On The Factorizations of Complete Graphs*

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ABSTRACT. The support size of a factorization is the sum over the factors of the number of distinct edges in factor. The spectrum of support sizes of $l\lambda$-factorizations of λK_n and $\lambda K_{n,n}$ are completely determined for all λ and n.

1 Introduction

A λ-factor in a multigraph G is a submultigraph F which is spanning and λ-regular. A λ-factorization of a multigraph G is a partition of edges of G into λ-factors. For a simple graph G, the multigraph λG is obtained by repeating each edge λ times. A $m\lambda$-factorization $\Gamma = \{F_1, \ldots, F_d\}$ of $l\lambda G$ is called Completely Decomposable (or briefly, C.D.), if these conditions are satisfied: (i) lG is m-factorable, (ii) for every common divisor k of m and l, $(l/k)G$ is not (m/k)-factorable, (iii) there exist λ m-factorizations of lG, e.g. $\Gamma_i = \{F_1^i, \ldots, F_d^i\}$, $1 \le i \le \lambda$, such that for every $1 \le j \le d$, F_j is the union of F_j^i's $(1 \le i \le \lambda)$. The support size of a λ-factor is the number of distinct edges in the factor, and the support size of a λ-factorization is the sum of the support sizes of its factors. We denote by $S(G, l\lambda, m\lambda)$ the set of the support sizes of $m\lambda$-factorizations of $l\lambda G$, and by $CS(G, l\lambda, m\lambda)$ the set of the support sizes of C.D. $m\lambda$-factorizations of $l\lambda G$. For simplicity, $S(K_n, l\lambda, m\lambda)$, $CS(K_n, l\lambda, m\lambda)$, and $CS(K_{n,n}, \lambda, \lambda)$ will be denoted by $S(n, l\lambda, m\lambda)$, $CS(n, l\lambda, m\lambda)$, and $C(n, \lambda)$, respectively.

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Let G be a simple ld-regular graph on n vertices which is l-factorable. Let $\mathcal{F} = \{F_1, \dots, F_d\}$ and $\mathcal{G} = \{G_1, \dots, G_d\}$ be two l-factorizations of G. Then $\mathcal{F}$ and $\mathcal{G}$ are said to have k edges in common if $\sum_{i=1}^{d} |F_i \cap G_i| = k$. Let $J(G, l)$ be the set of all k's such that there exists a pair of l-factorizations of G having exactly k edges in common. It is clear that determination of $J(G, l)$ is equivalent to the determination of $CS(G, 2, 2l)$, and so support size problem for C.D. factorizations can be considered as a generalization of intersection problem for factorizations of the complete graphs.

The factorizations of complete graphs (and complete bipartite graphs) occur in a very natural way in the recursive constructions of triple systems, e.g. $v \to 2v + 4$ and $v \to 3v$ constructions. In each of these constructions, repeated edges give rise to repeated blocks in related designs. For this reason, $C(n, \lambda)$, and $S(n, \lambda, l\lambda)$ have had some interesting features, and have been instrumental in determination of the set $SS(v, \lambda)$ of the support sizes of triple systems for $\lambda = 2$ [17], $\lambda = 6$, $v > 14$ [7] and for $v \equiv 0 \pmod 4$ [12] with one possible omission for every $v \equiv 8 \pmod{12}$. Recently, Colbourn and Lindner utilizing some tripling constructions and well known results on $S(n, \lambda, \lambda)$ have determined $SS(v, \lambda)$ for $v > 14$ (again with one possible omission) [6]. Also, in [3] utilizing graph factorization techniques $SS(v, \lambda)$ has been completely determined for $v > 14$, $v \not\equiv 1 \pmod{12}$ with one possible omission for every $v \equiv 8 \pmod{12}$. Similarly, completely decomposable factorizations of complete graphs can be utilized in the intersection problem for simple triple systems, and in the problem of determination of the set $CSS(v, \lambda)$ of the support sizes of completely decomposable triple systems of order v and index λ.

In [14], Lindner and Wallis have completely determined the set of all integers k such that there exist two 1-factorizations of K_{2n} intersecting in exactly k edges. In other words, they have completely determined $CS(2n, 2, 2)$. In [11], Fu has completely determined the set of all integers k such that there exist two Latin squares of order n which agree in exactly k cells. In other words, he has completely determined $C(n, 2)$. $S(n, 2, 2)$, $S(n, 3, 3)$ and $S(K_{n,n}, 3, 3)$ have been completely determined by Colbourn and Rosa [8], Colbourn [4], and Colbourn [5], respectively. In [9] utilizing results of [4,8] Colbourn and Rosa completely determined the set $S(n, \lambda, \lambda)$.

In this paper, $C(n, \lambda)$ and $CS(n, \lambda, l\lambda)$ are completely determined. The main idea of our proofs is an effective use of embedding theorems for factorizations of complete graphs. In Section 2, we develop some general methods to find spectrum of a graph G, $CS(G, \lambda, l\lambda)$, from spectrum of a special class of its subgraphs, and in Section 3 utilizing this methods we prove our main results. It should be noted that in all proofs completely decomposability arises in a natural way, and even some of our proofs are not true in general case. In fact, in all proofs we strongly use this assumption that we have a C.D. factorization of a subgraph to construct a factorization of the original graph which is obviously completely decomposable. Therefore, by restricting ourselves to the case of C.D. factorizations we can prove some of our lemmas in an easier way, while the result is the same as the general case.

Throughout this paper, we shall use $G_1 + G_2$ to denote the union of two multigraph G_1 and G_2.

2 Some General Results

In this section, we develop some recursive constructions to find spectrum of a graph G from the spectrum of some special class of its subgraphs. To apply recursive constructions we must find a partial determination of $CS(G, \lambda, l\lambda)$ for all l-factorable graphs. To do this our main tools are (p, λ)-patterns. A (p, λ)-pattern is a $p \times p$ matrix, $P = (p_{ij})$, with nonnegative integral entries such that $\sum_{i=1}^{p} p_{ij} = \sum_{i=1}^{p} p_{ji} = \lambda$ for $1 \leq j \leq p$. A $(p, 1)$-pattern is a permutation matrix. The support size of a (p, λ)-pattern is the number of its nonzero entries. Let $S_p(p, \lambda)$ denote the set of support sizes of (p, λ)-patterns. In [9] Colbourn and Rosa introduced (p, λ)-pattern and their support sizes in an equivalent form and they proved the following lemma.

Lemma 2.1. *If $p \geq 2$, then*

$$S_p(p, \lambda) = \begin{cases} \{p, \dots, \min(\lambda p, p^2)\} \setminus \{p+1\}, & \text{if } \lambda \neq p, \\ \{p, \dots, \min(\lambda p, p^2)\} \setminus \{p+1, p^2-1\}, & \text{otherwise.} \end{cases}$$

□

The following lemma is proved in [15].

Lemma 2.2. *Let P be a (p, λ)-pattern, then there exist λ permutation matrices, $P_1, \dots, P_\lambda$, such that $P = \sum_{i=1}^{\lambda} P_i$.* □

The following lemma demonstrates the patterns in graph factorization.

Lemma 2.3. *Let G be a simple ld-regular graph on n vertices which is l-factorable. If $r \in S_p(d, \lambda)$, then $rln/2 \in CS(G, \lambda, l\lambda)$.*

Proof: Let $\{F_1, \dots, F_d\}$ be any l-factorization of G, and let $P = (p_{ij})$ be a (d, λ)-pattern with support size r. Put

$$G_i = \sum_{j=1}^{d} p_{ij} F_j, \quad 1 \leq i \leq d.$$

Then $\mathcal{G} = \{G_1, \dots, G_d\}$ is a $l\lambda$-factorization with support size $rln/2$ of λG. Complete decomposability of $\mathcal{G}$ is a straightforward consequence of Lemma 2.1. □

Two following lemmas are trivial and so their proofs are left.

Lemma 2.4. *Let $G_1, \ldots, G_n$ be edge-disjoint simple graphs such that* (i) *G_i is $l_i d$-regular and l_i-factorable* (ii) *their union $G = \cup_{i=1}^n G_i$ is ld-regular and l-factorable, and* (iii) *if F_i is any l_i-factor of G_i $(1 \leq i \leq n)$, then $\cup_{i=1}^n F_i$ is a l-factor of G. If $r_i \in CS(G_i, \lambda, \lambda)$, $(1 \leq i \leq n)$, then $\sum_{i=1}^n r_i \in CS(G, \lambda, l\lambda)$.* □

Lemma 2.5. *Let G_1 and G_2 be two edge-disjoint l-factorable graphs, $r \in CS(G_1, \lambda, l\lambda)$ and $s \in CS(G_2, \lambda, l\lambda)$. If either vertex sets of G_1 and G_2 are the same or these sets are disjoint but G_1 and G_2 are of the same degree, then $r + s \in CS(G_1 + G_2, \lambda, l\lambda)$.* □

Lemma 2.6. *Let G_1 and G_2 be two edge-disjoint simple graphs such that* (i) *G_1 is m-regular and 1-factorable* (ii) *$G = G_1 + G_2$ is l-factorable* (iii) *G_2 has a partition $\{F_1, \ldots, F_m\}$ into spanning subgraphs such that if K is any 1-factor of G_1 then for every $1 \leq i \leq m$ $F_i + K$ is a l-factor of G. If $r \in CS(G_1, \lambda, \lambda)$ and $s \in S_p(m, \lambda)$, then $r + sn \in CS(G, \lambda, l\lambda)$ in which $n = |F_1|$.*

Proof: Let $\{M_1, \ldots, M_m\}$ be a C.D. λ-factorization with support size r of G_1 and let $P = (p_{ij})$ be a (m, λ)-pattern with support size s. Let $K_i = M_i + \sum_{j=1}^m p_{ij} F_j$ for $1 \leq i \leq m$. Then $\{K_1, \ldots, K_m\}$ is a C.D. λ-factorization with support size $r + sn$ of G. □

Now, let G be a simple d-regular graph on $2n$ vertices which is 1-factorable, and let G_1 be a subgraph of G such that both G_1 and $G_2 = G \setminus G_1$ are 1-factorable. Let $k = \deg(G_1)$ and suppose that $2 < 2k \leq d$ and denote $l = d - k = \deg(G_2)$. Let B be any Latin square of order k. It is well-known that B can be embedded in a Latin square A of order d (i.e. $b_{ij} = a_{ij}$, $1 \leq i, j \leq k$) [10]. By assumption on G_1, we can form a 1-factorization $\{F_1, \ldots, F_d\}$ of G such that $\{F_1, \ldots, F_k\}$ is a 1-factorization of G_1. For $1 \leq i, j \leq d$ denote $F_i^j = F_{a_{ij}}$. Clearly, if $1 \leq i \leq k$, then $\{F_i^j | j > k\}$ is a 1-factorization of G_2.

Lemma 2.7. *If $r \in CS(G_1, \lambda, \lambda)$, $s_1 \in CS(G_2, \lambda, \lambda)$, $s_2 \in CS(G_2, \lambda + 1, \lambda + 1)$, $0 \leq t \leq l$, and $1 \leq j \leq \min\{k, \lambda\}$, then*

(i) *$b_1 = r + tdn + jnl \in CS(G, \lambda + t, \lambda + t)$.*

(ii) *if $d = 2k$, then $b_2 = r + tdn + s_1 \in CS(G, \lambda + t, \lambda + t)$.*

(iii) *if $d = 2k + 1$, and $\lambda = k$, then $r + s_1 + t(d-1)n \in CS(G, \lambda + t, \lambda + t)$, and if also $l(kn - 1) < s_1 < lkn$ then $r + s_1 + dn \in CS(G, \lambda + 1, \lambda + 1)$, and $r + s_1 + (2d - 1)n \in CS(G, \lambda + 2, \lambda + 2)$.*

(iv) *If $d = 2k + 1$, then $r + s_2 + l(d - 1)n \in CS(G, \lambda + l, \lambda + l)$.*

Proof: Let $\{K_1, \ldots, K_k\}$ be a C.D. λ-factorization with support size r of λG_1, Let $\{K_{k+1}, \ldots, K_d\}$ be a C.D. λ-factorization with support size s_1 of

λG_2, and let $\{H_i | k < i \le d\}$ be a C.D. $(\lambda+1)$-factorization with support size s_2 of $(\lambda+1)G_2$. and let

$$L_i = K_i, \qquad 1 \le i \le k,$$

$$L_i = \begin{cases} \lambda F_1^i, & \text{if } j = 1, \\ \sum_{p=1}^{j-1} F_p^i + (\lambda - j + 1)F_j^i, & \text{otherwise,} \end{cases} \qquad k+1 \le i \le d,$$

$$N_i = L_i + \sum_{p=k+1}^{k+t} F_p^i, \qquad 1 \le i \le d.$$

It is easy to see that $\mathcal{L} = \{L_1, \ldots, L_n\}$ is a C.D. λ-factorization with support size $r + jnl$ of λG, and for every i, $1 \le i \le n$, L_i and $\sum_{k=m+1}^{n} F_k^i$ are edge-disjoint. Thus $\mathcal{N} = \{N_1, \ldots, N_n\}$ is a C.D. $(\lambda+t)$-factorization with support size b_1 of $(\lambda+l)G$. To prove (ii) in the definition of N_i's simply replace L_i with K_i. Now, let $d = 2k+1$. Then we can assume that $a_{ii} = i$. Trivially if $\{F_i | i = 1, \ldots, d\}$ is a 1-factorization of G such that $\{F_i | i \le k\}$ is a 1-factorization of G_1, then for $k < i$, $1 \le j \le d$, K_j and $F_{a_{ij}}$ are edge-disjoint except possibly for $j = i$. Let

$$N_j^t = K_j + \sum_{i=k+1}^{k+t} F_{a_{ij}}, \text{ for } 1 \le j \le d,$$

then $\mathcal{N}^t = \{N_j^t | j \ge 1\}$ is a C.D. $(\lambda+t)$-factorization of $(\lambda+t)G$. Since $\mathcal{K} = \{K_i | i \ge 1\}$ is C.D., we can form F_i's in such a way that F_i is a subgraph of K_i, and in this case support size of $\mathcal{N}^t$ is $r + s + t(d-1)n$. If $l(kn-1) < s_1 < lkn$, then support size of one of K_i's $(k < i)$, say K_{k+1} is exactly kn, and so it is simple. If we choose F_i's in such a way that $F_{k+1} = G_2 \setminus K_{k+1}$, then support size of $\mathcal{N}^1$ is $r + s_1 + dn$. Now let σ be a permutation on $\{1, \ldots, d\}$ which fixes only $k+1$ and $\sigma(i) \in \{1, \ldots, k\}$ if $1 \le i \le k$. Then $\{N_j^1 + F_{\sigma(a_{(k+1)j})} | j \ge 1\}$ is a $(k+2)$-factorization with support size $r + s_1 + (2d-1)n$ of $(k+2)G$. This completes proof of (iii). (iv) Since $\mathcal{H}$ is C.D., without loss of generality we can suppose that for $k < i$ F_i is a subgraph of H_i. Let

$$M_j = \begin{cases} \sum_{i=k+1}^{d} F_{a_{ij}} + K_j & \text{for } 1 \le j \le k \\ \sum_{i=k+1}^{d} F_{a_{ij}} + (H_j \setminus F_j) & \text{otherwise.} \end{cases}$$

It is an easy exercise to show that $\{M_i | 1 \le i \le d\}$ is a C.D. $(\lambda+l)$-factorization with support size $r + s_2 + l(d-1)n$ of $(\lambda+l)G$ (Notice that $\{H_j \setminus F_j | j > k\}$ is a C.D. λ-factorization of λG_2, and $\{\sum_{i=k+1}^{d} F_{a_{ij}} | 1 \le j \le d\}$ is a C.D. l-factorization of lG). □

To determine $CS(n, \lambda, l\lambda)$ and $C(n, \lambda)$ for small n's the following lemmas which demonstrate relation between intersection of factorizations and support sizes of C.D. factorizations will be of some use.

Lemma 2.8. *Let G be a simple ld-regular graph on n vertices which is l-factorable. If $r \in J(G,l)$, then $nld^2/2 - r \in CS(G,d,ld)$.*

Proof: Let $\mathcal{F} = \{F_1,\dots,F_d\}$ and $\mathcal{G} = \{G_1,\dots,G_d\}$ be two l-factorizations of G having exactly r edges in common. For every $1 \le i \le d$, denote $G'_i = \sum_{j\neq i} G_j$. It is easy to check that $\{G'_1,\dots,G'_d\}$ is a C.D. $l(d-1)$-factorization of $(d-1)G$. Let $H_i = G'_i + F_i$, for $1 \le i \le d$. Now, $\mathcal{H} = \{H_1,\dots,H_d\}$ is a C.D. ld-factorization with support size $nld^2/2-r$ of dG. □

Lemma 2.9. *Let G be a simple ld-regular graph on n vertices which is l-factorable. Then $k \in J(G,l)$ if and only if $lnd - k \in CS(G,2,2l)$.* □

Lemma 2.10. [14] *If $n > 3$, then $J(k_{2n},1) = \{0,\dots,n(2n-1)\} \setminus \{n(2n-1) - i | i = 1,2,3,5\}$.* □

Lemma 2.11. [11] *If $n > 4$, then $J(K_{n,n},1) = \{0,\dots,n^2\} \setminus \{n^2 - i | i = 1,2,3,5\}$.* □

3 Main Results

In this section utilizing methods which are developed in Section 2 we completely determine $C(n,\lambda)$ and $CS(n,\lambda,l\lambda)$. For this reason, firstly we obtain some necessary conditions on the support sizes of a $l\lambda$-factorization of λG, where G is a simple graph such that λG is $l\lambda$-factorable, and then we show that if G is a complete graph, these conditions are also sufficient. For simplicity, we define the following notation. For any ld-regular l-factorable graph on n vertices we define

$$PS(G,\lambda,l\lambda) = \begin{cases} \{m,\dots,M\} \setminus A & \text{if } \lambda \neq d, \\ \{m,\dots,M\} \setminus (A \cup B) & \text{otherwise,} \end{cases}$$

in which $m = ldn/2$, $M = \min\{\lambda,d\}m$, $A = \{m + i | i = 1,2,3,5\}$ and $P = \{M - i | i = 1,2,3,5\}$.

For even λ and odd n, we define

$$PS(K_n,\lambda,\lambda) = \begin{cases} \{l,\dots,M\} & \text{if } \lambda \neq n-1 \\ \{l,\dots,M\} \setminus B & \text{otherwise,} \end{cases}$$

in which $l = (n-1)(n+2)/2$, $M = \min\{\lambda, n-1\}n(n-1)/2$, and $B = \{M - i | i = 1,2,3,5\}$. For odd λ, let $PS(K_n,\lambda,\lambda) = \emptyset$.

In [9], Colbourn and Rosa have proved that $S(n,\lambda,\lambda) = PS(K_n,\lambda,\lambda)$. An argument similar to the proof of necessity of conditions in [9] can be used to prove the following lemmas.

Lemma 3.1. *Let G be a simple ld-regular graph on n vertices which is l-factorable. Then $CS(G,\lambda,l\lambda) \subseteq PS(G,\lambda,l\lambda)$.* □

Lemma 3.2. *For odd n, $CS(K_n, \lambda, \lambda) \subseteq PS(K_n, \lambda, l\lambda)$.* □

In the remainder of this section, we show that in the case of complete graphs except for some small cases we have $CS(G, \lambda, l\lambda) = PS(G, \lambda, l\lambda)$.

Proposition 3.1. *If $n \geq 5$, then $C(n, \lambda) = PS(K_{n,n}, \lambda, \lambda)$. $PS(K_{4,4}, 2, 2) \setminus C(n, 2) = \{22, 25, 27\}$, and for $\lambda \geq 3$. $PS(K_{4,4}, \lambda, \lambda) \setminus C(n, \lambda) = \{22, 25\}$.*

Proof: The main idea of proof is an effective use of Lemma 2.7. Let $G = K_{n,n}$. We must find a subgraph G_1 such that both G_1 and $K_{n,n} \setminus G_1$ are 1-factorable. First of all, it is well known that an 1-factorization of $K_{m,m}$ can be embedded in a 1-factorization of $K_{n,n}$ if and only if $n \geq 2m$. Hence, we can form G_1 in such a way that G_1 is m-regular contains a copy of $K_{m,m}$ as a subgraph. By applying Lemmas 2.3, 2.5, we can find some partial determinations for $CS(G_1, \lambda, \lambda)$, and then applying Lemma 2.7 give a partial determination of $CS(G, \lambda, \lambda)$. In particular, it is easy to see that if assertion is true for $n = m$ it is also true for $n = 2m$, $2m + 1$. Hence we have to prove the assertion only for $4 \leq n \leq 9$. Verification of assertion for $n = 4, 5$ is an easy exercise [2] and so it is left. For $n = 6$ we can choose G_1 as vertex-disjoint union of either three copies of $K_{2,2}$ or two copies of $K_{3,3}$, and then by applying Lemma 2.7 we obtain a partial determination for $C(6, \lambda)$. This partial determination together with Table 1 prove the assertion. For $n = 7$, we can choose G_1 to be vertex disjoint union of either two copies of $K_{2,2}$ and a 2-regular bipartite graph on 6 vertices or a copy of $K_{3,3}$ and a 3-regular bipartite graph on 8 vertices. It is an easy exercise to see that if H is a 3-regular bipartite graph on 8 vertices, then

$$
\begin{aligned}
&CS(H, 1, 1) = \{12\}, \\
&\{12, 16, 19, 20, 22, 24\} \subseteq CS(H, 2, 2), \\
&\{22 - 30, 32, 36\} \subseteq CS(H, 3, 3), \\
&34, 35 \in CS(H, \lambda, \lambda), \text{ for } \lambda > 3.
\end{aligned}
$$

Now, Lemmas 2.5–2.9 give a strong partial determination of $C(7, \lambda)$ and then all we have to prove is $49l - 1 \in C(7, \lambda)$ for $1 \leq l \leq 6$ and it is done in Table 2. For $n = 8, 9$ we can choose G_1 to be vertex-disjoint union of a copy of $K_{4,4}$ and a 4-regular bipartite graph on $2(n - 4)$ vertices, and by applying Lemmas 2.5, 2.7, 2.9 and 2.11 we obtain the result. □

Proposition 3.2. *For given n and l, $n \geq 5$, $CS(K_{ln,ln}, \lambda, l\lambda) = PS(K_{ln,ln}, \lambda, l\lambda)$.*

Proof: $K_{ln,ln}$ is edge-disjoint union of l^2 copies of $K_{n,n}$ which satisfy conditions of Lemma 2.4. Now assertion is an immediate consequence of Lemma 2.4 and Proposition 3.1. □

Proposition 3.3. *For given n and λ, $n \geq 4$, $CS(2n, \lambda, \lambda) = PS(2n, \lambda, \lambda)$.*

Proof: It is well known that a 1-factorization of K_{2m} can be embedded in a 1-factorization of K_{2n} if and only if $n \geq 2m$. Now an argument similar to the proof of Proposition 3.1 shows that to prove the assertion, it suffices to verify it only for $4 \leq n \leq 7$. First of all, note that Lemma 2.3 gives a partial determination for $CS(2n, \lambda, \lambda)$ for all n. For $n = 4, 6$, let $n = 2m$ and choose G_1 to be vertex-disjoint union of two copies of K_{2m} and so $G_2 = \overline{G}_1$ is a copy of $K_{m,m}$. Now applying parts (iii) and (iv) of Lemma 2.7 together with Lemmas 2.8 and 2.10 and Proposition 3.1 give rise to result. For $n = 5$, notice that K_{10} has a 3-factorization $\{F_1, F_2, F_3\}$ in which each F_i is vertex-disjoint union of a copy of K_4 and a copy of $K_{3,3}$. Now if $P = (p_{ij})$ is any $(3, \lambda)$-pattern and we let $G_i = \sum_{j=1}^{3} p_{ij} F_j$, then $\{G_1, G_2, G_3\}$ is a 3λ-factorization of $3\lambda K_{10}$. Applying Lemma 2.4 gives partial determination for $CS(G_i, 1, \lambda)$, and clearly if $\{G_i^j | j = 1, 2, 3\}$ is a λ-factorization of G_i, then $\{G_i^j | 1 \leq i, j \leq 3\}$ is a λ-factorization of λK_{10}. By taking different patterns and factorizations we obtain a strong partial determination of $CS(K_{10}, \lambda, \lambda)$, and in view of it and Lemmas 2.8–2.10, to complete proof we only have to show that $45l - 1 \in CS(K_{10}, l, l)$ for $l \leq 8$ and $404 \in CS(10, 10, 10)$. To prove this, Let G_1 be 4-regular bipartite graph on 10 vertices, and apply Lemma 2.7 (it is an easy exercise to check that $39 \in CS(G_1, 2, 2)$, $59 \in CS(G_1, 3, 3)$ and $79 \in CS(G_1, 5, 5)$) [2]. Finally, for $n = 7$ let G_1 be vertex-disjoint union of a copy of $K_{4,4}$ and a 4-regular graph on 6 vertices and apply Lemma 2.7 together with Lemmas 2.5, 2.8, 2.10 and Proposition 3.1. □

Proposition 3.4. *Let u and l be two positive integers, u is odd and $u > 1$. If $(u, l) \neq (3, 1), (5, 1)$, then $CS(lu + 1, \lambda, l\lambda) = PS(K_{lu+1}, \lambda, l\lambda)$.*

Proof: Clearly for $m \geq 1$ $K_{(m+1)u+1}$ is edge-disjoint union of a copy of K_{u+1} a copy of K_{mu+1} and a copy of $K_{u,(m+1)u}$ and latter is edge-disjoint union of $(m + 1)$ copies of $K_{u,u}$. Applying Lemma 2.4 with this partition of $K_{(m+1)u+1}$ gives an inductive method to determine $CS(lu + 1, \lambda, l\lambda)$ and in particular it shows that if assertion is true for $l = m$ then it is also true for $l = m + 1$. Therefore, we must only prove the assertion for $(u, l) = (3, 2), (5, 2)$ and for $u > 5$, $l = 1$. The latter is proved in Proposition 3.3, for $(u, l) = (3, 2)$ assertion is an easy exercise and for $(u, l) = (5, 2)$ applying Lemma 2.4 ($K_{11} = K_6 + K_6 + K_{5,5}$) together with partial determination of $CS(6, \lambda, \lambda)$ which is obtained from Lemma 2.3 give rise to the result. □

Proposition 3.5. *Let u and l be two positive integers and $u \geq 2$. Then*

$$CS(4lu + 1, \lambda, 2l\lambda) = PS(K_{4lu+1}, \lambda, 2l\lambda).$$

Proof: An argument similar to the proof of Proposition 3.4 shows that we can reduce our problem to the case $l = 1$. Now, let A and B be two disjoint

$2u$-sets, $\infty \notin A \cup B$ and $X = A \cup B \cup \{\infty\}$. Let G be a copy of K_{4u+1} on X, G_1 be a copy of $K_{2u,2u}$ on bipartition (A, B) and let $G_2 = \overline{G}_1$. If $u = 2$, then it is an easy exercise to find a partition $\{F_1, \ldots,, F_4\}$ of G_2 which satisfies conditions of Lemma 2.6. For $u > 2$, let $\mathcal{H} = \{H_1, \ldots, H_{2u-1}\}$ and $\mathcal{G} = \{L_2, \ldots, L_{2u}\}$ be two 1-factorizations of K_{2u} on A and B, respectively, and let H_{2u} and L_1 be two 1-factors of K_{2u} on A and B, such that for every i, $1 \leq i \leq u$, H_{2u} and H_i, as well as L_1 and L_{u+i}, have exactly one edge in common (notice that if $u > 3$, then a Room square of order $2u$, and so a pair of orthogonal 1-factorizations of K_{2u} exists [16]). Denote

$$H_i \cap H_{2u} = \{\{a_i, b_i\}\},\ 1 \leq i \leq u,$$
$$L_1 \cap L_{u+i} = \{\{c_i, d_i\}\},\ 1 \leq i \leq u.$$

For $1 \leq i \leq u$, let

$$F_i = (H_i \cup L_i \cup \{\{\infty, a_i\}, \{\infty, b_i\}\}) \setminus \{\{a_i, b_i\}\},$$
$$F_{u+i} = (H_i \cup L_i \cup \{\{\infty, c_i\}, \{\infty, d_i\}\} \setminus \{\{c_i, d_i\}\}.$$

Now, applying Lemma 2.6 and Proposition 3.1 give rise to the result. □

Proposition 3.6. *Let l and n be two positive integers such that K_{ln+1} is l-factorable. If $n \geq 3$ and $(l, n) \neq (1, 3), (1, 5)$, then*

$$J(K_{ln+1}, l) = \{0, \ldots, ln(ln+1)/2\} \setminus \{ln(ln+1)/2 - i | i = 1, 2, 3, 5\}.$$

□

Proof: Assertion is an immediate consequence of Lemma 2.9 and Propositions 3.3–3.5. □

Now, we deal with the case $G = K_n$ for some odd n and $l = 1$. Clearly, a λ-factorization of λK_n exists if and only if λ is even. Let $\lambda = 2t$. Applying Lemma 2.5 with $G = 2K_n$, $G_1 = G_2 = K_n$ shows that one can essentially reduce problem to the case $t = 1$. It should be noticed that by definition, each 2-factorization of $2K_{2u+1}$ is completely decomposable. Thus we have $CS(2u + 1, 2, 2) = S(2u + 1, 2, 2)$, and consequently it is completely determined in [8]. However, for the sake of completeness and attaining some uniformity in proofs, we give an inductive method to determine $CS(2u + 1, 2, 2)$ but instead we don't prove the result for some small cases and for these we refer to [8].

Let m be an odd integer, $l \in \{0, 1\}$ and $n = m + 3 - 2l$. Let $X_1 = Z_n = \{1, \ldots, n\}$, $X_2 = \{x_1, \ldots, x_m\}$ be an m-set disjoint from X_1, and $X = X_1 \cup X_2$. Let $F_1 = \{\{2i, 2i+1\} | i \in X_1\}$ and $F_2 = \{\{i, i+2 | i \in X_1\}$. Let $G = \overline{F}_1$ if $n = m + 1$ and $G = \overline{(F_1 + F_2)}$ otherwise. (Notice that in both cases G is 1-factorable [17].)

Lemma 3.3. *If $r \in CS(n, 2t, 2t)$, $s \in CS(G, 2t, 2t)$, and $u \in S_p(m-l, 2t)$, then $b = r + s + (u+3)n \in CS(m+n, 2t, 2t)$.*

Proof: Let $\mathcal{H} = \{H_1, \ldots, H_{m-1}\}$ be a C.D. $2t$-factorization with support size r of $2tK_m$ on X_2, and let $\mathcal{K} = \{K_1, \ldots, K_{m-1}\}$ be a C.D. $2t$-factorization with support size s of $2tG$, and for $1 \leq i \leq m_1$ define $L_i = H_i + K_i$. Let $P_1, \ldots, P_{2t}$ be permutation matrices of order $m - l$ such that support size of $P = \sum_{i=1}^{2t} P_i$ is u, and let $\sigma_1, \ldots, \sigma_{2t}$ be the corresponding permutations on $\{1, \ldots, m-l\}$. Let $A = (a_{ij})$ be any latin square of order n on X_1 whose last three rows are

3	n	5	2	...	1	n-2
1	2	3	4	...	n-1	n
2	1	4	3	...	n	n-1

For $1 \leq i \leq n$ define $D_i^0 = \{(a_{(n-2)i}, a_{(n-1),i}, a_{ni})\}$ and $D_i^1 = \{(x_m, a_{(n-1),i}, a_{ni})\}$, and let

$$L_{m-1+i} = \{\{x_j, a_{\sigma_k(j)i}\} | 1 \leq j \leq m-l,\ 1 \leq k \leq 2t\} + tD_i^l.$$

It is an easy exercise to check that $\{L_1, \ldots, L_{m+n-1}\}$ is a $2t$-factorization with support size b of $2tK_{m+n}$ on X. □

Proposition 3.7. *For every odd n, $n \geq 7$,*

$$CS(n, 2t, 2t) = \begin{cases} \{m, \ldots, M\}, & \text{if } t \neq (n-1)/2, \\ \{m, \ldots, M\} \setminus A, & \text{otherwise}, \end{cases}$$

where $m = (n-1)(n+3)/2$, $M = \min\{n-1, 2t\}.n(n-1)/2$, and $A = \{M - i | i = 1, 2, 3, 5\}$.

Proof: Applying Lemma 2.5 with $G = 2K_n$, $G_1 = G_2 = K_n$ and $l = 2$ shows to prove the assertion it suffices to show that (i) $CS(n, 2, 2) = \{(n-1)(n+3)/2, \ldots, n(n-1)\}$ and (ii) $\{n(n-1)+i | i = 1, 2, 3, 5\} \subset CS(n, 4, 4)\}$. By Lemma 3.1 if (i) and (ii) are true for $n = m$ they are also true for $n = 2m+1$, $2m+3$, so we have to prove them only for $n \in \{7, 9, 11, 13\}$, but this is an easy exercise (To see a direct proof which proves (i) as well one can consult [7,8]). □

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Table 1. A partial determination of $C(6,\lambda)$

$$A_1 = \begin{bmatrix} 2 & 1 & 5 & 3 & 4 & 6 \\ 3 & 4 & 2 & 1 & 6 & 5 \\ 1 & 2 & 3 & 6 & 5 & 4 \\ 4 & 3 & 6 & 5 & 1 & 2 \\ 5 & 6 & 1 & 4 & 2 & 3 \\ 6 & 5 & 4 & 2 & 3 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 4 & 5 & 6 & 2 & 3 & 1 \\ 5 & 6 & 4 & 3 & 1 & 2 \\ 6 & 4 & 5 & 1 & 2 & 3 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 6 & 4 & 5 & 2 & 3 & 1 \\ 5 & 6 & 4 & 1 & 2 & 3 \\ 4 & 5 & 6 & 3 & 1 & 2 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 6 & 2 & 4 & 5 \\ 4 & 5 & 2 & 6 & 3 & 1 \\ 5 & 6 & 4 & 3 & 1 & 2 \\ 6 & 4 & 5 & 1 & 2 & 3 \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 1 & 2 & 3 & 5 & 6 & 4 \\ 2 & 3 & 1 & 4 & 5 & 6 \\ 3 & 1 & 6 & 2 & 4 & 5 \\ 4 & 5 & 2 & 6 & 3 & 1 \\ 5 & 6 & 4 & 3 & 1 & 2 \\ 6 & 4 & 5 & 1 & 2 & 3 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 1 & 2 & 5 & 3 & 4 & 6 \\ 2 & 4 & 3 & 1 & 6 & 5 \\ 3 & 1 & 2 & 6 & 5 & 4 \\ 4 & 3 & 6 & 5 & 1 & 2 \\ 5 & 6 & 1 & 4 & 2 & 3 \\ 6 & 5 & 4 & 2 & 3 & 1 \end{bmatrix}.$$

Supp.	Name	λ	Construction
36	$A(36)$	1	A_6
72	$A(72)$	2	$A(36) + (5\,6\,4\,2\,3\,1)A_1$
108	$A(108)$	3	$A(72) + (6\,4\,5\,3\,1\,2)A_1$
144	$A(144)$	4	$A(108) + (4\,5\,6\,1\,2\,3)A_1$
$36k+7$	$A(k,7)$	$k+1$	$A(36k) + A_1$
46	$A(46)$	2	$A_2 + A_5$
$36k+11$	$A(k,11)$	$k+1$	$A(36k) + (1\,3\,2\,4\,5\,6)A_1$
$36k+13$	$A(k,13)$	$k+1$	$A(36k) + (3\,1\,2\,4\,5\,6)A_1$
$36k+14$	$A(k,14)$	$k+1$	$A(36k) + (2\,1\,3\,4\,5\,6)A_1$
$36k+16$	$A(k,16)$	$k+1$	$A(36k) + (2\,3\,1\,4\,5\,6)A_1$
53	$A(53)$	$k+2$	$A_2 + (1\,3\,2\,4\,5\,6)A_5$
$36k+19$	$A(k,19)$	$k+1$	$A(36k) + (1\,2\,3\,4\,6\,5)A_1$
58	$A(58)$	2	$A(36) + A_2$
$36k+23$	$A(k,23)$	$k+1$	$A(36k) + (1\,3\,2\,4\,6\,5)A_1$
$36k+25$	$A(k,25)$	$k+1$	$A(36k) + (3\,1\,2\,4\,6\,5)A_1$
$36k+26$	$A(k,26)$	$k+1$	$A(36k) + (2\,1\,3\,4\,6\,5)A_1$
$36k+29$	$A(k,29)$	$k+1$	$A(36k) + (3\,2\,1\,5\,6\,4)A_1$
$36k+31$	$A(k,31)$	$k+1$	$A(36k) + (3\,1\,2\,5\,6\,4)A_1$

Remark: $1 \le k \le 4$.

Table 1. (continue)

Supp.	Name	λ	Construction
$36k+34$	$A(k,34)$	$k+1$	$A(36k)+(2\,3\,1\,5\,6\,4)A_1$
71	$A(71)$	2	$A_4+(2\,3\,1\,5\,6\,4)A_3$
$36k+37$	$A(k,37)$	$k+2$	$A(k,25)+(3\,1\,2\,6\,5\,4)A_1$
$36k+38$	$A(k,38)$	$k+2$	$A(k,26)+(2\,1\,3\,6\,5\,4)A_1$
$36k+41$	$A(k,41)$	$k+2$	$A(k,25)+(2\,3\,1\,4\,5\,6)A_1$
$36k+46$	$A(k,46)$	$k+2$	$A(k,29)+(2\,3\,1\,6\,5\,4)A_1$
$36k+53$	$A(k,53)$	$k+2$	$A(k,34)+(3\,1\,2\,6\,5\,4)A_1$
$36k+56$	$A(k,56)$	$k+2$	$A(k,31)+(1\,2\,3\,6\,4\,5)A_1$
$36k+58$	$A(k,58)$	$k+2$	$A(k,29)+(2\,3\,1\,6\,4\,5)A_1$
107	$A(107)$	3	$A(71)+(3\,1\,2\,6\,4\,5)A_3$
143	$A(143)$	4	$A(107)+(2\,3\,1\,5\,6\,4)A_2$
179	$A(143)$	5	$A(143)+(3\,1\,2\,6\,4\,5)A_2$
209	$A(209)$	6	$A(4,34)+(3\,1\,2\,6\,4\,5)A_1$
211	$A(211)$	7	$A(209)+(1\,3\,2\,4\,5\,6)A_1$
214	$A(214)$	7	$A(209)+(6\,2\,3\,4\,5\,1)A_1$
215	$A(215)$	7	$A(209)+(2\,3\,1\,4\,5\,6)A_1$

Remark: $1 \leq k \leq 4$.

Table 2. A partial determintation of $C(7,\lambda)$

$$A_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 6 & 4 & 7 & 5 \\ 3 & 1 & 2 & 7 & 6 & 5 & 4 \\ 4 & 5 & 6 & 2 & 7 & 3 & 1 \\ 5 & 6 & 7 & 1 & 2 & 4 & 3 \\ 6 & 7 & 4 & 5 & 3 & 1 & 2 \\ 7 & 4 & 5 & 3 & 1 & 2 & 6 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 7 & 6 & 5 & 4 \\ 3 & 1 & 2 & 6 & 4 & 7 & 5 \\ 4 & 5 & 6 & 2 & 7 & 3 & 1 \\ 5 & 6 & 7 & 1 & 2 & 4 & 3 \\ 6 & 7 & 4 & 5 & 3 & 1 & 2 \\ 7 & 4 & 5 & 3 & 1 & 2 & 6 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 7 & 6 & 5 & 4 \\ 3 & 1 & 6 & 2 & 4 & 7 & 5 \\ 4 & 5 & 2 & 6 & 7 & 3 & 1 \\ 5 & 6 & 7 & 1 & 2 & 4 & 3 \\ 6 & 7 & 4 & 5 & 3 & 1 & 2 \\ 7 & 4 & 5 & 3 & 1 & 2 & 6 \end{bmatrix}.$$

Table 2. (continue)

Supp.	Name	λ	Construction
97	A_{97}	2	$A_3 + (5\,7\,6\,2\,4\,1\,3)A_1$
146	A_{146}	3	$A_{97} + (7\,4\,5\,6\,1\,3\,2)A_1$
196	A_{195}	4	$A_{146} + (4\,6\,7\,1\,3\,2\,5)A_1$
244	$A_{244^\bullet}$	5	$A_{195} + (2\,3\,1\,5\,7\,4\,6)A_2$
293	A_{293}	6	$A_{244} + (3\,1\,2\,7\,6\,5\,4)A_2$

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