Finite Unitary Geometries and Block Designs

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ABSTRACT. Taking as blocks some subspace pairs in a finite unitary geometry we construct a number of new BIB designs and PBIB designs, and also give their parameters.

1 Introduction

Let q be a prime power, and F_{q^2} finite field with q^2 elements. We denote by $UV_n(F_{q^2})$ the n-dimensional unitary geometry over F_{q^2}, and by $U_n(F_{q^2})$ the finite unitary group of degree n over F_{q^2}.

Wan et al. [1] have given the formula for the number $N(m, s; n)$ of (m, s)-type subspaces in $UV_n(F_{q^2})$, the formula for the number $N(m_1, s_1; m, s; n)$ of (m_1, s_1)-type subspaces included in a given (m, s)-type subspace, and some theorems on the transitivity of $U_n(F_{q^2})$. Based on these results, they constructed a number of BIB designs and PBIB designs with parameters computed. The authors of the present paper have investigated the conjugation relation among subspaces in $UV_n(F_{q^2})$, and proved that the conjugation P^* of an (m, s)-type subspace P in $UV_n(F_{q^2})$ is an $(n-m, n+s-2m)$-type subspace. Using these results and choosing some subspace pairs as blocks, we will construct some new BIB designs and PBIB designs with two or more associate classes.

The following known results will be needed.

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Lemma 1. *Let P_1 and P_2 be two m-dimensional subspaces of $UV_n(F_{q^2})$, Then there is a $T \in U_n(F_{q^2})$ such that $P_1 = AP_2T$, where A is an $m \times m$ nonsingular matrix over F_{q^2}, if and only if P_1 and P_2 are of the same type. In other words, $U_n(F_{q^2})$ acts transitively on each set of subspaces of the same type. (See [1], Theorem 5.8).*

Lemma 2. *Let P_1 and P_2 be two $m \times n$ matrices of rank m. Then there exists an element $T \in U_n(F_{q^2})$ such that $P_1 = P_2T$ if and only if $P_1\overline{P_1'} = P_2\overline{P_2'}$. (See [1], Theorem 5.12).*

Lemma 3. *Let $n = 2\nu + \delta$, $\nu \geq 2$ and $\delta = 0$ or 1. Take as treatments the $(\nu, 0)$-type subspaces of the $(2\nu + \delta)$-dimensional unitary geometry $UV_n(F_{q^2})$, and define two treatments to be the ith associates of each other if their join is a $(\nu + i)$-dimensional subspace $(i = 1, 2, \ldots, \nu)$. Then one obtains an association scheme with ν associate classes. Its parameters are*

$$v = \frac{\prod_{\ell=\delta+1}^{2\nu+\delta}(q^\ell - (-1)^\ell)}{\prod_{\ell=1}^{\nu}(q^{2\ell} - 1)}$$

$$n_i = \frac{\prod_{\ell=\nu-i+1}^{\nu}(q^{2\ell} - 1)}{\prod_{\ell=1}^{i}(q^{2\ell} - 1)} q^{i^2+2i\delta}$$

$$p_{jk}^i = \sum \frac{\prod_{\ell=i-k+\rho+1}^{i}(q^{2\ell} - 1)\prod_{\ell=\rho+1}^{\nu-i}(q^{2\ell} - 1)}{\prod_{\ell=1}^{\nu-i-\rho}(q^{2\ell} - 1)\prod_{\ell=1}^{i-j-\rho}(q^{2\ell} - 1)\prod_{\ell=1}^{j+k-i-2\rho-\sigma-\sigma_1}(q^\ell - (-1)^\ell)} \cdot q^\omega$$

Where $\omega = \rho^2 + 2\rho(i + \delta) + \frac{1}{2}(j + k - i - 2\rho - \sigma - \sigma_1)\ (j + k - i - 2\rho - \sigma - \sigma_1 - 1) + \sigma[2(j + k - i - 2\rho - \sigma - \sigma_1) + 1]$ and the range of the sum is $\max(0, j - i, k - i) \leq \rho \leq \min(v - i, \left\lceil \frac{j+k-i}{2} \right\rceil)$, $0 \leq \sigma_1 \leq \sigma \leq \delta$. (See [2], Theorem 12 in Chapter 8).

Lemma 4. *Let $n \geq 4$. Take the (1,0)-type subspaces as treatments, and define two treatments to be the first (resp. second) associates of each other if they as subspaces are orthogonal (resp. nonorthogonal). Then one obtains an association scheme with parameters*

$$v = [q^n - (-1)^n][q^{n-1} - (-1)^{n-1}]/(q^2 - 1),$$
$$n_1 = (q^{n-2} - (-1)^{n-2})(q^{n-3} - (-1)^{n-3})q^2/(q^2 - 1),$$
$$p_{11}^1 = q^2 - 1 + (q^{n-4} - (-1)^{n-4})(q^{n-5} - (-1)^{n-5})q^4/(q^2 - 1),$$
$$p_{11}^2 = (q^{n-2} - (-1)^{n-2})(q^{n-3} - (-1)^{n-3})/(q^2 - 1).$$

(See [3]).

2 Construction of BIB designs

Throughout this section we conduct our discussion in $UV_3(F_{q^2})$, and will construct some BIB designs by taking the (1,0)-type subspaces in $UV_3(F_{q^2})$ as treatments and taking some subspace pairs as blocks.

We first prove

Theorem 1. *Let treatments be the (1,0)-type-subspaces in $UV_3(F_{q^2})$, and let the blocks be the (1,1)-type subspace pairs each consisting of two orthogonal subspaces, and define a treatment V to be arranged in a block (W_1, W_2) if $V \perp W_1$, and $V \not\perp W_2$. Then we obtain a BIB design with parameters*

$$v = q^3 + 1, \quad b = (q^2 - q + 1)(q - 1)q^3, \quad r = (q - 1)q^3$$
$$k = q + 1, \quad \lambda = (q - 1)q.$$

Proof: We know that the unitary group $U_3(F_{q^2})$ acts transitively on the set of (1,0)-type subspaces (see Lemma 1). According to Lemma 2, it is easy to see that $U_3(F_{q^2})$ acts transitively on the set of ordered (1,0)-type subspace pairs (note that two (1,0)-type subspaces must be nonorthogonal), and also acts transitively on the set of ordered (1,1)-type subspaces pairs each consisting of two orthogonal subspaces. Thus we certainly obtain a BIB design. We now compute its parameters.

Clearly, the number of treatments is

$$v = N(1, 0; 3) = q^3 + 1.$$

Let (W_1, W_2) be a block. Then W_1 has $N(1, 1; 3)$ choices. For each chosen W_1, the (1,1)-ye subspaces which are orthogonal to W_1 are those (1,1)-type subspaces which are included in W_1^*. On the other hand,the conjugate subspace of a (1,1)-type subspacei is a (2,2)-type subspace. Therefore

$$b = N(1, 1; 3)N(1, 1; 2, 2; 3) = \frac{(q^3 + 1)q^2}{q + 1} \cdot (q^2 + 1 - (q + 1))$$
$$= (q^2 - q + 1)(q - 1)q^3.$$

Let V be a given treatment. For the blocks (W_1, W_2) in which V is arranged, the number of ways of choosing W_2 is the number of (1,1)-type subspaces that are nonorthogonal to V. As the conjugation of a (1,0)-type subspace in $UV_3(F_{q^2})$ is a (2,1)-type subspace, this number is

$$N(1, 1; 3) - N(1, 1; 2, 1; 3) = (q^2 - q + 1)q^2 - (q^2 + 1 - 1) = (q - 1)q^3.$$

For every such W_2, the number of ways of choosing W_1 is the number of the (1,1)-type subspaces that are orthogonal to V as well as to W_2. As

$V \cup W_2$ is a (2,2)-type subspace and $(V \cup W_2)^*$ is a (1,1)-type subspace, this number is 1. Thus

$$r = (q-1)q^3.$$

By the basic relations $bk = rv$ and $r(k-1) = \lambda(v-1)$, we easily obtain

$$k = \frac{rv}{b} = q+1,$$
$$\lambda = \frac{r(k-1)}{v-1} = (q-1)\cdot q$$

This comletes the proof.

Theorem 2. *Take the (1,0)-type subspaces in $UV_3(F_{q^2})$ as treatments. Take as blocks the elements of*

$$\mathcal{B} = \{(W_1, W_2) \mid W_1 \text{a (1,1)-type subspace, } W_2 \text{ a (1,0)-type subspace, and } W_1 \perp W_2\},$$

and define a treatment V to be arranged in a block (W_1, W_2) if $V \perp W_1$ and $V \not\perp W_2$. Then we obtain a BIB design with parameters

$$v = q^3+1,\ b = (q^3+1)q^2,\ r = q^3,\ k = q,\ \lambda = q-1.$$

Proof: By Theorem 1, the unitary group $U_3(F_{q^2})$ acts transitively on the set of treatments, and transitively on the set of treatment-pairs. According to Lemma 2, $U_3(F_{q^2})$ also acts transitively on $\mathcal{B}$. Thus we certainly obtain a BIB design. Its parameters can be computed as follows:

$$v = N(1,0;3) = q^3+1$$
$$b = N(1,0;3)N(1,1;2,1:3) = (q^3+1)q^2$$
$$r = [N(1,0;3) - N(1,0;2,1;3)]N(1,1;1,1;3) = q^3$$
$$k = \frac{rv}{b} = q$$
$$\lambda = \frac{r(k-1)}{v-1} = q-1.$$

This completes the proof.

3 PBIB designs with more than two associate classes

Theorem 3. *Let $n = 2v+\delta$, $v \geq 2$, and $\delta = 0$ or 1. Adopt the association scheme in Lemma 3. Take as blocks the subspace pairs (W_1, W_2) where W_1 is an (m,s)-type subspace ($s = 0$ or 1) and W_2 is a $(1,0)$-type subspace of $UV_n(F_{q^2})$ and $W_2 \subseteq W_1^* \backslash W_1$, and define a treatment V to be arranged in* *a block* (W_1, W_2) *if* $V \perp W_1$ *and* $V \not\perp W_2$. *Then we obtain a PBIB design with* ν *associate classes. Its parameters besides those of the association scheme in Lemma 3 are*

$$\begin{aligned} b &= N(m,s;n)[N(,1,0;n-m,n+s-2m;n) - N(1,m-s)], \\ r &= [N(1,0;n) - N(1,0;\nu+\delta,\delta;n)]N(m,s;\nu-1+\delta,\delta;n), \\ \lambda_i &= N(1,0;\nu+i,2i;n) - 2N(1,\nu) + N(1,\nu-i)] \\ &\quad \cdot N(m,s;\nu-i+\delta,\delta;n) \\ &\quad + [N(1,0;n) - N(1,0;\nu+i+\delta,2i+\delta;n)] \cdot N(m,s;\nu-i-1+\delta,\delta;n) \\ &\quad + [N(1,0;\nu+i+\delta,2i+\delta;n) - N(1,0;\nu+i,2i;n)] \\ &\quad \cdot N(m,s;\nu-i-1+\delta,\delta-1;n). \end{aligned}$$

Proof: Let (W_1, W_2) be a block. Then $\binom{W_1}{W_2}$ is an $(m+1)\times n$ matrix with rank $m+1$. Without loss of generally, we may assume that

$$W_1\overline{W_1'} = \begin{pmatrix} I^{(s)} & 0 \\ 0 & 0^{(m-s)} \end{pmatrix}$$

Then

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \overline{\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}}' = \begin{matrix} \\ \text{s} \\ \text{m-s} \\ 1 \end{matrix} \begin{matrix} \begin{matrix} \text{s} & \text{m-s} & 1 \end{matrix} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

Thus, According to Lemma 2, the unitary group $U_n(F_{q^2})$ acts transitively on the set of blocks. Therefore, we certainly obtain a PBIB design with ν associate classes. We now compute its parameters.

Let (W_1, W_2) be a block, Then there are $N(m,s;n)$ choices for W_1, and for each chosen W_1, the number of ways of choosing W_2 is the number of (1,0)-type subspaces in $W_1^*\backslash W_1$. Since the conjugation of the (m,s)-type subspace W_1 is an $(n-m, n+s-2m)$-type subspace, and $W_1 \cap W_1^*$ is an $(m-s, 0)$-type subspace (cf. Theorem 2 in [4]), W_2 has

$$N(1,0;n-m,n+s-2m;n) - N(1,m-s)$$

choices. Therefore,

$$b = (m,s;n)[N(1,0;n-m,n+s-2m;n) - N(1,m-s)].$$

Let V be a given treatment arranged in a block (W_1, W_2). Since V^* is a $(\nu+\delta,\delta)$-type subspace, it follows that W_2 has

$$N(1,0;n) - N(1,0;\nu+\delta,\delta;n)$$

choices. For each chosen W_2, the number of ways of choosing W_1 is the number of (m, s)-type subspace which are orthogonal to both V and W_2. Since $V \cup W_2$ is a $(\nu+1, 2)$-type subspace and $(V \cup W_2)^*$ is a $(\nu-1+\delta, \delta)$-type subspace, W_1 has

$$N(m, s; \nu - 1 + \delta, \delta; n)$$

choices, Then

$$r = [N(1,0;n) - N(1,0;\nu + \delta, \delta; n)] \cdot N(m, s; \nu - 1 + \delta, \delta; n)$$

Let V_1 and V_2 be two treatments which are the ith associates of each other, and let them be arranged in a block (W_1, W_2). There are two cases to be considered.

Case 1. $W_2 \subseteq V_1 \cup V_2$.

Since $V_1 \cup V_2$ is a $(\nu + i, 2i)$-type subspace, there are $N(1, 0; \nu + i, 2i; n)$ (1,0)-type subspace in $V_1 \cup V_2$ among which those included in V_1 are orthogonal to V_1, while those included in V_2 are orthogonal to V_2; none of them can be taken as W_2. We claim that the (1,0)-type subspace α in $V_1 \cup V_2$ that are included neither in V_1 nor in V_2 must be orthogonal neither to V_1 nor V_2. In fact, let $\alpha = \beta_1 + \beta_2$, $\beta_1 \subseteq V_1 \backslash V_2$, $\beta_2 \subseteq V_2 \backslash V_1$. If $\alpha \perp V_1$, then $(\beta_1 + \beta_2)\overline{\gamma}' = 0$ for all $\gamma \subseteq V_1$. Note that $\beta_1 \overline{\gamma}' = 0$. Then $\beta_2 \overline{\gamma}' = 0$ for all $\gamma \subseteq V_1$. Hence

$$\begin{pmatrix} V_1 \\ \beta_2 \end{pmatrix}$$

is a $(\nu + 1, 0)$-type subspace which is impossible. Similarly, $\alpha \perp V_2$ is also impossible. Therefore, such α can be taken as W_2. By the inclusion-exclusion principle, such α has

$$N(1, 0; \nu + i, 2i; n) - 2N(1, \nu) + N(1, \nu - i)$$

choices. For each chosen W_2, the number of ways of choosing W_1 is the number of (m, s)-type subspace that are orthogonal to the $(\nu + i, 2i)$-type subspace $V_1 \cup V_2$. Since the conjugation of.a $(\nu + i, 2i)$-type subspace is a $(\nu - i + \delta, \delta)$-type subspace, it follows that W_1 has

$$N(m, s; \nu - i + \delta, \delta; n)$$

choices. In this case, therefore, such a block has

$$[N(1, 0; \nu + i, 2i; n) - 2N(1, \nu) + N(1, \nu - i)] \cdot N(m, s; \nu - i + \delta, \delta; n)$$

choices.

Case 2. $W_2 \not\subseteq V_1 \cup V_2$.

We claim that a (1,0)-type subspace α that is not included in $V_1 \cup V_2$ cannot be orthogonal to V_1. Otherwise there would be a $(\nu+1, 0)$-type subspace $\binom{V_1}{\alpha}$ which is impossible. Similarly, such an α must not be orthogonal to V_2. Thus α can be taken as W_2. For a chosen W_2, the number of ways of choosing W_1 is the number of (m, s)-type subspace that are orthogonal to $V_1 \cup V_2 \cup W_2$. To calculate this number we need to know of which type the $(\nu + i + 1)$-dimensional subspace $V_1 \cup V_2 \cup W_2$ should be. We further consider two subcases;

(1) $W_2 \not\subseteq (V_1 \cap V_2)^*$

We claim that in this case $V_1 \cup V_2 \cup W_2$ must be a $(\nu+i+1, 2i+2)$-type subspace. In fact write $D = V_1 \cap V_2$, $V_1 = \binom{D_1}{D}$, $V_2 = \binom{D}{D_2}$, and

$$\begin{pmatrix} D_1 \\ D \\ D_2 \\ W_2 \end{pmatrix} \overline{\begin{pmatrix} D_1 \\ D \\ D_2 \\ W_2 \end{pmatrix}}' = \begin{matrix} \\ i \\ \nu - i \\ i \\ 1 \end{matrix} \begin{matrix} \begin{matrix} i & \nu - i & i & 1 \end{matrix} \\ \begin{pmatrix} 0 & 0 & \overline{A}' & \overline{B_1}' \\ 0 & 0 & 0 & \overline{B_2}' \\ A & 0 & 0 & \overline{B_3}' \\ B_1 & B_2 & B_3 & 0 \end{pmatrix} \end{matrix} \tag{1}$$

For W_2 is not orthogonal to $D = V_1 \cap V_2$, we have $B_2 \neq 0$. Thus the rank of the matrix on the right hand side of (1) is $2i + 2$. Since $(V_1 \cap V_2)^*$ is a $(\nu + i + \delta, 2i + \delta)$-type subspace, the (1,0)-type subspace W_2 has

$$N(1, 0; n) - N(1, 0; \nu + i + \delta, 2i + \delta; n)$$

choices. For a chosen W_2, since $V_1 \cup V_2 \cup W_2$ is a $(\nu + i + 1, 2i + 2)$-type subspace and its conjugation is a $(\nu - i - 1 + \delta, \delta)$-type subspace, W_1 has

$$N(m, s; \nu - i - 1 + \delta, \delta; n)$$

choices. Therefore, the number of blocks in this case is

$$[N(1, 0; n) - N(1, 0; \nu + i + \delta, 2i + \delta; n)] \cdot N(m, s; \nu - i - 1 + \delta, \delta; n)$$

(2) $W_2 \subseteq (V_1 \cap V_2)^* \backslash (V_1 \cup V_2)$.

We will claim that in this case $V_1 \cup V_2 \cup W_2$ must be a $(\nu+i+1, 2i+1)$-type subspace. If we still adopt the symbols in the subcase (1), then $B_2 = 0$, $B_1 \neq 0$ and $B_3 \neq 0$. Noting that A is an invertible matrix of order i, we can write $\widetilde{W_2} = W_2 - B_1 \cdot A^{-1} \cdot D_2 - B_3 \cdot (\overline{A}')^{-1} \cdot D_1$. Then

$$\begin{pmatrix} D_1 \\ D \\ D_2 \\ \widetilde{W}_2 \end{pmatrix} \overline{\begin{pmatrix} D_1 \\ D \\ D_2 \\ \widetilde{W}_2 \end{pmatrix}}' = \begin{pmatrix} I^{(i)} & 0 & 0 & 0 \\ 0 & I^{(\nu-i)} & 0 & 0 \\ 0 & 0 & I^{(i)} & 0 \\ -B_3(\overline{A}')^{-1} & 0 & -B_1A^{-1} & 1 \end{pmatrix} \begin{pmatrix} D_1 \\ D \\ D_2 \\ W_2 \end{pmatrix} \overline{\begin{pmatrix} D)1 \\ D \\ D_2 \\ W_2 \end{pmatrix}}'$$

$$\cdot \overline{\begin{pmatrix} I^{(i)} & 0 & 0 & 0 \\ 0 & I^{(\nu-i)} & 0 & 0 \\ 0 & 0 & I^{(i)} & 0 \\ -B_3(\overline{A}')^{-1} & 0 & -B_1A^{-1} & 1 \end{pmatrix}}'$$

$$= \begin{pmatrix} I^{(i)} & 0 & 0 & 0 \\ 0 & I^{(\nu-i)} & 0 & 0 \\ 0 & 0 & I^{(i)} & 0 \\ -B_3(\overline{A}')^{-1} & 0 & -B_1A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \overline{A}' & \overline{B_1}' \\ 0 & 0 & 0 & 0 \\ A & 0 & 0 & \overline{B_3}' \\ B_1 & 0 & B_3 & 0 \end{pmatrix}$$

$$\cdot \overline{\begin{pmatrix} I^{(i)} & 0 & 0 & 0 \\ 0 & I^{(\nu-i)} & 0 & 0 \\ 0 & 0 & I^{(i)} & 0 \\ -B_3(\overline{A}')^{-1} & 0 & -B_1A^{-1} & 1 \end{pmatrix}}'$$

$$= \begin{pmatrix} 0 & 0 & \overline{A}' & 0 \\ 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{W}_2\overline{\widetilde{W}_2}' \end{pmatrix} \tag{2}$$

Since $\widetilde{W}_2$ is not included in $V_1 \cup V_2$, it follows that $\widetilde{W}_2 \cdot \overline{\widetilde{W}_2}' \neq 0$. Otherwise $\binom{V_2}{\widetilde{W}_2}$ is a $(\nu+1,0)$-type subspace by (2). But this is impossible. Since $\widetilde{W}_2 \cdot \overline{\widetilde{W}_2}' \neq 0$, (2) implies that the rank of

$$\begin{pmatrix} D_1 \\ D \\ D_2 \\ W_2 \end{pmatrix} \overline{\begin{pmatrix} D_1 \\ D \\ D_2 \\ W_2 \end{pmatrix}}'$$

is $2i+1$. Then $V_1 \cup V_2 \cup W_2$ is a $(\nu+i+1, 2i+1)$-type subspace. We know that $(V_1 \cap V_2)^*$ is a $(\nu+i+\delta, 2i+\delta)$-type subspace, $V_1 \cup V_2$ is a $(\nu+i, 2i)$-type subspace, and

$$(V_1 \cap V_2)^* = V_1^* \cup V_2^* \supseteq V_1 \cup V_2.$$

Then the (1,0)-type subspaces W_2 that are included in $(V_1 \cap V_2)^* \backslash (V_1 \cup V_2)$ have

$$N(1,0;\nu+i+\delta, 2i+\delta; n) - N(1,0;\nu+i, 2i; n)$$

choices. For a chosen W_2, since $V_1 \cup V_2 \cup W_2$ is a $(\nu+i+1, 2i+1)$-type subspace and its conjugation is a $(\nu-i-1+\delta, \delta-1)$-type subspace, W_1 has

$$N(m, s; \nu-i-1+\delta, \delta-1; n)$$

choices. Therefore, the number of blocks in this case is

$$[N(1,0;\nu+i+\delta, 2i+\delta; n) - N(1,0;\nu+i, 2i; n)] \cdot N(m, s; \nu-i-1+\delta, \delta-1; n)$$

Combining these cases, we have

$$\begin{aligned}\lambda_1 = {} & [N(1,0;\nu+i,2i;n) - 2N(1,\nu) + N(1,\nu-i)] \\ & \cdot N(m,s;\nu-i+\delta,\delta;n) \\ & + [N(1,0;n) - N(1,0;\nu+i+\delta,2i+\delta;n)] \\ & \cdot N(m,s;\nu-i-1+\delta,\delta;n) \\ & + [N(1,0;\nu+i+\delta,2i+\delta;n) - N(1,0;\nu+i,2i;n)] \\ & \cdot N(m,s;\nu-i-1+\delta,\delta-1;n)\end{aligned}$$

This completes the proof.

Theorem 4. *Let $n = 2\nu+\delta$, $\nu \geq 2$, and $\delta = 0$ or 1. Adopt the association scheme in Lemma 3. Take as blocks the subspace pairs (W_1, W_2) where W_1 is an (m,s)-type subspace ($s = 0$ or 1) and W_2 is a (1,1)-type subspace of $UV_n(F_{q^2})$ and $W_2 \perp W_1$. Define a treatment V to be arranged in a block (W_1, W_2) if $V \perp W_1$ and $V \not\perp W_2$. Then we obtain a PBIB design with ν associate classes. Its parameters besides those of the association scheme in Lemma 3 are*

$$\begin{aligned} b &= N(m,s;n) \cdot N(1,1;n-m,n+s-2m;n), \\ r &= [N(1,1;n) - N(1,1;\nu+\delta,\delta;n)] \cdot N(m,s;\nu-1+\delta,\delta;n), \\ \lambda_i &= [N(1,1;\nu+i,2i;n) \cdot N(m,s;\nu-i+\delta,\delta;n) \\ &\quad + [N(1,1;n) - N(1,1;\nu+i+\delta,2i+\delta;n)] \cdot N(m,s;\nu-i-1+\delta,\delta;n) \\ &\quad + [N(1,1;\nu+i+\delta,2i+\delta;n) - 2N(1,1;\nu+\delta,\delta;n) \\ &\quad + N(1,1;\nu-i+\delta,\delta;n) - N(1,1;\nu+i,2i;n)] \\ &\quad \cdot N(m,s;\nu-i-1+\delta,\delta-1;n). \end{aligned}$$

Proof: We only need to compute λ_i because the proof is similar to that of Theorem 3.

Let V_1 and V_2 be two treatments, the ith associates of each other, and be arranged in a block (W_1, W_2). There are three cases to be considered.

Case 1. $W_2 \subseteq V_1 \cup V_2$.

The (1,1)-type subspace in $V_1 \cup V_2$ are neither included in V_1 nor included in V_2. Thus an argument similar to that in the proof of Theorem 3 gives that the (1,1)-type subspaces in $V_1 \cup V_2$ are orthogonal neither to V_1 nor to V_2, and then all of them can be taken as W_2. Hence in this case W_2 has

$$N(1,1;\nu+i,2i;n)$$

choices. For a chosen W_2, the number of ways of choosing W_1 is the number of (m,s)-type subspace in $(V_1 \cup V_2)^*$, i.e. $N(m,s;\nu-i+\delta,\delta;n)$. Therefore, the number of blocks in this case is

$$N(1,1;\nu+i,2i;n)\cdot N(m,s;\nu-i+\delta,\delta;n).$$

Case 2. $W_2 \not\subseteq V_1^* \cup V_2^*$.

Because the (1,1)-type subspaces not in $V_1^* \cup V_2^*$ are orthogonal neither to V_1 nor to V_2, they can be taken as W_2. Since $V_1^* \cup V_2^* = (V_1 \cap V_2)^*$ is a $(\nu+i+\delta, 2i+\delta)$-type subspace, W_2 has

$$N(1,1;n) - N(1,1;\nu+i+\delta,2i+\delta;n)$$

choices. For a chosen W_2, the number of ways of choosing W_1 is the number of (m,s)-type subspaces that are orthogonal to all of V_1, V_2 and W_2. An argument similar to that in the proof of Theorem 3 gives that $V_1 \cup V_2 \cup W_2$ is a $(\nu+i+1, 2i+2)$-type subspace and its conjugation is a $(\nu-i-1+\delta,\delta)$-type subspace, so W_1 has $N(m,s;\nu-i-1+\delta,\delta;n)$ choices. Therefore the number of blocks in this case is

$$[N(1,1;n) - N(1,1;\nu+i+\delta,2i+\delta;n)]\cdot N(m,s;\nu-i-1+\delta,\delta;n)$$

Case 3. $W_2 \subseteq (V_1 \cap V_2)^* \backslash (V_1 \cup V_2)$.

We know that $(V_1 \cap V_2)^*$ includes $N(1,1;\nu+i+\delta,2i+\delta;n)$ (1,1)-type subspaces. Among them there are $N(1,1;\nu+\delta,\delta;n)$ (1,1)-type subspaces in V_1^* and these subspaces are orthogonal to V_1, and there are $N(1,1;\nu+\delta,\delta;n)$ (1,1)-type subspaces in V_2^* and these subspaces are orthogonal to V_2. All these subspaces can not be taken as W_2. Noting that $V_1^* \cap V_2^* = (V_1 \cup V_2)^*$ is a $(\nu-i+\delta,\delta;n)$-type subspace, we obtain by the inclusion-exclusion principle the number of (1,1)-type subspaces that are orthogonal neither to V_1 nor to V_2 to be

$$N(1,1;\nu+i+\delta,2i+\delta;n) - 2N(1,1;\nu+\delta,\delta;n) + N(1,1;\nu-i+\delta,\delta;n).$$

Since $(V_1 \cap V_2)^* = V_1^* \cup V_2^* \supseteq V_1 \cup V_2$ and the (1,1)-type subspaces in $V_1 \cup V_2$ are orthogonal neither to V_1 nor to V_2, the number of (1,1)-type subspaces in $(V_1 \cap V_2)^* \backslash (V_1 \cup V_2)$ that are orthogonal neither to V_1 nor to V_2 is

$$\begin{aligned} &N(1,1;\nu+i+\delta,2i+\delta;n) - 2N(1,1;\nu+\delta,\delta;n) \\ &\quad + N(1,1;\nu-i+\delta,\delta;n) - N(1,1;\nu+i,2i;n). \end{aligned}$$

For a chosen W_2, an argument similar to that in the proof of Theorem 3 gives that $V_1 \cup V_2 \cup W_2$ is a $(\nu + i + 1, 2i + 1)$-type subspace and its conjugation is a $(\nu - i - 1 + \delta, \delta - 1)$-type subspace, so W_1 has

$$N(m, s; \nu - i - 1 + \delta, \delta - 1; n)$$

choices. Therefore, the number of blocks in this case is

$$[N(1,1;\nu+i+\delta, 2i+\delta; n) - 2N(1,1;\nu+\delta,\delta;n) + N(1,1;\nu-i+\delta,\delta;n) \\ - N(1,1;\nu+i,2i;n)] \cdot N(m,s;\nu-i-1+\delta,\delta-1;n).$$

Combining all the three cases, we have

$$\begin{aligned}\lambda_i = {} & [N(1,1;\nu+i,2i;n) \cdot N(m,s;\nu-i+\delta,\delta;n) \\ & + N(1,1;n) - N(1,1;\nu+i+\delta,2i+\delta;n)] \cdot N(m,s;\nu-i-1+\delta,\delta;n) \\ & + [N(1,1;\nu+i+\delta,2i+\delta;n) - 2N(1,1;\nu+\delta,\delta;n) \\ & + N(1,1;\nu-i+\delta,\delta;n) - N(1,1;\nu+i,2i;n)] \\ & \cdot N(m,s;\nu-i-1+\delta,\delta-1;n).\end{aligned}$$

This proves the theorem.

4 PBIB designs with two associate classes

In this section we will construct PBIB designs with two associate classes by using the association scheme in Lemma 4 and also by specializing Theorems 3 and 4.

Theorem 5. *Adopt the association scheme in Lemma 4. Take as blocks the subspace pairs (W_1, W_2) of $UV_n(F_{q^2})$ where W_1 is an (m,s)-type subspace, W_2 is a (1,0)-type subspace and $W_2 \subseteq W_1^* \backslash W_1$. Define a treatment V to be arranged in a block (W_1, W_2) if $V \perp W_1$ and $V \not\perp W_2$. Then we obtain a PBIB design with two associate classes and with the parameters given in Lemma 4 and by the following:*

$$\begin{aligned} b &= N(m,s;n) \cdot [N(1,0;n-m,n+s-2m;n) - N(1,m-s)]. \\ r &= n_2 \cdot N(m,s;n-2,n-2;n), \\ k &= \frac{rv}{b} \\ \lambda_1 &= P^1_{22} \cdot N(m,s;n-3,n-4;n), \\ \lambda_2 &= \frac{r(k-1) - \lambda_1 n_1}{n_2}. \end{aligned}$$

Proof: An argument similar to the proof of Theorem 3 shows that the structure so obtained is a PBIB design.

The parameter b has the same value as in Theorem 3.

Let V be a given treatment and be arranged in a block (W_1, W_2). Then there are n_2 choices for W_2, and for each chosen W_2, the number of ways of choosing W_1 is the number of (m, s)-type subspaces that are orthogonal to both V and W_2. Note that such an (m, s)-type subspace must not include W_2 for W_2 is not orthogonal to V. Since $V \cup W_2$ is a (2,2)-type subspace and its conjugation is an $(n-2, n-2)$-type subspace, it follows that W_1 has $N(m, s; n-2, n-2; n)$ choices and then

$$r = n_2 \cdot N(m, s; n-2, n-2; n).$$

Let V_1 and V_2 be a pair of orthogonal (1,0)-type subspaces, and be arranged in a block (W_1, W_2). Then W_2 has P_{22}^1 choices. Noting that $V_1 \cup V_2 \cup W_2$ is a (3,2)-type subspace whose conjugation is an $(n-3, n-4)$-type subspace, it follows that W_1 has $N(m, s; n-3, n-4; n)$ choices, and then

$$\lambda_1 = P_{22}^1 \cdot N(m, s; n-3, n-4; n).$$

The other parameters can be computed from the basic parameter relations. This proves the Theorem.

Similarly, we have

Theorem 6. *Adopt the association scheme in Lemma 4. Take as blocks the subspace pairs (W_1, W_2) where W_1 is an (m, s)-type subspace and W_2 is a (1,1)-type subspace orthogonal to W_1, and define a treatment V to be arranged in a block (W_1, W_2) if $V \perp W_1$ and $V \not\perp W_2$. Then we obtain a PBIB design with two associate classes, and its parameters are those in Lemma 4 and as in the following:*

$$\begin{aligned}
b &= N(m, s; n) \cdot N(1, 1; n-m, n+s-2m; n),\\
r &= [N(1, 1; n) - N(1, 1; n-1, n-2; n)] \cdot N(m, s; n-2, n-2; n),\\
\lambda_1 &= [N(1, 1; n) - 2N(1, 1; n-1, n-2; n) + N(1, 1; n-2, n-4; n)]\\
&\quad \cdot N(m, s; n-3, n-4; n),\\
k &= \frac{rv}{b}\\
\lambda_2 &= \frac{r(k-1) - \lambda_1 n_1}{n_2}.
\end{aligned}$$

Taking $\nu = 2$ in Theorems 3 and 4 and suitably taking the value of δ, m and s, we obtain a number of PBIB designs with two associate classes.

If $\nu = 2$ and $\delta = 0$, then the association scheme in Lemma 3 becomes one with two associate classes and with parameters

$$\nu = (q^3+1)(q+1), \quad n_1 = (q^2+1)q, \quad p_{11}^1 = q-1, \quad p_{11}^2 = q^2+1. \quad (3)$$

If $\nu = 2$ and $\delta = 1$, then the association scheme in lemma 3 also becomes the one with two associate classes and with parameters

$$v=(q^5+1)(q^3+1), \quad n_1=(q^2+1)q^3, \quad p_{11}^1=q^3-1, \quad p_{11}^2=q^2+1. \tag{4}$$

Then we have the following PBIB designs with two associate classes.

Theorem 7. *Taking $\nu = 2$, $\delta = 0$, $m = 1$ and $s = 0$ in Theorem 3, we obtain a PBIB design wish two associate classes. Its parameters are those in (3) and as in the following*

$$\begin{aligned} b &= (q^3+1)(q^2+1)(q+1)q^2 \\ r &= (q^2+1)q^3 \\ k &= q \\ \lambda_1 &= (q-1)q^2 \\ \lambda_2 &= 0. \end{aligned}$$

Theorem 8. *Taking $\nu = 2$, $\delta = 1$, $m = 1$ and $s = 0$ in Theorem 3, we obtain a PBIB design with two associate classes. Its parameters are those in (4) and as in the following*

$$\begin{aligned} b &= (q^5+1)(q^3+1)(q^2+1)q^2 \\ r &= (q^2+1)q^5 \\ k &= q^3 \\ \lambda_1 &= (q^3-1)q^2 \\ \lambda_2 &= 0. \end{aligned}$$

Theorem 9. *Taking $\nu = 2$, $\delta = 1$, $m = 1$ and $s = 1$ in Theorem 3, we obtain a PBIB design with two associate classes. It parameters are those in (4) and as in the following*

$$\begin{aligned} b &= (q^5+1)(q^2+1)(q^2-q+1)q^4 \\ r &= (q^2+1)q^7 \\ k &= (q+1)q^3 \\ \lambda_1 &= (q^3+q-1)q^4 \\ \lambda_2 &= (q^3-1)(q^2+1). \end{aligned}$$

Theorem 10. *Taking $\nu = 2$, $\delta = 1$, $m = 2$ and $s = 1$ in Theorem 3, we obtain a PBIB design with two associate classes. Its parameters are those* *in (4) and as in the following*

$$\begin{aligned} b &= (q^5+1)(q^3+1)(q^2+1)q^4 \\ r &= (q^2+1)q^5 \\ k &= q \\ \lambda_1 &= (q-1)q^2 \\ \lambda_2 &= 0. \end{aligned}$$

Theorem 11. *Taking $\nu = 2$, $\delta = 0$, $m = 1$ and $s = 0$ in Theorem 4, we obtain a PBIB design with two associate classes. Its parameters are those in (3) and as in the following*

$$\begin{aligned} b &= (q^3+1)(q^2+1)(q-1)q^3 \\ r &= (q^2+1)(q-1)q^3 \\ k &= q+1 \\ \lambda_1 &= (q-1)q^2 \\ \lambda_2 &= 0. \end{aligned}$$

Theorem 12. *Taking $\nu = 2$, $\delta = 1$, $m = 1$ and $s = 0$ in Theorem 4, we obtain a PBIB design with two associate classes. Its parameters are those in (4) and as in the following*

$$\begin{aligned} b &= (q^5+1)(q^2+1)(q^2-q+1)q^4 \\ r &= (q^2+1)(q-1)q^5 \\ k &= (q^2-1)q \\ \lambda_1 &= (q-1)(q^3-q-1)q^2 \\ \lambda_2 &= 0. \end{aligned}$$

Theorem 13. *Taking $\nu = 2$, $\delta = 1$, $m = 1$ and $s = 1$ in Theorem 4, we obtain a PBIB design win two associate classes. Its parameters are those in (4) and as in the following*

$$\begin{aligned} b &= (q^2+1)(q-1)(q^4-q^3+q^2-q+1)q^7 \\ r &= (q^2+1)(q-1)q^7 \\ k &= (q^3+1)(q+1) \\ \lambda_1 &= (q^2+1)(q-1)q^5 \\ \lambda_2 &= (q^2+1)(q-1)q^3. \end{aligned}$$

Theorem 14. *Taking $\nu = 2$, $\delta = 1$, $m = 2$ and $s = 1$ in Theorem 4,we obtain a PBIB design with two associate classes. Its parameters are those in (4) and as in the following*

$$
\begin{aligned}
b &= (q^5+1)(q^2+1)(q^2-q+1)(q-1)q^5 \\
r &= (q^2+1)(q-1)q^5 \\
k &= q+1 \\
\lambda_1 &= (q-1)q^3 \\
\lambda_2 &= 0.
\end{aligned}
$$

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