More frames with block size four

C.J. Colbourn
Department of Combinatorics and Optimization
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

D.R. Stinson
Department of Computer Science and Engineering
University of Nebraska
Lincoln, NE 68588–0115, U.S.A.

L. Zhu
Department of Mathematics
Suzhou University
Suzhou, 215006, China

ABSTRACT. We present a new construction to obtain frames with block size four using certain skew Room frames. The existence results of Rees and Stinson for frames with block size four are improved, especially for hole sizes divisible by 6. As a by-product of the skew Room frames we construct, we are also able to show that a resolvable $(K_4 - e)$-design with $60t + 16$ points exists if $t \geq 0$ and $t \neq 8, 12$. Additionally, we give a new construction for holey self-orthogonal Latin squares with symmetric orthogonal mates (HSOLSSOMs) from frames: if there is a 4-frame of type h^u, then there is a HSOLSSOM of type h^u. We use this to construct some new HSOLSSOMs of types 3^u and 6^u.

1 Introduction

A *group divisible design* (or GDD), is a triple $(X, \mathbf{G}, \mathbf{B})$ which satisfies the following properties:

(1) G is a partition of a set X (of *points*) into subsets called *groups*,

(2) B is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point,

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(3) every pair of points from distinct groups occurs in a unique block.

The group type of the GDD is the multiset $\{|G|: G \in \mathbf{G}\}$. A GDD $(X, \mathbf{G}, \mathbf{B})$ will be referred to as a K-GDD if $|B| \in K$ for every block B in $\mathbf{B}$.

A *frame* is a group divisible design $(X, \mathbf{G}, \mathbf{B})$ whose block set admits a partition into holey parallel classes, each holey parallel class being a partition of $X \setminus G$ for some $G \in \mathbf{G}$. The groups of a frame are usually referred to as *holes*. The *type* of the frame is defined to be the group type of the GDD. A frame of block size k is denoted by k-frame. A frame of type h^u has u holes of size h and is called *uniform*.

It is known [10] that if there is a k-frame of type h^u (with $u > 1$), then $u \geq k+1$, $h \equiv 0 \bmod (k-1)$ and $h(u-1) \equiv 0 \bmod k$. The necessary conditions for the existence of a uniform k-frame have been proved to be sufficient for $k = 3$ [13]. A partial solution for $k = 4$ has been provided in [10].

Theorem 1.1. [10] *There is a 4-frame of type h^u if and only if $u \geq 5$, $h \equiv 0 \bmod 3$ and $h(u-1) \equiv 0 \bmod 4$, except possibly where*

(i) $h = 9$ *and* $u \in \{13, 17, 29, 33, 93, 113, 133, 153, 173, 193\}$;

(ii) $h \equiv 0 \bmod 12$ *and* $u \in \{8, 12\}$,
$h = 36$ *and* $u \in \{7, 18, 23, 28, 33, 38, 43, 48\}$,
$h = 24$ *or* 120 *and* $u \in \{7\}$,
$h = 72$ *and* $u \in 2Z^+ \cup \{n: n \equiv 3 \bmod 4$ *and* $n \leq 527\} \cup \{563\}$; *or*

(iii) $h \equiv 6 \bmod 12$ *and* $u \in \{17, 29, 33, 563\} \cup \{n: n \equiv 3$ *or* $11 \bmod 12$ *and* $n \leq 527\} \cup \{n: n \equiv 7 \bmod 12$ *and* $n \leq 259\}$,
$h = 18$.

In this paper, we present a new construction for 4-frames using certain skew Room frames. Then we improve Theorem 1.1, especially for hole sizes $h = 6$, 18 and 72, so that for any h there are finitely many possible exceptions of u. More specifically, we shall show the existence of 4-frames of type h^u when

$h = 9$ and $u \in \{33, 133\}$,
$h = 24$ and $u = 8$,
$h = 36$ and $u = 33$,
$h = 72$ and $u \geq 5$,
$h = 6$ and $u \in \{17, 29, 33, 563\} \cup \{n: n \equiv 3,$ or $11 \bmod 12$ and $n \geq 275\}$,
$h = 18$ and $u \in \{n: n \equiv 1 \bmod 4, n \neq 17\} \cup \{n: n \equiv 3 \bmod 4$ and $n \geq 271\}$.

As a by-product of the skew Room frames we construct, we show that a resolvable $(K_4 - e)$-design with $60t + 16$ points exists if $t \geq 0$ and $t \neq 8$, 12. Additionally, we give a new construction for holey self-orthogonal Latin squares with symmetric orthogonal mates (HSOLSSOMs) from frames: if there is a 4-frame of type h^u, then there is a HSOLSSOM of type h^u. We use this to construct some new HSOLSSOMs of types 3^u and 6^u.

We use [1] as our standard design theory reference. We also follow [10] for notation of pairwise balanced designs (PBDs), group divisible designs (GDDs), transversal designs (TDs) and other types of designs. We shall use the following basic constructions [10].

Lemma 1.2 (Inflation). *If there exists a 4-frame of type h^u and a resolvable $TD(4, m)$, then there exists a 4-frame of type $(mh)^u$.*

Lemma 1.3 (Weighting). *Let $(X, \mathbf{G}, \mathbf{B})$ be a GDD, and let $w: X \to Z^+ \cup \{0\}$ be a weight function on X. Suppose that for every block $B \in \mathbf{B}$ there exists a 4-frame of type $\{w(x): x \in B\}$. Then there exists a 4-frame of type $\{\sum_{x \in G} w(x): G \in \mathbf{G}\}$.*

Lemma 1.4 (Filling in holes). *Let $d = 0$ or 1. If there exist a 4-frame of type $(s_1, s_2, \ldots, s_n)$ and a 4-frame of type $h^{d+s_j/h}$ for $1 \leq j \leq n$, then there exists a 4-frame of type $h^{d+s/h}$, where $s = \sum_{1 \leq j \leq n} s_j$.*

Lemma 1.5 (PBD closure). *For any fixed hole size h, the set $\{u: \exists$ 4-frame of type $h^u\}$ is PBD-closed.*

2 A skew Room frame construction

Recently a very useful construction of BIBDs and GDDs with block size four has been found by making use of skew Room frames [11]. This construction has been employed in solving the existence problem for weakly 3-chromatic BIBDs with block size four [12]. We shall now adapt it to construct frames of block size four.

For $1 \leq i \leq n$ define $H_i = \{x_{j+h(i-1)}: 0 \leq j \leq h-1\}$ and let $\mathbf{H} = \{H_i: 1 \leq i \leq n\}$; H_i is called a *hole*. A *Room frame of type h^n with hole set* $\mathbf{H}$ is a $hn \times hn$ array $\mathbf{F}$, indexed by $X = \{x_0, x_1, \ldots, x_{hn-1}\}$, in which

(a) for $1 \leq i \leq n$, the cells $(s, t) \in H_i \times H_i$ are empty.

(b) each 2-element subset of X that is not a 2-element subset of H_i occurs in exactly one cell of $\mathbf{F}$, and each cell of $\mathbf{F}$ either contains a pair of symbols from X or is empty.

(c) each row and each column of $\mathbf{F}$ that intersects H_i contains each element from $X \setminus H_i$ exactly once.

A *skew* Room frame is a Room frame in which cell (i,j) is occupied if and only if cell (j,i) is empty.

From a skew Room frame of type h^n one can get a 4-GDD of type $(6h)^n$ [11]. The 4-GDD is based on $X \times Z_6$ with groups $H_i \times Z_6$, $1 \le i \le n$. The block set B contains all blocks $\{(a,j),(b,j),(c,1+j),(r,4+j)\}$, where $j \in Z_6$, $\{a,b\} \in \mathbf{F}$, $\{a,b\}$ occurs in column c and row r.

If all the quadruples (a,b,c,r) can be partitioned into sets such that each set forms a partition of $X \setminus H_i$ for some i, and each H_i corresponds to $2h$ of the sets, we call the skew Room frame *partitionable*. It is clear that from a partitionable skew Room frame, each set which partitions $X \setminus H_i$ will result in a holey parallel class in the resulting 4-GDD. We state this fact in the following lemma.

Lemma 2.1. *If there is a partitionable skew Room frame of type h^n, then there exists a 4-frame of type $(6h)^n$.*

Room frames of type h^n are often constructed using an abelian group of order h^n. Let G be an abelian group, written additively, and let H be a subgroup of G. Denote $g = |G|$, $h = |H|$ and suppose that $g-h$ is even. A *frame starter* in $G \setminus H$ is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \le i \le (g-h)/2\}$ satisfying

(1) $\cup_{1 \le i \le (g-h)/2}(\{s_i\} \cup \{t_i\}) = G \setminus H$, and

(2) $\cup_{1 \le i \le (g-h)/2}\{\pm(s_i - t_i)\} = G \setminus H$.

An *adder* for S is an injection $A\colon S \to G \setminus H$, such that

$$\cup_{1 \le i \le (g-h)/2}(\{s_i + a_i\} \cup \{t_i + a_i\}) = G \setminus H,$$

where $a_i = A(s_i, t_i)$, $1 \le i \le (g-h)/2$. An adder A is *skew* if, further,

$$\cup_{1 \le i \le (g-h)/2}(\{a_i\} \cup \{-a_i\}) = G \setminus H.$$

From a starter S and a skew adder A, we can construct a skew Room frame $\mathbf{F}$ in which the cell $(j, -a_i + j)$ is occupied by $\{s_i + j, t_i + j\}$ for $1 \le i \le (g-h)/2$ and any $j \in G$. To obtain a 4-frame, it suffices to partition the quadruples: $(s_i + j, t_i + j, -a_i + j, j)$.

Lemma 2.2. *There exists a partitionable skew Room frame of type* 3^5 *and a 4-frame of type* 18^5.

Proof: From [5] we have a starter S and a skew adder A of type 3^5 as follows.

$G = Z_{15}$ and $H = \{0,5,10\}$.
$S = \{\{1,7\},\{2,3\},\{4,8\},\{6,14\},\{9,12\},\{11,13\}\}$.
$A = \{12,14,4,8,9,13\}$.

We can partition all the quadruples $(s_i + j, t_i + j, -a_i + j, j)$ as follows. First, translate the initial quadruples:

1,	7,	3,	0	--- add 1 ---	2,	8,	4,	1
2,	3,	1,	0	--- add 1 ---	3,	4,	2,	1
4,	8,	11,	0	--- add 3 ---	7,	11,	14,	3
6,	14,	7,	0	--- add 2 ---	8,	1,	9,	2
9,	12,	6,	0	--- add 2 ---	11,	14,	8,	2
11,	13,	2,	0	--- add 1 ---	12,	14,	3,	1

It is easily seen that each quadruple (u, v, w, z) on the right covers the four non-zero residues modulo 5, and hence will give one partition of $G \setminus H$: $\{(u+j, v+j, w+j, z+j) : j \in H\}$. Altogether we get six partitions of $G \setminus H$. Under the action of group G, we get further partitions so that **F** is partitionable. Hence, there exists a 4-frame of type 18^5. □

Lemma 2.3. *There exists a partitionable skew Room frame of type* 3^9 *and a 4-frame of type* 18^9.

Proof: Using a computer program we found a starter S and a skew adder A of type 3^9 as follows.

$G = Z_{27}$ and $H = \{0, 9, 18\}$.
$S = \{\{1,2\},\{3,5\},\{4,7\},\{8,12\},\{14,19\},\{15,21\},\{13,20\},$
$\{17,25\},\{16,26\},\{22,6\},\{11,23\},\{24,10\}\}$.
$A = \{1,14,3,12,21,11,2,8,22,17,20,4\}$.

First, translate the initial quadruples:

1,	2,	26,	0	--- add 2 ---	3,	4,	1,	2
3,	5,	13,	0	--- add 1 ---	4,	6,	14,	1
4,	7,	24,	0	--- add 4 ---	8,	11,	1,	4
8,	12,	15,	0	--- add 4 ---	12,	16,	19,	4
14,	19,	6,	0	--- add 2 ---	16,	21,	8,	2
15,	21,	16,	0	--- add 8 ---	23,	2,	24,	8
13,	20,	25,	0	--- add 3 ---	16,	23,	1,	3
17,	25,	19,	0	--- add 7 ---	24,	5,	26,	7
16,	26,	5,	0	--- add 7 ---	23,	6,	12,	7
22,	6,	10,	0	--- add 4 ---	26,	10,	14,	4
11,	23,	7,	0	--- add 6 ---	17,	2,	13,	6
24,	10,	23,	0	--- add 6 ---	3,	16,	2,	6

We pair the quadruples on the right as follows:

$$\begin{array}{rr} (3,4,1,2), & (24,5,26,7); \\ (4,6,14,1), & (16,21,8,2); \\ (8,11,1,4), & (23,6,12,7); \\ (12,16,19,4), & (23,2,24,8); \\ (16,23,1,3), & (17,2,13,6); \\ (26,10,14,4), & (3,16,2,6). \end{array}$$

It is easily seen that each pair P of quadruples covers the eight non-zero residues modulo 9, and hence gives one partition of $G \setminus H$: $\{(u+j, v+j, w+j, z+j) : j \in H, (u,v,w,z) \in P\}$. Altogether we get six partitions of $G \setminus H$. Under the action of group G, we get further partitions so that $\mathbf{F}$ is partitionable. Hence, there exists a 4-frame of type 18^9. □

Lemma 2.4. *There exists a partitionable skew Room frame of type 1^{17} and a 4-frame of type 6^{17}.*

Proof: Let $G = Z_{17}$ and $H = \{0\}$. From a starter and a skew adder of type 1^{17} we translate the initial quadruples as follows:

1,	4,	5,	0	- - - add 6 - - -	7,	10,	11,	6
11,	10,	4,	0	- - - add 15 - - -	9,	8,	2,	15
16,	7,	6,	0	- - - add 14 - - -	13,	4,	3,	14
13,	6,	2,	0	- - - add 16 - - -	12,	5,	1,	16
2,	8,	10,	0	- - - add 12 - - -	14,	3,	5,	12
12,	14,	9,	0	- - - add 4 - - -	16,	1,	13,	4
15,	3,	1,	0	- - - add 8 - - -	6,	11,	9,	8
9,	5,	14,	0	- - - add 10 - - -	2,	15,	7,	10

It is easily seen that each of the first and the last four quadruples gives one partition of $G \setminus H$. Under the action of group G, we get other partitions so that $\mathbf{F}$ is partitionable. Hence, there exists a 4-frame of type 6^{17}. □

Lemma 2.5. *There exists a partitionable skew Room frame of type 1^u and a 4-frame of type 6^u, where $u = 13, 29$.*

Proof: Let $G = GF(u)$ and $H = \{0\}$. Let C_0 be the multiplicative subgroup of order G having index 4. Denote the other cosets by C_1, C_2, C_3, such that $C_1 \cup C_3$ consists of all the quadratic non-residues. Take an element $a = 4$ in C_2. Let $c = 4$ or 6 for $u = 13$ or 29, respectively. It is readily checked that $1 + c$, $a + c$, $a + 1 + c$, and c are in distinct cosets. Then we have a starter $S = \{\{x, ax\} : x \in C_0 \cup C_1\}$ and a skew adder A, where $A(\{x, ax\}) = -(a+1)x$ for $x \in C_0 \cup C_1$. Use cx to get the translate of the quadruple $(x, ax, (a+1)x, 0)$. We get two partitions of $G \setminus H$:

$$((1+c)x, (a+c)x, (a+1+c)x, cx), x \in C_0;$$
$$((1+c)x, (a+c)x, (a+1+c)x, cx), x \in C_1.$$

Under the action of group G, we get further partitions so that $\mathbf{F}$ is partitionable. Hence, there exists a 4-frame of type 6^u. □

Lemma 2.6. *There exist 4-frames of types 24^8 and 6^{33}.*

Proof: Using a computer program we found a starter S and a skew adder A of type 4^8 as follows.

$G = Z_{32}$ and $H = \{0, 8, 16, 24\}$.
$S = \{\{1,4\},\{2,7\},\{9,23\},\{12,31\},\{14,25\},\{15,17\},\{18,30\},$
$\{5,20\},\{10,3\},\{13,19\},\{28,27\},\{6,29\},\{11,21\},\{26,22\}\}$.
$A = \{1,7,13,26,4,11,17,22,9,20,3,14,2,27\}$.

Translate the initial quadruples:

1,	4,	-1,	0	--- add 3 ---	4,	7,	2,	3
2,	7,	-7,	0	--- add 19 ---	21,	26,	12,	19
14,	25,	-4,	0	--- add 13 ---	27,	6,	9,	13
18,	30,	-17,	0	--- add 31 ---	17,	29,	14,	31
5,	20,	-22,	0	--- add 5 ---	10,	25,	15,	5
10,	3,	-9,	0	--- add 20 ---	30,	23,	11,	20
26,	22,	-27,	0	--- add 28 ---	22,	18,	1,	28
9,	23,	-13,	0	--- add 3 ---	12,	26,	22,	3
12,	31,	-26,	0	--- add 11 ---	23,	10,	17,	11
15,	17,	-11,	0	--- add 4 ---	19,	21,	25,	4
13,	19,	-20,	0	--- add 1 ---	14,	20,	13,	1
28,	27,	-3,	0	--- add 2 ---	30,	25,	31,	2
6,	29,	-14,	0	--- add 9 ---	15,	6,	27,	9
11,	21,	-2,	0	--- add 7 ---	18,	28,	5,	7

It is easily seen that the first 7 quadruples on the right forms a partition of $G \setminus H$, so do the last 7 quadruples. From the partitionable skew Room frame of type 4^8 we get a 4-frame of type 24^8. Adding 6 new points and filling in holes with a 4-frame of type 6^5 gives the desired 4-frame of type 6^{33}. □

3 Some new 4-frames

First, we show that for any $u \geq 5$ there exists a 4-frame of type 72^u. To do this, we mainly use the weighting construction. Since a 4-frame of type 12^n exists for all $n \in K = \{k\colon k \geq 5 \text{ and } k \neq 8, 12\}$, we start with a K-GDD of type 6^u.

Lemma 3.1. *There exists a K-GDD of type 6^u for $u \geq 5$ and $u \neq 5, 33$, where $K = \{k\colon k \geq 5, k \neq 8, 12\}$.*

Proof: If there are 4 idempotent MOLS(k), then we have a $\{k, 6\}$-GDD of type 6^k. Deleting one group from the GDD gives a $\{k-1, 5, 6\}$-GDD of type 6^{k-1} From Tables 2.58 and 2.59 in [3], we get the required 4 MOLS and hence a K-GDD of type 6^u exists for $u \geq 5$ and $u \neq 5, 8, 12, 14, 33, 34, 38, 44$.

It is known [7] that there exists a resolvable $\{9\}$-GDD of type 3^{33} and a $\{9, 10\}$-GDD of type 3^{34}.

If there is a TD$(7, t)$, we may delete $t - 6$ points in one group to get a $\{t, 6, 7\}$-GDD of type 6^{t+1}. This takes care of the remaining cases. □

Lemma 3.2. *There exists a 4-frame of type* 72^u *for all* $u \geq 5$.

Proof: For $u = 5$ and $u = 33$, a 4-frame of type 72^u is known from Theorem 1.1. For other u, start with a K-GDD of type 6^u from Lemma 3.1 and apply the weighting construction giving weight 12 to each point. Since from Theorem 1.1 (ii) a 4-frame of type 12^n exists for all $n \in K = \{k\colon k \geq 5, k \neq 8, 12\}$, we get the desired 4-frames of type 72^u. □

Lemma 3.3. *There exists a 4-frame of type* 18^u *for* $u \in \{n\colon n \equiv 1 \bmod 4, n \neq 17\} \cup \{n\colon n \equiv 3 \bmod 4, n \geq 271\}$.

Proof: Apply Lemma 1.4 with $s_j = 72$, $h = 18$ and $d = 1$. Since a 4-frame of type 18^5 exists from Lemma 2.2, we get a 4-frame of type 18^u for all $u \geq 21$, $u \equiv 1 \bmod 4$. A 4-frame of type 18^9 exists from Lemma 2.3. From [1, Proposition 9.17], there exists a $\{q^2\}$-GDD of type $(q^2 - q)^m$ for every prime power $q \geq 2$, where $m = q^2 + q + 1$. Taking $q = 3$ gives a $\{9\}$-GDD of type 6^{13}. Give weight 3 to each point and apply Lemma 1.3. Since a 4-frame of type 3^9 exists from Theorem 1.1, we get a 4-frame of type 18^{13}. This takes care of the case $u \equiv 1 \bmod 4$.

We now deal with the case $u \equiv 3 \bmod 4$. Start with a TD$(7, 6t)$, $t \geq 11$, which exists from Tables 2.58 and 2.59 in [3]. Delete $6t - 6s$ points in a group, $s = 5, 6, 7, 8, 9$. Delete $6t - 45$ points in another group. We get a $\{5, 6, 7\}$-GDD of type $(6t)^5(6s)^1 45^1$. Give weight 12 to each point of the GDD and use 4-frames of types 12^n as input designs, where $n = 5, 6, 7$, which all exist from Theorem 1.1 (ii). We get a 4-frame of type $(72t)^5(72s)^1 540^1$. Apply Lemma 1.4 with $s_j = 72t$, or $72s$, or 540, $h = 18$ and $d = 1$. Since a 4-frame of type 18^{31} exists from [10] and a 4-frame of type 18^u exists for all $u \geq 21$, $u \equiv 1 \bmod 4$, we get a 4-frame of type 18^u, $u = 20t + 4s + 31$, $s = 5, 6, 7, 8, 9$. This gives a 4-frame of type 18^u for all $u \geq 271$, $u \equiv 3 \bmod 4$. The proof is complete. □

Lemma 3.4. *There exist 4-frames of types* 9^{33}, 9^{133}, 36^{33}.

Proof: It is known [7] that a 9-GDD of type 3^{33} exists. Give weight 3 to each point and use a 4-frame of type 3^9 as input design. We get a 4-frame of type 9^{33}. Further applying Lemma 1.2 with $m = 4$ gives a 4-frame of type 36^{33}. Still further apply Lemma 1.4 with $s_j = 36$, $h = 9$ and $d = 1$.

Since a 4-frame of type 9^5 exists from Theorem 1.1 (i), we get a 4-frame of type 9^{133}. □

Next, we deal with the hole size 6.

Lemma 3.5. *Suppose there is a GDD such that* (1) $|G| \in 2N \cup \{15\}$ *for all groups* G, *and* (2) $|B| \in N \setminus \{1,2,3,4,8,12\}$ *for all blocks* B. *Then there is a 4-frame of type* 6^u, *where* $u = 2|X| + 1$.

Proof: Give every point weight 12 and apply Lemma 1.3. Then adjoin 6 points at infinity to fill in the holes. □

Corollary 3.6. *Suppose there is a TD(7,n), where* n *is even,* $n \geq 16$, $0 \leq t \leq n$ *and* t *is even. Then there is a 4-frame of type* 6^u, *where* $u = 10n + 2t + 31$.

Proof: Delete $n - 15$ points from one group and $n - t$ points from another group. We get a GDD on $5n + 15 + t$ points, having group type $n^5 15^1 t^1$ and blocks of size 5, 6 and 7. Apply Lemma 3.5. □

Lemma 3.7. *For* $u \equiv 3, 11 \bmod 12$, $351 \leq u \leq 527$, *and for* $u = 563$, *there is a 4-frame of type* 6^u.

Proof: Apply Corollary 3.6 with $n = 50, 48, 40, 36, 32$. □

We can do further examples using the following construction for GDDs that uses the idea of thwarts [4, Section 5.7].

Lemma 3.8. *Suppose* n *is a prime power,* $k \leq n+1$, $0 \leq s, t \leq n-1$, *and* $s + t \geq n + k - 2$. *Then there is a GDD having group type* $(n-1)^k s^1 t^1$ *and block sizes* k, $k+1$ *and* $k+2$.

Proof: Suppose we begin with any TD$(k+2, n)$ and delete the points on one block B, and then delete $n - s - 1$ further points from one group and $n - t - 1$ further points from another group. This will yield a GDD of the desired type, having blocks of size $k-1$, k, $k+1$, $k+2$. We want to make sure that there are no blocks of size $k-1$. We can do this for the TD obtained from the Desarguesian plane if $s + t \geq n + k - 2$.

Let A be a primitive element of $GF(n)$. For $1 \leq i \leq k$, let $a_i = -A^{-i+1}$. Let the groups of the TD be $G_1, \ldots, G_k$, H_1, H_2, where $G_i = \{i\} \times GF(n)$, $H_1 = \{h\} \times GF(n)$ and $H_2 = \{h'\} \times GF(n)$. The blocks of the TD are given by the formula

$$B_{xy} = \{(h,y),(h',x)\} \cup \{(i, x + a_i y) \colon 1 \leq i \leq k\}, x, y \in GF(n).$$

Suppose $B = B_{00}$ is the block whose points are deleted, and suppose $T_2 = \{(h', A^j) \colon 1 \leq j \leq n - t - 1\}$ are the further deleted points from H_2. S_1 will consist of s points, to be specified later, that are not deleted from H_1. The property we require is the following: any block that contains a point of B intersecting G_i ($1 \leq i \leq k$) and a point of T_2, intersects H_1 in a point of S_1.

Now, the points $(i, 0)$ and (h', A^j) are in the block B_{xy}, where $y = A^{i+j-1}$, so the intersection of this block with H_1 is (h, A^{i+j-1}). Since $1 \leq i \leq k$ and $1 \leq j \leq n-t-1$, we need all points $(h, A^1), \ldots, (h, A^{n-t+k-2})$ to be in S_1. This is possible if $s \geq n - t + k - 2$, which is equivalent to $s + t \geq n + k - 2$. □

We will use the following corollary.

Corollary 3.9. *Suppose n is an odd prime power, $n \geq 17$, $0 \leq t \leq n - 1$, t even, and $n - t \leq 12$. Then there is a 4-frame of type 6^u, where $u = 10n + 2t + 21$.*

Proof: Take $k = 5$ and $s = 15$ in Lemma 3.8. There is a GDD on $5n+10+t$ points, having group type $(n-1)^5 15^1 t^1$ and blocks of sizes 5, 6 and 7. Apply Lemma 3.5. □

Lemma 3.10. *For $u \equiv 3, 11 \bmod 12$, $u \geq 275$, there is a 4-frame of type 6^u.*

Proof: We need only consider $275 \leq u \leq 347$. Apply Corollary 3.9 with $n = 29, 27, 25, 23$ to do these cases. □

Lemma 3.11. *For $u \equiv 3, 11 \bmod 12$, $u \geq 275$, there is a 4-frame of type h^u, where $h \equiv 6 \bmod 12$, $h \neq 18$.*

Proof: Let $h = 6g$ where g is odd, $g \neq 3$. So we can multiply the 4-frames in Lemma 3.10 by g. □

With the newly constructed 4-frames we can now update the results in Theorem 1.1 as follows.

Theorem 3.12. *There exists a 4-frame of type h^u, if and only if $u \geq 5$, $h \equiv 0 \bmod 3$ and $h(u-1) \equiv 0 \bmod 4$, except possibly where*

(i) $h = 9$ *and* $u \in \{13, 17, 29, 93, 113, 153, 173, 193\}$;

(ii) $h \equiv 12 \bmod 24$ *and* $u \in \{8, 12\}$,
$h = 36$ *and* $u \in \{7, 18, 23, 28, 38, 43, 48\}$;

(ii') $h \equiv 0 \bmod 24$ *and* $u \in \{12\}$,
$h = 48, 144$ *or* 240 *and* $u \in \{8\}$,
$h = 24$ *or* 120 *and* $u \in \{7\}$; *or*

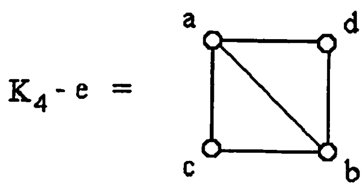
(iii) $h \equiv 6 \bmod 12$ *and* $u \in \{n : n \equiv 3 \bmod 4$ *and* $n \leq 267\}$,
$h = 18$ *and* $u \in \{17\}$.

We remark that for some $u \equiv 3 \bmod 4$ and $u \leq 267$ it is possible to use the above constructions to get 4-frames of type 6^u. However, the bound in Theorem 3.12 (iii) still remains. So, we do not intend to list those possible values of u.

4 Resolvable $(K_4 - e)$-designs

In this section we shall present another application of the partitionable skew Room frames given in Section 2 to show the existence of certain resolvable $(K_4 - e)$-designs.

A $(K_4 - e)$-*design* of order n is a pair $(X, \mathbf{B})$, where $\mathbf{B}$ is an edge-disjoint decomposition of the edge set of K_n (the complete undirected graph on n vertices) with vertex set X, into copies of



We shall denote $K_4 - e$ by any one of (a, b, c, d), (a, b, d, c), (b, a, c, d), or (b, a, d, c) and call it a *block* of the design. It is well known that a $(K_4 - e)$-design of order n exists for all $n \equiv 0$ or $1(\bmod 5)$, $n \geq 6$. If $(X, \mathbf{B})$ is a $(K_4 - e)$-design of order n, then $\mathbf{B}$ contains $n(n-1)/10$ blocks. A $(K_4 - e)$-design is called *resolvable* if the block set $\mathbf{B}$ can be partitioned into *parallel classes*, each forming a partition of X. Simple counting shows that a resolvable $(K_4 - e)$-design of order n exists only if $n \equiv 16(\bmod 20)$ and $n \geq 16$. A. Street posed the existence question at Auburn conference (1994). A. Rosa (private communication) provided the first example of order 16 as follows.

Lemma 4.1. *There exists a resolvable $(K_4 - e)$-design of order* 16.

Proof: Let $X = \{\infty, 0, 1, \ldots, 14\}$. Let $\mathbf{B}$ be the set of 24 blocks shown below, which is partitioned into six parallel classes, where each row forms a parallel class.

$$(1,\infty,0,2),(5,14,4,6),(9,10,3,7),(11,13,8,12);$$
$$(5,10,0,\infty),(4,8,1,2),(13,6,7,9),(3,14,11,12);$$
$$(6,11,\infty,1),(2,5,3,9),(8,14,7,10),(0,4,12,13);$$
$$(7,12,\infty,2),(4,6,3,10),(0,9,8,11),(1,13,5,14);$$
$$(\infty,3,8,13),(7,11,4,5),(1,12,9,10),(0,2,6,14);$$
$$(\infty,9,4,14),(8,12,5,6),(2,10,11,13),(3,7,0,1).$$

Then $(X, \mathbf{B})$ is the desired resolvable $(K_4 - e)$-design of order 16. □

A K_4-e *group divisible design* of type h^n is a triple $(X, \mathbf{G}, \mathbf{B})$, where X is the vertex set, $\mathbf{G}$ is a partition of X into groups (*holes*) of size h each, $\mathbf{B}$ is an edge-disjoint decomposition of the edge set of $K_{n,n,\ldots,n}$ (the multipartite complete undirected graph with $\mathbf{G}$ as the partition of the vertex set X) into copies of (*blocks*) $K_4 - e$. A $(K_4 - e)$*-frame* of type h^n is a $K_4 - e$ group divisible design of type h^n $(X, \mathbf{G}, \mathbf{B})$ in which the blocks can be partitioned into *Holey parallel classes*, each forming a partition of $X \setminus G$ for some $G \in \mathbf{G}$.

Lemma 4.2. *If there is a partitionable skew Room frame of type h^n, then there is a $(K_4 - e)$-frame of type $(5h)^n$.*

Proof: Let $\mathbf{F}$ be a given partitionable skew Room frame of type h^n with hole set $\mathbf{H}$ and let $\{x, y\}$ be a pair in the cell (r, c) of $\mathbf{F}$. We first construct a (K_4-e)-GDD of type $(5h)^n$ $(X, \mathbf{G}, \mathbf{B})$, where $\mathbf{G} = \{H \times Z_5 \colon H \in \mathbf{H}\}$, $X = \cup_{G \in \mathbf{G}} G$ and $\mathbf{B}$ contains all the 5 graphs $((x, g), (y, g), (c, 1 + g), (r, 2 + g))$ for $\{x, y\} \in \mathbf{F}$ and $g \in Z_5$. This construction has been used implicitly in [6]. Since $\mathbf{F}$ is partitionable, for each $H' \in \mathbf{H}$ the quadryples which partition the set $\cup_{H \in \mathbf{H} \setminus H'} H$ will result in a holey parallel class in the resulting $(K_4 - e)$-GDD of type $(5h)^n$. Therefore, the $(K_4 - e)$-GDD is actually a $(K_4 - e)$-frame. □

Lemma 4.3. *There exist $(K_4 - e)$-frames of type 15^5, 15^9, 5^{17}, 5^{13} and 5^{29}.*

Proof: From Lemmas 2.2–2.5, there are partitionable skew Room frames of types 3^5, 3^9, 1^{17}, 1^{13} and 1^{29}. The conclusion follows from Lemma 4.2. □

For $(K_4 - e)$-frames, constructions such as Inflation, Weighting and PBD closure are valid. We state them below.

Lemma 4.4 (Inflation). *If there exists a $(K_4 - e)$-frame of type h^u and a resolvable TD$(3, m)$, then there exists a $(K_4 - e)$-frame of type $(mh)^u$.*

Lemma 4.5 (Weighting). *Let $(X, \mathbf{G}, \mathbf{B})$ be a GDD, and let $w \colon X \to Z^+ \cup \{0\}$ be a weight function on X. Suppose that for every block $B \in \mathbf{B}$ there exists a $(K_4 - e)$-frame of type $\{w(x) \colon x \in B\}$. Then there exists a $(K_4 - e)$-frame of type $\{\sum_{x \in G} w(x) \colon G \in \mathbf{G}\}$.*

Lemma 4.6 (PBD closure). *For any fixed hole size h, the set $\{u \colon \exists (K_4 - e)$-frame of type $h^u\}$ is PBD-closed.*

We may use these recursive constructions to get a certain class of (K_4-e)-frames.

Lemma 4.7. *There exists a $(K_4 - e)$-frame of type 15^u for $u \in \{n \colon n \geq 5, n \equiv 1 (\bmod 4), n \neq 33, 49\}$.*

Proof: From Lemma 4.3, there are $(K_4 - e)$-frames of types 15^5, 15^9, 5^{17}, 5^{13} and 5^{29}. Applying Lemma 4.4 with $m = 3$, we have $(K_4 - e)$-frames of types 15^{17}, 15^{13} and 15^{29}. For $u \in \{n: n \geq 5, n \equiv 1 (\bmod 4)$, and $n \neq 33, 49\}$, it is known [10, Lemma 3.4] that $u \in B(5, 9, 13, 17, 29)$. Then the conclusion follows from Lemma 4.6. □

We are now in a position to show the main result of this section.

Lemma 4.8. *There exists a resolvable $(K_4 - e)$-design of order n for $n \equiv 16 (\bmod 60)$, $n \geq 16$, $n \neq 496, 736$.*

Proof: From Lemma 4.7, there exists a $(K_4 - e)$-frame of type 15^u for $u \in \{n: n \geq 5, n \equiv 1 (\bmod 4)$, and $n \neq 33, 49\}$. Add one new point ∞ to the $(K_4 - e)$-frame and construct a resolvable $(K_4 - e)$-design of order 16 on set $H \cup \{\infty\}$ for each hole H of the $(K_4 - e)$- frame. The resolvable $(K_4 - e)$-design of order 16 from Lemma 4.1 has six parallel classes. The $(K_4 - e)$-frame of type 15^u has $u(u-1)15^2/10$ blocks and $6u$ holey parallel classes, since each holey parallel class contains $15(u-1)/4$ blocks. For each hole H, combine the six parallel classes of the resolvable $(K_4 - e)$-design of order 16 and the six holey parallel classes to form six parallel classes of a resolvable $(K_4 - e)$-design of order $15u + 1$. The proof is complete. □

5 A new construction for HSOLSSOMs

In [14], 4-frames have been used to construct three mutually orthogonal Latin squares with holes (3 HMOLS). In this section, we show how to use 4-frames to construct HSOLSSOMs, a special kind of 3 HMOLS.

Suppose A, B and C are three mutually orthogonal Latin squares of order v such that $B = A^T$ and $C = C^T$. We say that they form a *self-orthogonal Latin square* (SOLS) *with a symmetric orthogonal mate* (SOM) of order v, denoted by SOLSSOM(v). If C contains the same entries on the main diagonal, the SOLSSOM is called *unipotent.* A unipotent SOLSSOM of order 4 based on $\{a, b, c, d\}$ is shown in Table 5.1.

a	c	d	b
d	b	a	c
b	d	c	a
c	a	b	d

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

Table 5.1. A unipotent SOLSSOM(4)

If the SOLSSOM contains u holes of size h which are disjoint and spanning, we denote the holey design by HSOLSSOM(h^u), where h^u is the type of the design. The existence of a HSOLSSOM(h^u) has been investigated in [8], [9], [15], [2], we state the most recent result (see [2]) as follows.

Theorem 5.1. (1) *If h is an odd positive integer, then a HSOLSSOM(h^u) exists if and only if u is odd and $u \geq 5$, except possibly when $h = 3$ and*

$u \in \{11, 13, 15, 17, 19, 23, 27, 33, 39, 51, 59, 87\}$. (2) *If h is an even positive integer, then a HSOLSSOM(h^u) exists if and only if $u \geq 5$, except possibly when $h = 6$, or $h \equiv 2 (\bmod 4)$ and $u \in \{8, 10, 12, 14, 15, 16, 18, 20, 22, 24, 28, 32\}$.*

Theorem 5.2. *If there exists a 4-frame of type h^u, then there exists a HSOLSSOM of type h^u.*

Proof: Let $(X, \mathbf{G}, \mathbf{B})$ be a 4-frame of type h^u, where $\mathbf{G} = \{G_1, G_2, \ldots, G_u\}$, $|G_j| = h$ for $1 \leq j \leq u$. We shall construct a HSOLSSOM of type h^u on X having $\mathbf{G}$ as a hole set.

We construct the holey SOLS of order h^u first. Suppose the rows and columns of the array are indexed by X and the subarrays $G_j \times G_j$ are empty for $1 \leq j \leq u$. For any block $B = \{a, b, c, d\}$, delete the diagonal entries of the first square in Table 1 and put it as the subarray $B \times B$. It is clear that each cell (x, y) not in any subarray $G_j \times G_j$ is occupied with some entry since the two elements x and y are in different G_j's and must be contained in a unique block of the 4-frame. It is easily seen that the array constructed above is a holey SOLS of type h^u. It remains to construct its symmetric orthogonal mate.

For the symmetric orthogonal mate, the construction is similar. Suppose the rows and columns of the array are indexed by X in the same way as the holey SOLS does. Let the subarrays $G_j \times G_j$ be empty for $1 \leq j \leq u$.

For any $G \in \mathbf{G}$, partition G into triples $T_1, T_2, \ldots, T_{h/3}$ such that T_j corresponds to the j-th holey parallel class with hole G, $1 \leq j \leq h/3$. For any block $B = \{a, b, c, d\}$, B must appear in a unique, say j-th, holey parallel class with certain hole $G \in \mathbf{G}$. Suppose $T_j = \{x, y, z\}$. Delete the diagonal entries of the second square in Table 1, replace b, c, d by x, y, z, respectively and put it as the subarray $B \times B$. The resulting array is the required SOM. First, it is easily seen that each cell not in any of $G_j \times G_j$ is occupied with certain entry. If two cells (r, x) and (r, y) are occupied with the same entry z, where $r \in G_i$, $x, y \notin G_i$, and $z \in G_j$. Suppose r and x are contained in block B and r and y in B'. Suppose z is in the s-th triple from G_j. According to the construction, B and B' must both in the s-th holey parallel class with the hole G_j. But, B and B' have a common element r, a contradiction. Therefore, no two entries in the same row can be equal. Also, z is not in G_i. Otherwise, $i = j$ and r is in G_j, a contradiction. This means that each row intersecting $G_i \times G_i$ contains all elements in $X \setminus G_i$. Similarly, each column has the same property. We have shown that the array is a holey Latin square. It is also symmetric since each ingredient 4×4 subarray is symmetric.

It remains to show the orthogonality. Suppose the holey SOLS and the SOM are not orthogonal. Then there must be two cells (a, b) and (a', b') such that the entries in the holey SOLS are both c and the entries in the

SOM are both x. Suppose block B contains a, b, c and block B' contains a', b', c. Suppose x is in triple T_s of G_j. According to the construction of the SOM, B and B' are both in s-th holey parallel class with hole G_j. But, B and B' have a common element c, a contradiction. The proof is complete. □

Example 5.3. A HSOLSSOM of type 3^5 constructed from a 4-frame. Start from a 4-frame of type 3^5, based on $X = \{1, 2, \ldots, 15\}$ with holes and holey parallel classes indicated below.

holey parallel class	hole
$\{4,9,10,13\}, \{6,8,11,14\}, \{5,7,12,15\}$	$\{1,2,3\}$
$\{2,8,12,13\}, \{1,7,10,14\}, \{3,9,11,15\}$	$\{4,5,6\}$
$\{1,5,11,13\}, \{3,4,12,14\}, \{2,6,10,15\}$	$\{7,8,9\}$
$\{1,4,8,15\}, \{2,5,9,14\}, \{3,6,7,13\}$	$\{10,11,12\}$
$\{2,4,7,11\}, \{3,5,8,10\}, \{1,6,9,12\}$	$\{13,14,15\}$

A holey SOLS of type 3^5 and its symmetric orthogonal mate are shown in Table 5.2 and Table 5.3, respectively.

			8	11	9	10	15	12	14	13	6	5	7	4
			7	9	10	11	12	14	15	4	13	8	5	6
			12	8	7	13	10	11	5	15	14	6	4	9
15	11	14				2	1	10	13	7	3	9	12	8
13	14	10				12	3	2	8	1	15	11	9	7
12	15	13				3	11	1	2	14	9	7	8	10
14	4	6	11	15	13				1	2	5	3	10	12
4	13	5	15	10	14				3	6	2	12	11	1
6	5	15	13	14	12				4	3	1	10	2	11
7	6	8	9	3	15	14	5	13				4	1	2
5	7	9	2	13	8	4	14	15				1	6	3
9	8	4	14	7	1	15	13	6				2	3	5
11	12	7	10	1	3	6	2	4	9	5	8			
10	9	12	3	2	11	1	6	5	7	8	4			
8	10	11	1	12	2	5	4	3	6	9	7			

Table 5.2. A holey SOLS of type 3^5

We can employ Theorem 5.2 and Theorem 3.12 to improve Theorem 5.1, especially for hole sizes 3 and 6.

			10	7	13	4	11	14	5	8	15	9	6	12
			13	10	7	14	4	11	8	15	5	6	12	9
			7	13	10	11	14	4	15	5	8	12	9	6
10	13	7				15	12	1	2	14	9	3	8	11
7	10	13				1	15	12	14	9	2	8	11	3
13	7	10				12	1	15	9	2	14	11	3	8
4	14	11	15	1	12				6	13	3	10	5	2
11	4	14	12	15	1				13	3	6	5	2	10
14	11	4	1	12	15				3	6	13	2	10	5
5	8	15	2	14	9	6	13	3				1	4	7
8	15	5	14	9	2	13	3	6				7	1	4
15	5	8	9	2	14	3	6	13				4	7	1
9	6	12	3	8	11	10	5	2	1	7	4			
6	12	9	8	11	3	5	2	10	4	1	7			
12	9	6	11	3	8	2	10	5	7	4	1			

Table 5.3. A symmetric orthogonal mate

Theorem 5.4. *There exist HSOLSSOMs of type h^u when*

$h = 3$ *and* $u \in \{13, 17, 33\}$,

$h = 6$ *and* u *odd* $\geq 5, u \notin \{n\colon n \equiv 3 \bmod 4$ *and* $n \leq 267\}$.

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