A Census of $(9; 1; 3, 2)$ Balanced Ternary Designs

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ABSTRACT. A balanced ternary design of order nine with block size three, index two and $\rho_2 = 1$ is a collection of multi-subsets of size 3 (of type $\{x, y, z\}$ or $\{x, x, y\}$) called blocks, chosen from a 9-set, in which each unordered pair of distinct elements occurs twice, possibly in one block, and in which each element is repeated in just one block. So there are precisely 9 blocks of type $\{x, x, y\}$. We denote such a design by $(9; 1; 3, 2)$ BTD. In this note we describe the procedures we have used to determine that there are exactly 1475 non-isomorphic $(9; 1; 3, 2)$ BTDs.

1 Introduction

A *balanced ternary design* or BTD is a collection of multi-sets of size k (called *blocks*), chosen from a v-set in such a way that each pair of distinct elements $\{x, y\}$ occurs precisely λ times, each pair of non-distinct elements $\{x, x\}$ occurs precisely ρ_2 times, and each element occurs 0, 1 or 2 times in a block (hence ternary). Such a design is usually denoted by $(v; \rho_2; k, \lambda)$ BTD. The ordered pair $(V, \mathcal{B})$, may also be used to denote a BTD where V is the element set and where $\mathcal{B}$ is the collection of multi-subsets of V. See [1] for more details on BTDs. Of course a $(v; 0; k, \lambda)$ BTD is a (v, k, λ) BIBD. Where there can be no confusion we will often write the blocks of a BTD as xyz rather than $\{x, y, z\}$. Note that the block $\{x, x, y\}$ contains the pairs $\{x, x\}$ once and $\{x, y\}$ twice.

A *partial balanced ternary design* is a pair $(V, \mathcal{B})$ where $\mathcal{B}$ is a collection of multi-sets of size k (called blocks), chosen from a v-set V in such a way

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that each pair of distinct elements $\{x, y\}$ occurs at most λ times, each pair of non-distinct elements $\{x, x\}$ occurs at most ρ_2 times and each element occurs 0,1 or 2 times in a block. If $(V', \mathcal{B}')$ is a partial BTD, $(V, \mathcal{B})$ is a $(v; \rho_2; k, \lambda)$ BTD, $V' \subseteq V$ and $\mathcal{B}' \subseteq \mathcal{B}$, then $(V, \mathcal{B})$ is called a *completion* of $(V', \mathcal{B}')$.

This paper deals with $(9;1;3,2)$ BTDs. A $(9;1;3,2)$ BTD (with element set $\{1,2,\ldots,9\}$) has precisely nine blocks with repeated elements; $11x_1, 22x_2, \ldots, 99x_9$ where each $x_i \in \{1,2,\ldots 9\}$. The only partial BTDs which we are interested in are those consisting of the nine blocks which contain repeated elements and we will place a "*" on the word partial (that is, partial*) to signify that we are talking about a partial BTD which consists of the nine blocks $11x_1, 22x_2, \ldots, 99x_9$. Any partial* BTD, say with blocks $11x_1, 22x_2, 33x_3, 44x_4, 55x_5, 66x_6, 77x_7, 88x_8, 99x_9$, is completely described by the 9-tuple $x_1x_2x_3x_4x_5x_6x_7x_8x_9$. We refer to this 9-tuple as the *signature* of the partial* design. Note that not every partial* BTD can be completed to a $(9;1;3,2)$ BTD, for example the signature 231111113 cannot be completed to a $(9;1;3,2)$ BTD.

An *isomorphism* of two (partial) balanced ternary designs $(V_1, \mathcal{B}_1)$ and $(V_2, \mathcal{B}_2)$ is a bijection $\sigma : V_1 \to V_2$ such that $\sigma(\mathcal{B}_1) = \mathcal{B}_2$ where $\sigma(\mathcal{B}_1)$ is defined in the obvious way. If there exists such a bijection then we say that $(V_1, \mathcal{B}_1)$ and $(V_2, \mathcal{B}_2)$ are *isomorphic*.

In [2] Donovan has proved that there are two non-isomorphic $(6;1;3,2)$ BTDs and Morgan [5] and Mathon and Rosa [3] have found all the 36 non-isomorphic $(9;0;3,2)$ BTDs (that is, all the $(9,3,2)$ BIBDs). Clearly, the next enumeration problem to solve is that of $(9;1;3,2)$ BTDs. Here, we determine that there are exactly 1475 of these. This number is surprisingly large (at least to the authors) when compared with the results of BTD enumerations done previously.

2 Techniques

The following simple result is useful for proving that two BTDs are non-isomorphic. Let $(V_i, \mathcal{B}_i)$, $i = 1, 2$, be two $(9;1;3,2)$ BTDs and let σ be an isomorphism from $(V_1, \mathcal{B}_1)$ to $(V_2, \mathcal{B}_2)$. If $\mathcal{B}_i' = \{\{x,x,y\} | \{x,x,y\} \in \mathcal{B}_i\}$ then clearly $\sigma(\mathcal{B}_1') = \mathcal{B}_2'$. Hence if $(V_1, \mathcal{B}_1')$ and $(V_2, \mathcal{B}_2')$ are two non-isomorphic partial* $(9;1;3,2)$ BTDs then no completion of $(V_1, \mathcal{B}_1')$ is isomorphic to a completion of $(V_2, \mathcal{B}_2')$.

Example 2.1 Let $\mathcal{B}_1' = \{112, 223, 331, 441, 551, 662, 772, 884, 994\}$ and $\mathcal{B}_2' = \{112, 223, 334, 445, 556, 667, 778, 889, 991\}$, so $(\mathbb{Z}_9, \mathcal{B}_1')$ and $(\mathbb{Z}_9, \mathcal{B}_2')$ are not isomorphic. An exhaustive computer search shows that the partial design $(\mathbb{Z}_9, \mathcal{B}_1')$ has 4 completions (see Table 2.1) but only two of them are non-isomorphic (Designs 1 and 2 are isomorphic to Designs 3 and 4, respectively, both under the permutation (67)). The partial design $(\mathbb{Z}_9, \mathcal{B}_2')$ has 3 completions (see Table 2.2) all of which are non-isomorphic. The above result tells us that no design in Table 2.1 is isomorphic to a design in Table 2.2.

1	112	223	331	441	551	662	772	884	994
	169	169	178	178	245	245	289	289	346
	347	358	359	368	379	467	567	568	579
2	112	223	331	441	551	662	772	884	994
	168	169	178	179	245	245	289	289	346
	347	358	359	369	378	467	567	568	579
3	112	223	331	441	551	662	772	884	994
	168	168	179	179	245	245	289	289	346
	347	358	359	369	378	467	567	569	578
4	112	223	331	441	551	662	772	884	994
	168	169	178	179	245	245	289	289	346
	347	358	359	368	379	467	567	569	578

Table 2.1

1	112	223	334	445	556	667	778	889	991
	136	136	148	148	157	157	247	247	259
	259	268	268	358	358	379	379	469	469
2	112	223	334	445	556	667	778	889	991
	136	137	146	148	157	158	247	248	257
	259	268	269	358	359	368	379	469	479
3	112	223	334	445	556	667	778	889	991
	137	137	146	146	158	158	248	248	257
	257	269	269	359	359	368	368	479	479

Table 2.2

It turns out that there are 363 non-isomorphic partial* $(9;1;3,2)$ BTDs. Hence the above result means that the set of all non-isomorphic $(9;1;3,2)$ BTDs may be partitioned into 363 disjoint sets (corresponding to the 363 non-isomorphic partial* designs) with each set containing all the non-isomorphic completions of the corresponding partial* $(9;1;3,2)$ BTD.

We now describe the procedure used to find the 363 non-isomorphic partial* designs mentioned above. It is possible to use the software package *dreadnaut* [4] to determine precisely when two partial* BTDs are isomorphic. As *dreadnaut* determines graph isomorphisms it is first necessary to define, for any given partial* BTD an associated graph which characterises the design.

Definition 2.2. The *associated graph* $G(V,\mathcal{B})$ of the partial* BTD $(V,\mathcal{B})$ with signature $x_1x_2\ldots x_9$ has vertex set $A \cup B \cup C \cup D$ where:

(i) $A = \{1,2,3,4,5,6,7,8,9\}$ (these vertices correspond to the elements of the partial* design);

(ii) $B = \{11,22,33,44,55,66,77,88,99\}$ (these vertices correspond to the pairs of non-distinct elements);

(iii) $C = \{12,13,\dots,19,23,24,\dots,29,34,\dots,78,79,89\}$ (these vertices correspond to pairs of distinct elements);

(iv) $D = \{11x_1, 22x_2, \dots, 99x_9\}$ (these vertices correspond to the 9 blocks of the partial* design).

The edge set of the associated graph is then given by $AB \cup AC \cup AD \cup BC \cup BD \cup CD$ where

(i) $AB = \{\{1,11\},\{2,22\}\dots\{9,99\}\}$;

(ii) $AC = \{\{1,12\},\{1,13\},\dots,\{1,19\},\{2,12\},\{2,23\},\dots,\{2,29\},\dots,\{9,19\},\{9,29\},\dots,\{9,89\}\}$;

(iii) $AD = \{\{x,yyz\} | x = y \text{ or } x = z\}$;

(iv) $BC = \emptyset$;

(v) $BD = \{\{11,11x_1\},\{22,22x_2\},\dots,\{99,99x_9\}\}$;

(vi) $CD = \{\{wx,yyz\} | (w = y \text{ and } x = z) \text{ or } (w = z \text{ and } x = y)\}$. □

It is easy to see that:

Lemma 2.3. *The partial* BTDs $(V_1,\mathcal{B}_1)$ and $(V_2,\mathcal{B}_2)$ are isomorphic if and only if their associated graphs $G(V_1,\mathcal{B}_1)$ and $G(V_2,\mathcal{B}_2)$ are isomorphic.* □

This result means that we can use *dreadnaut* to determine precisely when two partial* BTDs are isomorphic. So now it is simply a matter of sorting the 9^9 possible partial* BTDs into isomorphism classes. Clearly 9^9 is an overestimate as the list of possible partial* BTDs can be greatly reduced. For example we do not need to consider blocks of the form xxx. Moreover, we can assume without loss of generality that the first two blocks are 112 and 223. This immediately reduces the number of possible partial* BTDs to less than 9^7. It turns out that there are 363 non-isomorphic partial* BTDs.

The next step after obtaining all the non-isomorphic partial* BTDs is to find all the non-isomorphic completions (if any) of each of them. This is achieved by first finding all completions of each, and then removing isomorphic copies. It is possible to find all completions of each non-isomorphic partial* BTD by using a simple backtracking computer search. Of the 363 non-isomorphic partial* designs there are 178 which cannot be completed to a $(9;1;3,2)$ BTD leaving 185 completable non-isomorphic partial* designs.

Now we use *dreadnaut* again to sort the completions of each partial* BTD into isomorphism classes. This requires the definition of an associated graph which characterises a complete $(9;1;3,2)$ BTD. This is done by extending the associated graph of the partial* BTD.

Definition 2.4. The associated graph $G(V,\mathcal{B})$ of a $(9;1;3,2)$ BTD $(V,\mathcal{B})$ with blocks $B_1, B_2, \dots, B_{18}$ not in its partial* design is the associated graph of its partial* design together with 18 new vertices $B_1, B_2, \dots, B_{18}$ and new edges $\{x, B_i\}$ if and only if $x \in B_i$, for each $x \in A$ and each $i \in \{1, 2, \dots, 18\}$; □

As before, it is easy to see that:

Lemma 2.5. *The* $(9;1;3,2)$ *BTDs* $(V_1,\mathcal{B}_1)$ *and* $(V_2,\mathcal{B}_2)$ *are isomorphic if and only if their associated graphs* $G(V_1,\mathcal{B}_1)$ *and* $G(V_2,\mathcal{B}_2)$ *are isomorphic.* □

Hence, we can use *dreadnaut* to determine precisely when two $(9;1;3,2)$ BTDs are isomorphic. So now all we need to do is, for each of the 185 completable non-isomorphic partial* BTDs, sort the set of all completions of the partial* design into isomorphism classes. The results of this are shown in Section 3. The set of all non-isomorphic $(9;1;3,2)$ BTDs can now be determined by taking the (disjoint) union over the 185 completable non-isomorphic partial* BTDs of the sets of non-isomorphic completions of each. The following theorem summarises this result.

Theorem 2.6. *There are precisely* 1475 *non-isomorphic* $(9;1;3,2)$ *BTDs.* □

3 Results

In this section we give the results referred to in the previous sections. This is conveniently done via the following table. The table contains for each of the 185 completable non-isomorphic partial* BTDs; its number (1 to 185), its signature, the number of completions and the number of non-isomorphic completions. These are given in columns 1 to 4 respectively.

No.	Signature	No.C.	No.N.D.	No.	Signature	No.C.	No.N.D.
1	231111111	4800	18	2	231111122	108	6
3	231111123	36	3	4	231111124	84	14
5	231111133	108	6	6	231111134	84	14
7	231111144	60	7	8	231111145	52	17
9	231111222	36	3	10	231111223	48	4
11	231111224	32	9	12	231111233	48	4
13	231111234	24	12	14	231111244	16	5
15	231111245	29	16	16	231111334	32	9
17	231111344	16	5	18	231111345	29	16
19	231111445	30	16	20	231111456	47	13
21	231112233	74	7	22	231112234	16	8
23	231112236	16	8	24	231112238	20	6
25	231112244	4	2	26	231112245	17	8
27	231112246	8	8	28	231112248	5	3
29	231112266	4	2	30	231112267	17	8
31	231112268	5	3	32	231112344	8	4
33	231112345	13	8	34	231112346	11	11
35	231112347	11	11	36	231112348	5	5
37	231112367	18	9	38	231112445	23	12
39	231112446	15	8	40	231112456	21	13
41	231112457	6	6	42	231112466	6	3
43	231112467	8	8	44	231112468	4	4
45	231113445	23	12	46	231113446	15	8
47	231113456	21	13	48	231113457	6	6
49	231113466	6	3	50	231113467	8	8
51	231113468	4	4	52	231114455	50	19
53	231114458	26	14	54	231114567	14	12
55	231123445	8	4	56	231123446	8	4
57	231123456	30	12	58	231123457	6	6
59	231123458	6	6	60	231124455	4	2
61	231124458	10	5	62	231124556	10	5
63	231124567	8	8	64	231144456	23	14
65	231144458	26	14	66	231144567	22	22
67	231564897	132	8	68	231567894	42	8
69	234111112	108	9	70	234111114	108	9
71	234111115	102	18	72	234111122	42	5
73	234111124	66	11	74	234111125	35	18
75	234111144	42	5	76	234111145	35	18
77	234111155	43	14	78	234111156	40	23
79	234111223	26	7	80	234111224	26	7

No.	Signature	No.C.	No.N.D.	No.	Signature	No.C.	No.N.D.
81	234111225	21	11	82	234111227	21	11
83	234111233	12	3	84	234111234	12	6
85	234111235	10	10	86	234111237	32	16
87	234111238	6	3	88	234111245	36	36
89	234111247	14	7	90	234111248	14	7
91	234111255	11	6	92	234111256	16	11
93	234111257	5	5	94	234111258	15	15
95	234111277	4	2	96	234111278	5	3
97	234111335	4	2	98	234111345	10	10
99	234111347	6	3	100	234111348	32	16
101	234111355	4	2	102	234111356	15	10
103	234111358	18	18	104	234111378	11	6
105	234111455	11	6	106	234111456	16	11
107	234111457	5	5	108	234111458	15	15
109	234111477	4	2	110	234111478	5	3
111	234111556	17	9	112	234111567	8	8
113	234112345	17	17	114	234112355	4	2
115	234112356	18	18	116	234112357	10	10
117	234112358	7	7	118	234112366	8	4
119	234112367	18	18	120	234112368	9	9
121	234112377	4	2	122	234112378	7	7
123	234112556	6	3	124	234112566	6	3
125	234112567	5	5	126	234112568	5	5
127	234113567	5	5	128	234511122	24	8
129	234511123	4	2	130	234511124	6	3
131	234511125	10	5	132	234511126	13	13
133	234511128	18	10	134	234511134	2	1
135	234511135	6	3	136	234511136	2	2
137	234511138	3	2	138	234511145	4	2
139	234511146	2	2	140	234511148	3	2
141	234511156	13	13	142	234511158	18	10
143	234511166	18	10	144	234511167	18	13
145	234511168	21	21	146	234511234	7	7
147	234511236	9	9	148	234511237	12	12
149	234511238	9	9	150	234511246	11	11
151	234511247	11	11	152	234511248	4	4
153	234511266	14	7	154	234511267	15	15
155	234511268	16	16	156	234511277	14	7
157	234511278	16	16	158	234511366	2	1
159	234511367	6	6	160	234511368	7	7

No.	Signature	No.C.	No.N.D.	No.	Signature	No.C.	No.N.D.
161	234511377	2	1	162	234511378	7	7
163	234561112	4	2	164	234561113	3	2
165	234561115	3	2	166	234561116	4	2
167	234561117	1	1	168	234561123	7	7
169	234561124	1	1	170	234561125	1	1
171	234561127	10	10	172	234561128	10	10
173	234561135	7	7	174	234561137	4	4
175	234561138	4	4	176	234561147	3	3
177	234561177	4	2	178	234561178	9	9
179	234567111	4	3	180	234567112	4	4
181	234567113	4	4	182	234567114	1	1
183	234567118	4	4	184	234567811	5	5
185	234567891	3	3				

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