SNARKS AND NON-HAMILTONIAN CUBIC 2-EDGE-CONNECTED GRAPHS OF SMALL ORDER

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October 26, 1995

Abstract

All non-Hamiltonian cubic 2-edge-connected graphs, including all snarks, on 16 or fewer vertices are listed, along with some of their properties. Questions concerning the existence of graphs with certain properties are posed.

1 Introduction

All graphs considered in this paper are finite and have no loops or multiple edges. By $V(G)$ and $E(G)$ we denote the vertex set and edge set, respectively, of the graph G.

A graph G is k-edge-connected (resp. k-vertex-connected) if there exist at least k edge-disjoint (resp. vertex-disjoint) paths between each pair of vertices of G.

A *cycle* is a 2-regular connected graph. A *Hamilton cycle* in a graph G is a 2-regular connected spanning subgraph of G. A *Hamilton decomposition* of a regular graph G consists of a set of Hamilton cycles (plus a 1-factor if $\Delta(G)$ is odd) of G such that these cycles (and

*Supported in part by NSERC (Canada) and a grant-in-aid of research from Sigma Xi.

the 1-factor when $\Delta(G)$ is odd) partition the edges of G. If G has a Hamilton decomposition, it is said to be *Hamilton decomposable*.

A graph is said to be *class 1* (resp. *class 2*) if it can be properly edge-coloured with $\Delta(G)$ (resp. $\Delta(G)+1$) colours. A *snark* is a 2-edge-connected cubic graph that is class 2. Alternative definitions, with additional requisite properties, exist for snarks [4, 9], but they are not relevant to the discussion within this paper.

The *line graph*, denoted by $L(G)$, of a graph G is defined to be the graph with vertex set $E(G)$, where two vertices of $L(G)$ are adjacent in $L(G)$ if and only if the corresponding edges in G are incident with a common vertex in G.

Definitions omitted in this paper can be found in [3].

This paper is motivated by the problem of determining necessary and sufficient conditions for a graph G to be Hamilton decomposable. An obvious necessary condition is that if G is a k-regular graph, then G must be $(2\lfloor \frac{k}{2} \rfloor)$-edge-connected. A computer search is under way to find graphs that possess this necessary condition, but which fail to be Hamilton decomposable. Further analysis of these graphs will hopefully reveal additional, but less obvious, conditions which Hamilton decomposable graphs must possess.

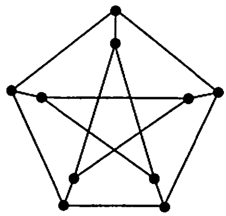
In this paper we focus on the results of this search with respect to cubic graphs of order up to 16. All 3-regular 2-edge-connected non-Hamiltonian graphs on 16 or fewer vertices are presented, along with some of their properties. Clearly no snark is Hamiltonian, and so this list includes all snarks of order up to 16.

2 Computer Search Results

The computer search determined that there exist exactly 1 non-Hamiltonian cubic 2-edge-connected graph of order 10 (the Petersen graph), 1 graph of order 12, 6 non-isomorphic graphs of order 14, and 33 non-isomorphic graphs of order 16. Each graph is listed in the following sections.

The Petersen graph is the only graph which possesses the properties of vertex-transitivity and edge-transitivity. All of the other graphs are neither vertex-transitive nor edge-transitive.

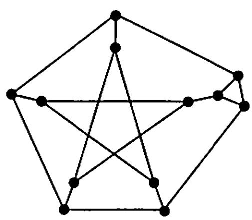
2.1 The Graph of Order 10



Graph 10.1
(The Petersen Graph)

Vertex-Connectivity	3
Class	2

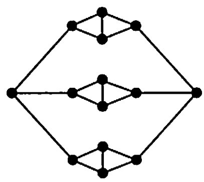
2.2 The Graph of Order 12



Graph 12.1

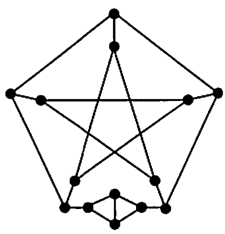
Vertex-Connectivity	3
Class	2

2.3 The Six Graphs of Order 14



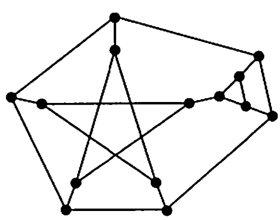
Graph 14.1

Vertex-Connectivity	2
Class	1



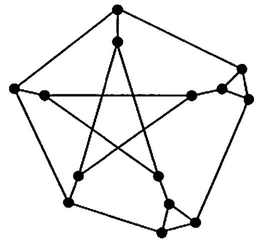
Graph 14.2

Vertex-Connectivity	2
Class	2



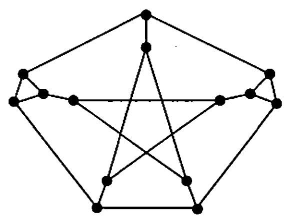
Graph 14.3

Vertex-Connectivity	3
Class	2



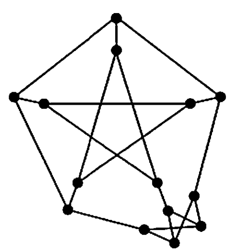
Graph 14.4

Vertex-Connectivity	3
Class	2



Graph 14.5

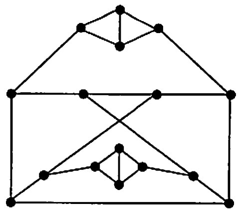
Vertex-Connectivity	3
Class	2



Graph 14.6

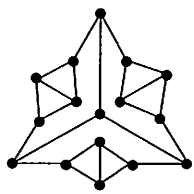
Vertex-Connectivity	3
Class	2

2.4 The Thirty-Three Graphs of Order 16



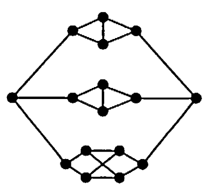
Graph 16.1

Vertex-Connectivity	2
Class	1



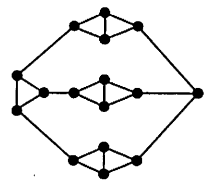
Graph 16.2

Vertex-Connectivity	2
Class	1



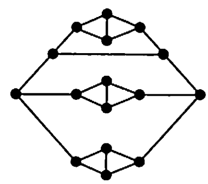
Graph 16.3

Vertex-Connectivity	2
Class	1



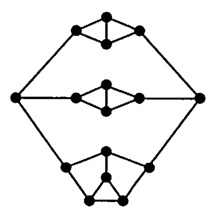
Graph 16.4

Vertex-Connectivity	2
Class	1



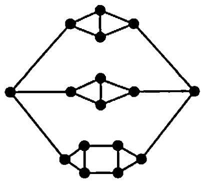
Graph 16.5

Vertex-Connectivity	2
Class	1



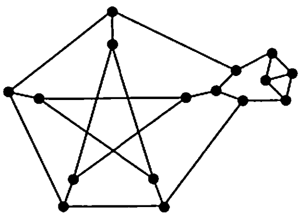
Graph 16.6

Vertex-Connectivity	2
Class	1



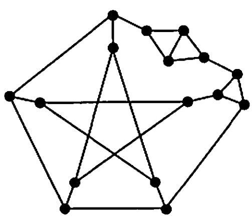
Graph 16.7

Vertex-Connectivity	2
Class	1



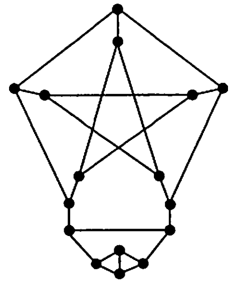
Graph 16.8

Vertex-Connectivity	2
Class	2



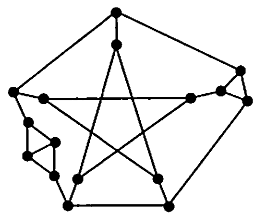
Graph 16.9

Vertex-Connectivity	2
Class	2



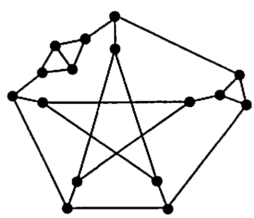
Graph 16.10

Vertex-Connectivity	2
Class	2



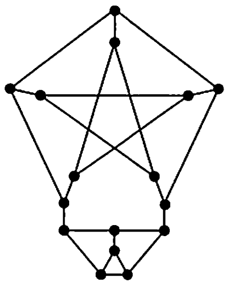
Graph 16.11

Vertex-Connectivity	2
Class	2



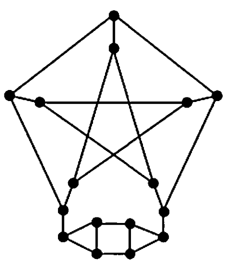
Graph 16.12

Vertex-Connectivity	2
Class	2



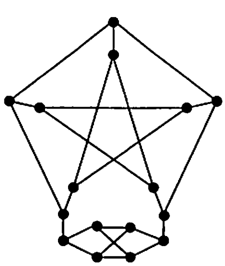
Graph 16.13

Vertex-Connectivity 2
Class 2



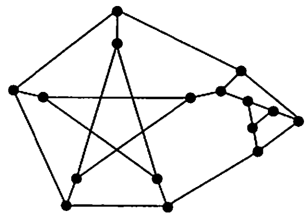
Graph 16.14

Vertex-Connectivity 2
Class 2



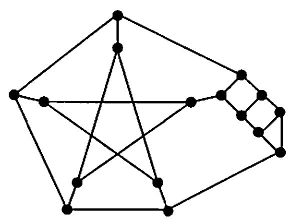
Graph 16.15

Vertex-Connectivity 2
Class 2



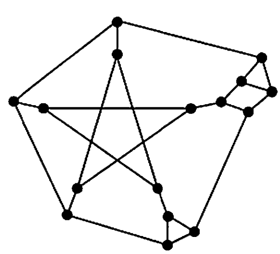
Graph 16.16

Vertex-Connectivity 3
Class 2



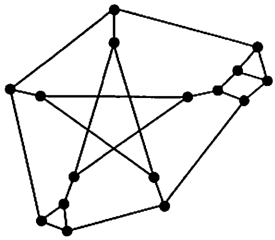
Graph 16.17

Vertex-Connectivity 3
Class 2



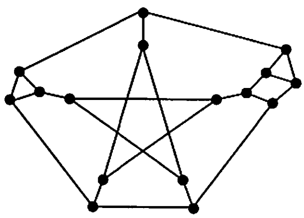
Graph 16.18

Vertex-Connectivity 3
Class 2



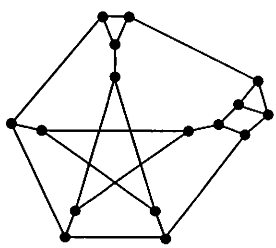
Graph 16.19

Vertex-Connectivity	3
Class	2



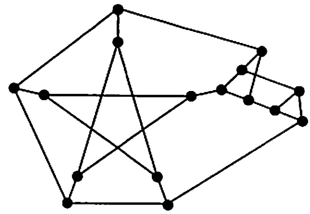
Graph 16.20

Vertex-Connectivity	3
Class	2



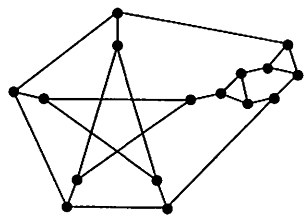
Graph 16.21

Vertex-Connectivity	3
Class	2



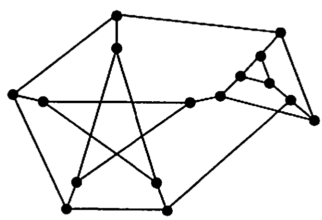
Graph 16.22

Vertex-Connectivity	3
Class	2



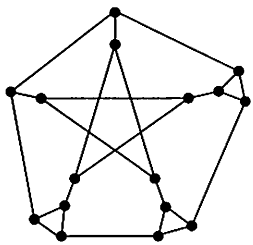
Graph 16.23

Vertex-Connectivity	3
Class	2



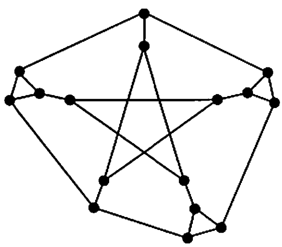
Graph 16.24

Vertex-Connectivity	3
Class	2



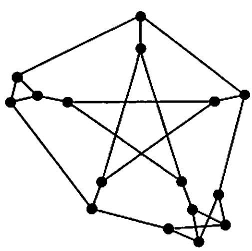
Graph 16.25

Vertex-Connectivity	3
Class	2



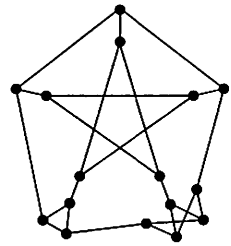
Graph 16.26

Vertex-Connectivity	3
Class	2



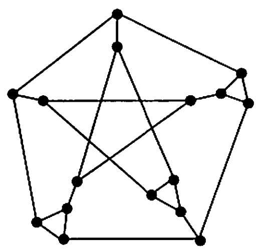
Graph 16.27

Vertex-Connectivity	3
Class	2



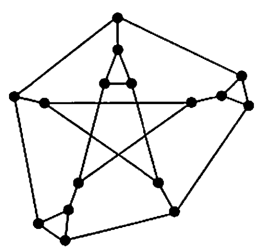
Graph 16.28

Vertex-Connectivity	3
Class	2



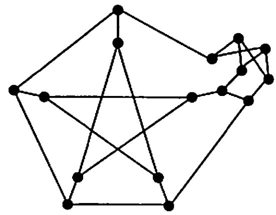
Graph 16.29

Vertex-Connectivity	3
Class	2



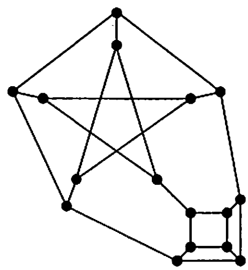
Graph 16.30

Vertex-Connectivity	3
Class	2



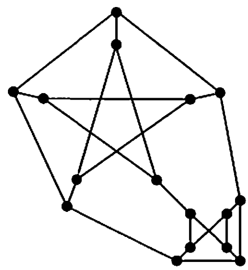
Graph 16.31

Vertex-Connectivity	3
Class	2



Graph 16.32

Vertex-Connectivity	3
Class	2



Graph 16.33

Vertex-Connectivity	3
Class	2

3 Discussion

We first note that many of the graphs presented are similar; in particular, many appear to have been derived from the Petersen graph. Indeed, graphs 10.1, 14.1, 16.1, 16.2 and 16.3 are basic in the sense that all of the remaining graphs can be obtained from them by means of the following three operations, each of which produces a cubic non-Hamiltonian graph:

1. Vertex-Splicing with a cubic graph

   Take a cubic non-Hamiltonian graph, G_1, and any cubic graph, G_2. Delete a vertex v_1 from G_1 and a vertex v_2 from G_2 and then arbitrarily pair the three neighbours of v_1 in G_1 with the three neighbours of v_2 in G_2. Form a single graph, G, by joining each set of paired vertices by an edge.

2. Edge-Splicing with a cubic graph

   Take a cubic non-Hamiltonian graph, G_1, and any cubic graph, G_2. Delete an edge e_1 from G_1 and an edge e_2 from G_2 and then arbitrarily pair the two end-vertices of e_1 in G_1 with the two end-vertices of e_2 in G_2. Form a single graph, G, by joining each set of paired vertices by an edge.

3. Subdividing a 2-edge-cut

   Take a cubic non-Hamiltonian graph with an edge-cut of size two, subdivide the two edges of a 2-edge-cut and then join the two new vertices by adding an edge between them.

Each of these three operations can be repeatedly applied to yield non-Hamiltonian cubic graphs of arbitrarily large order.

After conducting further analysis of these graphs, and of the preliminary output of other computer searches that are still in progress, various questions arise. For instance, it has been shown that vertex-transitive graphs having a prime number of vertices are Hamilton decomposable [8]. And connected vertex-transitive graphs of order p^2 or p^3, where p is a prime, have been shown to be Hamiltonian[7]. We wonder if these results can be extended, and so we ask:

Question 1 *Does there exist a connected vertex-transitive graph G of order p^n, where p is a prime and $n \geq 1$, such that G is not Hamilton decomposable?*

No answer is yet known for this first question, nor for the question which follows next; no such graphs have been found by the computer searches conducted thus far.

Question 2 *Does there exist a k-regular connected vertex-transitive bipartite graph G such that G is not Hamilton decomposable?*

Note that if, in the above question, we replace vertex-transitivity with the property of being $(2\lfloor\frac{k}{2}\rfloor)$-edge-connected, then the question is affirmatively answered by the existence of the Horton graph [3].

The computer results presented in this paper would also suggest the following question:

Question 3 *What further conditions are needed to ensure that a k-regular k-vertex-connected class 1 graph is Hamilton decomposable?*

The Horton graph [3] is one of many graphs [6] which are k-regular, k-vertex-connected, and class 1, yet which are not Hamilton decomposable.

Finally, Bermond [2] has conjectured that if G is Hamilton decomposable, then so is $L(G)$. One might thus ask the following related question:

Question 4 *What further conditions are needed to ensure that a k-regular $(2\lfloor \frac{k}{2} \rfloor)$-edge-connected non-Hamilton decomposable graph G has a line graph, $L(G)$, which is non-Hamilton decomposable?*

That additional conditions are necessary is illustrated by the line graph of the Petersen graph, which is 4-regular, 4-edge-connected, and has no Hamilton decomposition, yet its line graph is Hamilton decomposable.

4 Acknowledgement

The author wishes to extend thanks to the McNeil Consumer Products Company in Guelph, Ontario for their co-operation.

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