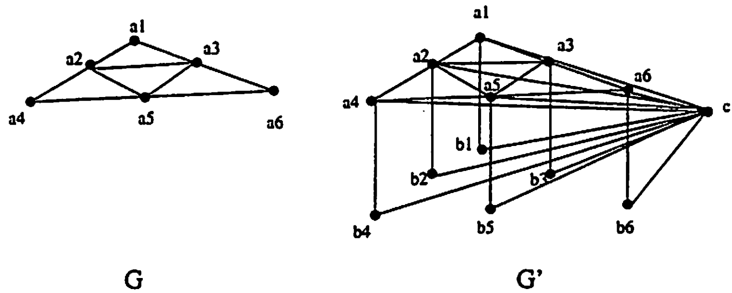


Figure 5: Transformation from the dominating set problem to the geodetic cover problem.

must have at least one vertex $v \in \{b_1, b_2, \ldots, b_p\} \cup \{c\}$. If $v \in \{b_1, b_2, \ldots, b_p\}$, then the cycle must also include $c$; and if $v \notin \{b_1, b_2, \ldots, b_p\}$, then $v = c$. Hence, the cycle must include $c$. But since $c$ is adjacent to every vertex in $G'$, it must be adjacent to every vertex in the cycle, and so the cycle has a chord, a contradiction. Therefore $G'$ is a chordal graph.

Next we show that if $G$ has a dominating set of size $k$, with $1 \le k \le p$, then $G'$ has a geodetic cover of size $p + k$. Suppose $D \subseteq V$ is a dominating set of $G$, with $|D| = k$. Let $D' = D \cup \{b_1, b_2, \ldots, b_p\}$ so that $|D'| = k + p$. Let $v$ be any vertex from $V' - D'$. Either $v = c$ or $v \in V$. If $v = c$, then $v$ lies on a geodesic between any pair of distinct vertices $b_i, b_j$. If $v \in V$, then $v = a_i$ for some $1 \le i \le p$. Since $D$ is a dominating set of $V$, there is some $a_j \in D'$ which is adjacent to $v = a_i$. Further, $b_i \in D'$, and $a_j b_i \notin E'$, so that the path $a_j a_i b_i$ is a $D'$-geodesic containing $v = a_i$. Since any vertex not in $D'$ is on some $D'$-geodesic, it follows that $D'$ is a geodetic cover of $G'$ of size $k + p$.

It remains to show that if $G'$ has a geodetic cover of size $k + p$, then $G$ has a dominating set of size $k$, with $1 \le k \le p$. We first prove some lemmas.

**Lemma 2.1** *Let $D'$ be any geodetic cover of $G'$. Then $\{b_1, b_2, \ldots, b_p\} \subseteq D'$.*

**Proof:** By the construction of $G'$, any $b_i$ is adjacent to only its corresponding $a_i$ and $c$. Since $a_i$ and $c$ are adjacent, $b_i$ is simplicial. Hence, by Theorem 1.1 $b_i$ must be in any geodetic cover. □

**Lemma 2.2** *The diameter of $G'$ is 2.*

**Proof:** Let $x, y$ be any two non-adjacent vertices in $G$. Then $xcy$ is a geodesic between them. □

**Lemma 2.3** *Let $D'$ be any geodetic cover of $G'$, and let $x \in V' - D'$. There must exist two vertices $y, z \in D'$ such that $xy \in E'$, $xz \in E'$, and $yz \notin E'$.*

**Proof:** Since $D'$ is a geodetic cover, $x$ must lie on a $D'$-geodesic. Any $D'$-geodesic which contains $x$ must have length 2, by the previous lemma. Let $xyz$ be a $D'$-geodesic containing $x$. If $yz \in E'$, then $xyz$ is not a geodesic. Therefore $yx, xz \in E'$ and $yz \notin E'$. □

We are now ready to show that if $G'$ has a geodetic cover $D'$ of size $p+k$, then $G$ has a dominating set $D$ of size $k$. Let $D = D' \cap V$, and let

$$a_i \in V - D = V - D' \subseteq V' - D'.$$

By Lemma 2.3, $a_i$ is adjacent to two non-adjacent vertices of $D'$, call them $y$ and $z$. At least one of $y$ or $z$ must be in $V$ (and therefore in $V \cap D' = D$); since the only vertices not in $V$ that are adjacent to $a_i$ are $b_i$ and $c$, if neither $y$ nor $z$ were in V, we would have $\{y, z\} = \{b_i, c\}$, implying that $y$ and $z$ are adjacent. Therefore any vertex of $V - D$ is adjacent to some vertex of $D$, and $D$ is a dominating set of $G$.

Now, if $c \in D'$, then

$$|D'| = |D| + p + 1 = k + p,$$

and

$$|D| = k - 1.$$

On the other hand, if $c \notin D'$, we have

$$|D'| = |D| + p = k + p,$$

and

$$|D| = k.$$

In either case, there is a dominating set of size $k$ in $G$.

We have given a polynomial transformation of the dominating set problem for a chordal graph $G = (V, E)$ to the geodetic cover problem for a chordal graph $G'$, and have proved that $G$ has a dominating set of size $k$ if and only if $G'$ has a geodetic cover of size $k + |V|$.

**Theorem 2.4** *The geodetic cover problem for chordal graphs is NP-Complete.* □

## 3 Geodetic Bases of Split Graphs

In this section, let $G = (V, E)$ be a split graph, with $V$ partitioned into a maximal clique $K$ and an independent set $I$. Also, let $S$ be the set of simplicial vertices of $G$.

**Lemma 3.1** *No vertex of $I$ is adjacent to every vertex of $K$.*

**Proof:** Assume some $i \in I$ was adjacent to every vertex of $K$. Then $K \cup \{i\}$ would be a clique, contradicting the maximality of $K$. □

**Lemma 3.2** *Every vertex of $I$ is simplicial.*

**Proof:** Let $i$ be any vertex in $I$, and let $x, y$ be any two vertices adjacent to $i$. Since I is an independent set, $x, y \notin I$. Therefore $x, y \in K$, and $x$ and $y$ are adjacent. □

**Lemma 3.3** *A vertex of $K$ is simplicial if and only if it is not adjacent to any vertex of $I$.*

**Proof:** Let $k \in K$ be adjacent to some $i \in I$. By Lemma 3.1, there must be some $k' \in K$ not adjacent to $i$. Since $K$ is a clique, $k$ is adjacent to $k'$, and since we have assumed $k$ is adjacent to $i$, it must be that $k$ is not simplicial. Conversely, assume $k \in K$ is not adjacent to any vertex of $I$. Then $k$ is adjacent only to the members of the clique $K$, and is therefore simplicial. □

Figure 6 shows a split graph with two simplicial vertices in its clique partition.

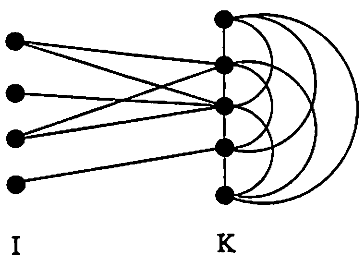


Figure 6: A split graph with simplicial vertices in $K$.

**Lemma 3.4** *If $K$ contains a simplicial vertex, then $S$ is the geodetic basis of $G$.*

**Proof:** Let $k$ be any simplicial vertex of $K$, and let $k'$ be any non-simplicial vertex of $G$. Since $k'$ is non-simplicial, $k' \in K$ by Lemma 3.2, and by Lemma 3.3 there is some $i \in I$ adjacent to $k'$. Furthermore, since $k$ is simplicial, $k$ is not adjacent to $i$, and so $kk'i$ is an $S$-geodesic containing $k'$. Since $S$ forms a geodetic cover of $G$, it follows from Theorem 1.1 that $S$ is the geodetic basis of $G$. □

By the above lemma, if $(S) \neq V$, then $K$ contains no simplicial vertices, and so $S = I$. Furthermore, by Lemma 3.2 and the fact that every set is a subset of its own closure, $V - (S) \subseteq K$. We assume that

$$(S) \neq V$$

and let

$$K' = V - (S)$$

in the remaining lemmas of this section.

**Lemma 3.5** *Every $k' \in K'$ is adjacent to exactly one vertex in $I$.*

**Proof:** Assume there is some $k' \in K'$ adjacent to two members of $I$, call them $i$ and $j$. Since $i$ and $j$ are not adjacent, $ik'j$ is an $S$-geodesic, contradicting the fact that $k' \notin (S)$. □

For the remaining lemmas, let $I' = \{i \mid i \in I \text{ and } i \text{ is adjacent to some } k' \in K'\}$, and define conditions 1 and 2 as follows.

**Condition 1** *There is some $k \in K$ which is adjacent to no vertex in $I'$.*

**Condition 2** *There is some $i' \in I'$ adjacent to exactly one vertex $k' \in K'$.*

**Lemma 3.6** *If Condition 1 holds, then $S \cup \{k\}$ is a geodetic basis.*

**Proof:** Let $k'$ be any vertex in $K'$, and let $i'$ be the vertex of $I'$ adjacent to $k'$. Since $k$ is adjacent to $k'$ and not adjacent to $i'$, it follows that $kk'i'$ is a geodesic between two vertices of the set $S \cup \{k\}$. Any vertex not in $K'$ is by definition on an $S$-geodesic. Since $S$ is not a geodetic cover and $S \cup \{k\}$ is, it must be the case that $S \cup \{k\}$ is a geodetic basis. □

Figure 7 illustrates Lemma 3.6. The single vertex from $K$ that is in the geodetic basis is enlarged.

**Lemma 3.7** *If Condition 2 holds, then $S \cup \{k'\}$ is a geodetic basis.*

**Proof:** As in Lemma 3.6, any vertex in $V - K'$ is on some $S$-geodesic. Let $k''$ be any vertex in $K'$. If $k'' = k'$, it is of course covered by the set $S \cup \{k'\}$. If $k'' \neq k'$, then let $i''$ be the one vertex of $I$ adjacent to $k''$. Since $k'' \neq k'$, and by Lemma 3.5, $k'$ is not adjacent to $i''$, it must be that $k'k''i''$ is a geodesic between members of $S \cup \{k'\}$. We have then that $S \cup \{k'\}$ is a geodetic cover, and so, again, a geodetic basis, since $S$ alone is not a cover. □

Figure 8 shows a split graph where Condition 2 holds.

**Lemma 3.8** *If neither Condition 1 nor Condition 2 hold, then there is no $x \in V$ such that $S \cup \{x\}$ is a geodetic cover.*

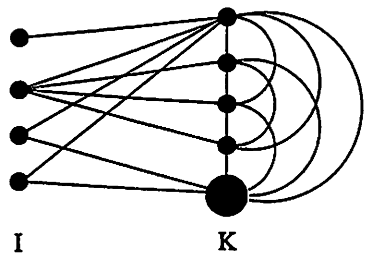


Figure 7: A split graph with one vertex of its geodetic basis in $K - K'$.

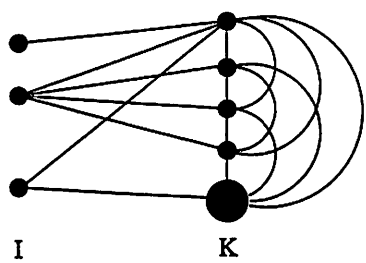


Figure 8: A split graph with one vertex of its geodetic basis in $K'$.

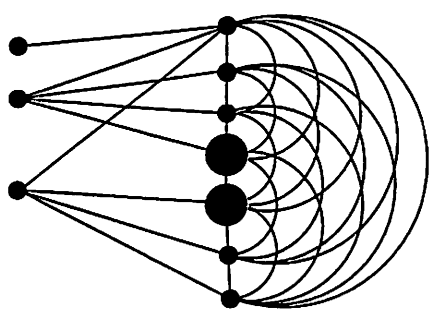

Figure 9: A split graph which requires two vertices from $K'$ in its geodetic basis.

**Proof:** Suppose neither condition holds but there is some $x$ such that $S \cup \{x\}$ is a geodetic cover. Since we are assuming that $(S) \neq V$, it must be that $x \notin S = I$, and so $x \in K$. Since Condition 1 doesn't hold, there must be some $i' \in I'$ adjacent to $x$, and since Condition 2 doesn't hold, there must be some $x' \in K'$ adjacent to $i'$, with $x \neq x'$. Since $x' \in K'$, $x'$ cannot lie on any $S$-geodesic. Since $x$ is adjacent to $i'$, $xx'i'$ is not a geodesic. Therefore there must be some $i'' \in S$ such that $x'$ lies on an $\{x, i''\}$-geodesic. Now, $x'$ is not adjacent to $i''$ by Lemma 3.5, so any $\{x, i''\}$-geodesic which contains $x'$ is of length at least three. On the other hand, let $x''$ be some member of $K$ adjacent to $i''$ ($x''$ exists, because $G$ is connected). The path $xx''i''$ is a geodesic of length two, a contradiction.

**Lemma 3.9** *If Condition 1 does not hold, then there are at least two members of $I'$, and therefore at least two members of $K'$.*

**Proof:** Assume Condition 1 does not hold, and that $I' = \{i'\}$. By the negation of Condition 1, every $k \in K$ is adjacent to $i'$. This contradicts Lemma 3.1.

**Lemma 3.10** *If neither Condition 1 nor Condition 2 hold, then let $i'$ and $i''$ be two distinct members of $I'$, and let $k', k'' \in K$ be such that $k'$ is adjacent to $i'$, and $k''$ is adjacent to $i''$. Then $S \cup \{k', k''\}$ is a geodetic basis of $G$.*

First, the existence of $k'$ and $k''$ is from Lemma 3.9. From Lemma 3.8, if $S \cup \{k', k''\}$ is a cover then it is a basis. We show that it is, in fact, a cover. Let $k$ be any member of $K'$. If $k \in \{k', k''\}$ it is trivially covered by $S \cup \{k', k''\}$. If $k$ is adjacent to $i'$, then $k''ki'$ is a geodesic which covers $k$. If $k$ is adjacent to $i''$, then $k'ki''$ is such a geodesic. Finally, if $k$ is adjacent to neither $i'$ nor $i''$, then it must be adjacent to some $i \in I = S$, and so both $k'ki$ and $k''ki$ are covering geodesics. □

Figure 9 shows a graph illustrating Lemma 3.10. The lemmas of this section prove the following theorem.

**Theorem 3.11** *Let $G = (V, E)$ be a split graph, and let $S \subseteq V$ be the simplicial vertices of $G$. Then the geodetic number of $G$ is bound so that*

$$|S| \leq g(G) \leq |S| + 2.$$

By using these lemmas, a polynomial time algorithm for finding a geodetic basis of a split graph can be developed as shown in Figure 10. The correctness of this algorithm follows from the above discussion and is omitted. The algorithm's first two steps take $O(|V|^3)$ to run, and the rest runs in $O(|V|)$, for a total running time of $O(|V|^3)$.

**Given** $G = (V, E)$, a split graph.

**Return** a geodetic basis of $G$.

1. Compute $S$, the simplicial vertices of $G$.
2. Compute $(S)$, the geodetic closure of $S$.
3. If $(S) = V$, then return $S$. Stop.
4. Compute $K' = V-(S)$, and $I' = \{i' \in S \mid i' \text{ is adjacent to some } k' \in K'\}$.
5. Partition $K'$ according to which single member of $I'$ each $k' \in K'$ is adjacent.
6. If there is a $k \in K$ adjacent to no member of $I'$, return $S \cup \{k\}$. Stop.
7. If there is an $i' \in I'$ adjacent to only one $k' \in K'$, return $S \cup \{k'\}$. Stop.
8. Let $i', i'' \in I'$ be two distinct vertices, and let $k'$ be some element of $K'$ adjacent to $i'$, and $k'' \in K'$ adjacent to $i''$. Return $S \cup \{k', k''\}$.

Figure 10: Algorithm for Geodetic Basis of a Split Graph

# 4 Conclusion

We have shown that, while the geodetic cover problem is NP-Complete for the class of chordal graphs, it becomes solvable in polynomial time when further restricted to the class of split graphs.

It is interesting to note that while chordal graphs may be the intersection graphs of subtrees of any tree, split graphs are the intersection graphs of subtrees of the star graph $K_{1,n}$ [8].

While we have shown that the geodetic cover problem for split graphs is in $P$, the dominating set problem for split graphs is NP-Complete[1]. Open questions include how hard it is to compute the geodetic basis for other classes of graphs.

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