New Designs

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June 2, 1995

Abstract

We study the maximal intersection number of known Steiner Systems and designs obtained from codes. By using a Theorem of Driessen, together with some new observations, we obtain many new designs.

1 Introduction

We assume familiarity with some basic facts from design theory and coding theory. The following definition of a design shall be used throughout this article.

Let $D = \{B_1, B_2, ..., B_b\}$ be a finite family of k-subsets (called **blocks**) of a v-set $X = \{1, 2, ..., v\}$ (with elements called **points**). Then D is a **t-(v, k, λ) design** if every t-subset of X is contained in exactly λ blocks of D.

A design without repeated blocks is called a **simple design**. A t-$(v, k, 1)$ design is called a **Steiner system**. The notation $S(t, k, v)$ is commonly adopted for such a design. It is well-known that if a t-(v, k, λ) design exists then the numbers λ_s, $1 \le s \le t$, where

$$\lambda_0 := b = |D|,\ \lambda_s = \frac{k-s+1}{v-s+1}\,\lambda_{s-1}$$

are integers. These are the so called **necessary conditions** for the existence of t-(v, k, λ) designs. Note that $\lambda_t = \lambda$.

JCMCC 23 (1997), pp. 212-220

Given integers $0 < t < k < v$, the smallest positive integer λ such that t, v, k, λ satisfy the above necessary condition is called the **minimal** λ and is denoted by λ_{min}. If a t-(v, k, λ) design exist, then necessarily λ_{min} divides λ. Also, if $X^{(k)}$ is the collection of all k-element subsets of X, then $X^{(k)}$ is a t-(v, k, λ_{max}) design, where $\lambda_{max} = \binom{v-t}{k-t}$. The value λ_{max} is called the **maximum** λ and $X^{(k)}$ is said to be the **trivial design.**

Given a t-(v, k, λ) design D and a point x, the blocks of D that contain x form a $(t-1)$-$(v-1, k-1, \lambda)$ design on $X\backslash\{x\}$ called the **derived design of D with respect to x.** The blocks of D that do not contain x form a $(t-1)$-$(v-1, k, \lambda_{t-1} - \lambda_t)$ design on $X\backslash\{x\}$ called the **residual design of D with respect to x.** Let $D = \{B_1, B_2, ..., B_b\}$ be a design with parameters t-(v, k, λ) and

$$p = \max_{1 \leq i < j \leq b} |B_i \cap B_j|.$$

Let $p^* = min\, p$, where the minimum is taken over all designs with these parameters. Designs for which $p = p^*$ we shall term as **designs with maximally different blocks**, or DMDB's. We call p the **maximal intersection number of D.**

The derived and residual design of a design with a maximal intersection number p have maximal intersection numbers $p - 1$ and p, respectively.

Designs with repeated blocks correspond to the case $p = k$; simple designs correspond to the cases $p \leq k - 1$; designs whose blocks cover only distinct $(k-1)$-subsets of X correspond to the cases $p \leq k - 2$ and so on. In a sense, DMDB's are generalization of simple designs as well as of supersimple designs (the latter being defined in [4]).

The following theorem, which is an equivalent form of the Johnson Bound for the maximal number of words in a constant weight code [7], can be used to obtain an estimate for p^*.

Theorem 1.1. Let $D = \{B_1, B_2, ..., B_b\}$, where B_i, $i = 1, 2, ..., b$, are k-element subsets of $X = \{1, 2, ..., v\}$ and $|B_i \cap B_j| \leq p < k$ for $1 \leq i < j \leq b$. Then

$$b \leq \left\lfloor \frac{v}{k} \left\lfloor \frac{v-1}{k-1} \cdots \left\lfloor \frac{v-p}{k-p} \right\rfloor \cdots \right\rfloor \right\rfloor.$$

2 Facts and observations

The proofs of existence are based on the following facts and observations.

A. The maximal intersection number of a Steiner system $S(t, k, v)$ is $p = t - 1$.

B. Assmus and Mattson [1] have constructed a 5-(24,9,6) design from the supports ofcodewords of weight 9 in the [24,12,9] extended quadratic residue code over $GF(3)$. If $\mathbf{x}$ and $\mathbf{y}$ are two codewords of weight 9 whose supports intersect in p coordinates, then either the distance between $\mathbf{x}$ and $\mathbf{y}$ or between $\mathbf{x}$ and $-\mathbf{y}$ is less than or equal to $2(9-p)+p/2$. But the minimal distance of the code is 9, and thus $p \le 6$. Similarly, $p \le 10$ for the 5-$(24,12,576)$ design obtained from the same code, and $p \le 8$ for the 5-$(24,12,48)$ design, obtained from the $[24,12,8]$ self-orthogonal code over $GF(2)$ (cf. the same paper of Assmus and Mattson).

C. MacWilliams, Odlyzko and Sloane [8] have constructed a 5-$(30,12,220)$ design from the subsets of coordinate places holding codewords of weight 12 in the even formally self-dual $[30,15,12]$ code over $GF(4)$. The code has minimal distance 12 (equal to the minimal weight of a codeword). There are three codewords for each support. Any other word of the code must be at a distance at least 12 from each of these three. Suppose $p > 9$. Then there must be a codeword which has at least 10 nonzero elements on positions occupied by the nonzero elements of the support. At least 4 of these elements must be identical. This gives two codewords at a distance at most 10, which is a contradiction. So, $p \le 9$ for the above set of parameters. Analogously, $p \le 12$ for the 5-$(30,14,5390)$ design and $p \le 14$ for the 5-$(30,16,123000)$ design, obtained in the same paper.

Remark. By using Theorem 1.1 we can prove that, in fact, all of the three designs from B, as well as, the 5-$(30,16,123000)$ design from C are DMDB's. For example, consider the 5-$(24,9,6)$ design from B. We have proved that $p \le 6$ for the last design. Suppose $p \le 5$. By Theorem 1.1 the number of blocks of the design should be at most 1349 while a 5-$(24,9,6)$ design has 2024 blocks, which is a contradiction. Therefore $p = 6$ for the 5-$(24,9,6)$ design found in B. Since $p = 6$ is the smallest possible intersection number for this set of parameters, the 5-$(24,9,6)$ design found in B is a DMDB. Analogously, the 5-$(24,12,576)$ design is a DMDB with $p = 10$, the 5-$(24,12,48)$ design is a DMDB with $p = 8$ and the 5-$(30,16,123000)$ design is a DMDB with $p = 14$.

D. Theorem 2.1 (DRIESSEN [3]) If D is a t-(v,k,λ) design with a maximal intersection number $p \le k-m-l-1$, for fixed integers $m, l \ge 0$, then

$$D_{ml} = \{(B \cup L) \setminus M : B \in D, M \subseteq B, |M| = m, L \subseteq (X \setminus B), |L| = l\}$$

is a

$$t-\left(v, k+l-m, \lambda\binom{v-k}{l}\binom{k+l-m}{t}\binom{k}{m}\Big/\binom{k}{t}\right)$$

design. (Note that $D_{00} = D$.)

We make the following two observations:

Corollary 2.2. Designs, obtained by Driessen's Theorem for pairs m_1, l_1

and m_2, l_2, where $m_1 \neq m_2$, but $m_1 - l_1 = m_2 - l_2$, have the same block-size, and are disjoint.

Corollary 2.3. If Driessen's Theorem produces simple nontrivial designs, then $m \leq k - t - 1$, and the initial design is not trivial.

The proofs follow directly from the definition of D_{ml}. The next two facts are well-known.

E. If D is a simple t-(v,k,λ) design, then $\{X \setminus B : B \in D\}$ is a simple t-$\left(v, v-k, \lambda\binom{v-k}{t} \Big/ \binom{k}{t}\right)$ design.

F. If D is a simple t-(v,k,λ) design, then $X^{(k)} \setminus D$ is a simple t-$(v,k,\lambda_{max} - \lambda)$ design.

3 Proofs

It is convenient to arrange the proofs of existence in a table. The proofs start with an initial design which has small maximal intersection number. An upper bound on the maximal intersection number is given in the fourth column of the table, although in most of the cases we know the exact value of p. We use the designs mentioned in A, B, and C, as well as their derived and residual designs, as initial designs. Then we apply Driessen's theorem and the corollaries to obtain new designs. In some cases we apply E and F.

As an example, let us obtain a 4-$(23,11,n6)$ design, $n = 2494$. Start with an initial 4-$(23,11,48)$ design D, with $p = 7$, the derived of the 5-$(24,12,48)$ design with $p = 8$, mentioned in B. By the theorem of Driessen we get a 4-$(23,12,52272)$ design, D_{12}, and a 4-$(23,12,864)$ design, D_{01}. According to the corollary 2.2, the last two designs are disjoint. Therefore, $D_{01} \cup D_{12}$ is a 4-(23,12,53136) design. Using E, we get a 4-(23,11,35424) design. Since λ_{max} is 50388 for the last set of parameters, using F, we finally obtain a 4-$(23,11,14964)$ design, i.e., a 4-$(23,11,n6)$ design, $n = 2494$. The proof can be described as the sequence $D_{01} \cup D_{12}$, E, F. The table below contains proofs of existence of 151 designs.

New design		The existence follows from		
Parameters	n	Initial design D	Upper bound on p	Proof
5-(24,8,n)	24	5-(24,9,6)	6	D_{10}
5-(24,9,n6)	135	5-(24,9,6)	6	D_{11}
	136	5-(24,9,6)	6	$D_{00} \cup D_{11}$
5-(24,10,n18)	10	5-(24,9,6)	6	D_{01}
	56	5-(24,12,48)	8	D_{02}, E
5-(24,11,n42)	8	5-(24,12,48)	8	D_{02}, E
	55	5-(24,9,6)	6	D_{02}
	96	5-(24,12,576)	10	D_{10}
	118	5-(24,12,48)	8	D_{21}, F
5-(24,12,n6)	1152	5-(24,12,48)	8	D_{11}
	1160	5-(24,12,48)	8	$D_{00} \cup D_{11}$
5-(28,7,n)	105	5-(28,7,1)	4	D_{11}, F
	106	5-(28,7,1)	4	$D_{00} \cup D_{11}$, F
5-(28,8,n7)	8	5-(28,7,1)	4	D_{01}
5-(28,9,n35)	36	5-(28,7,1)	4	D_{02}
5-(30,10,n42)	110	5-(30,12,220)	9	D_{20}
5-(30,11,n1540)	1	5-(30,12,220)	9	D_{10}
5-(30,12,n220)	216	5-(30,12,220)	9	D_{11}
	217	5-(30,12,220)	9	$D_{00} \cup D_{11}$
5-(30,13,n495)	13	5-(30,12,220)	9	D_{01}
	98	5-(30,14,5390)	12	D_{10}
	1025	5-(30,16,123000)	14	D_{01}, E
5-(30,14,n55)	1547	5-(30,12,220)	9	D_{02}
5-(30,15,n22)	5880	5-(30,14,5390)	12	D_{01}
	61500	5-(30,16,123000)	14	D_{10}
4-(23,7,n)	129	4-(23,8,4)	4	D_{21}, F
	180	4-(23,9,18)	6	D_{20}
4-(23,8,n2)	45	4-(23,9,18)	6	D_{10}
	240	4-(23,8,4)	4	D_{11}
	242	4-(23,8,4)	4	$D_{00} \cup D_{11}$
	360	4-(23,8,6)	5	D_{11}
	363	4-(23,8,6)	5	$D_{00} \cup D_{11}$
4-(23,9,n18)	9	4-(23,8,6)	5	D_{01}
	126	4-(23,9,18)	6	D_{11}
	127	4-(23,9,18)	6	$D_{00} \cup D_{11}$
	224	4-(23,11,48)	7	D_{03}, E
	304	4-(23,8,4)	4	$D_{01} \cup D_{12}$, F
	310	4-(23,8,4)	4	D_{12}, F

New design		The existence follows from		
Parameters	n	Initial design	Upper bound on p	Proof
4-(23,10,n42)	10	4-(23,9,18)	6	D_{01}
	45	4-(23,8,6)	5	D_{02}
	48	4-(23,11,48)	7	D_{02}, E
	158	4-(23,11,48)	7	$D_{10} \cup D_{21}$, F
	166	4-(23,11,48)	7	D_{21}, F
4-(23,11,n6)	672	4-(23,11,576)	9	D_{10}
	715	4-(23,9,18)	6	D_{02}
	1056	4-(23,11,48)	7	D_{11}
	1064	4-(23,11,48)	7	$D_{00} \cup D_{11}$
	2494	4-(23,11,48)	7	$D_{01} \cup D_{12}$, E, F
	2590	4-(23,11,48)	7	D_{12}, E, F
4-(27,6,n)	21	4-(27,7,7)	4	D_{10}
	126	4-(27,6,1)	3	D_{11}
4-(27,7,n7)	7	4-(27,6,1)	3	D_{01}
	112	4-(27,7,7)	4	$D_{00} \cup D_{11}$, F
	113	4-(27,7,7)	4	D_{11}, F
4-(27,8,n35)	8	4-(27,7,7)	4	D_{01}
	28	4-(27,6,1)	3	D_{02}
4-(27,9,n7)	684	4-(27,7,7)	4	D_{02}
4-(29,10,n140)	99	4-(29,12,495)	9	D_{20}
4-(29,11,n220)	18	4-(29,12,495)	9	D_{10}
	198	4-(29,11,220)	8	D_{11}
	199	4-(29,11,220)	8	$D_{00} \cup D_{11}$
4-(29,12,n495)	12	4-(29,11,220)	8	D_{01}
	204	4-(29,12,495)	9	D_{11}
	205	4-(29,12,495)	9	$D_{00} \cup D_{11}$
4-(29,13,n55)	221	4-(29,12,495)	9	D_{01}
	1326	4-(29,11,220)	8	D_{02}
	1568	4-(29,14,8624)	12	D_{10}
	16400	4-(29,15,123000)	13	D_{01}, E
4-(29,14,n22)	5488	4-(29,13,5390)	11	D_{01}
	6188	4-(29,12,495)	9	D_{02}
	57400	4-(29,16,143500)	14	D_{10}, E
3-(22,6,n)	180	3-(22,8,18)	5	D_{20}
	233	3-(22,7,4)	3	$D_{10} \cup D_{21}$, F
	249	3-(22,7,4)	3	D_{21}, F

New design		The existence follows from		
Parameters	n	Initial design	Upper bound on p	Proof
3-(22,7,n)	420	3-(22,7,4)	3	D_{11}
	424	3-(22,7,4)	3	$D_{00} \cup D_{11}$
	630	3-(22,7,6)	4	D_{11}
	636	3-(22,7,6)	4	$D_{00} \cup D_{11}$
	876	3-(22,8,12)	4	$D_{10} \cup D_{21}$, F
	936	3-(22,8,12)	4	D_{21}, F
3-(22,8,n6)	24	3-(22,7,6)	4	D_{01}
	42	3-(22,9,42)	6	D_{10}
	224	3-(22,8,12)	4	D_{11}
	226	3-(22,8,12)	4	$D_{00} \cup D_{11}$
	336	3-(22,8,18)	5	D_{11}
	339	3-(22,8,18)	5	$D_{00} \cup D_{11}$
	672	3-(22,11,72)	7	D_{30}
	784	3-(22,7,4)	3	D_{12}
	800	3-(22,7,4)	3	$D_{01} \cup D_{12}$
3-(22,9,n42)	9	3-(22,8,18)	5	D_{01}
	36	3-(22,7,6)	4	D_{02}
	117	3-(22,9,42)	6	D_{11}
	118	3-(22,9,42)	6	$D_{00} \cup D_{11}$
	176	3-(22,10,48)	6	D_{03}, E
	206	3-(22,10,48)	6	$D_{10} \cup D_{21}$, F
	214	3-(22,10,48)	6	D_{21}, F
	312	3-(22,8,12)	4	D_{12}
	318	3-(22,8,12)	4	$D_{01} \cup D_{12}$
3-(22,10,n6)	130	3-(22,9,42)	6	D_{01}
	528	3-(22,10,48)	6	D_{02}, E
	585	3-(22,8,18)	5	D_{02}
	960	3-(22,10,48)	6	D_{11}
	968	3-(22,10,48)	6	$D_{00} \cup D_{11}$
	3118	3-(22,11,72)	7	D_{12}, E, F
	3022	3-(22,11,72)	7	$D_{01} \cup D_{12}$, E, F
3-(22,11,n9)	88	3-(22,10,48)	6	D_{01}
	715	3-(22,9,42)	6	D_{02}
	968	3-(22,11,72)	7	D_{11}
	976	3-(22,11,72)	7	$D_{00} \cup D_{11}$
	3470	3-(22,10,48)	6	$D_{01} \cup D_{12}$, F
	3558	3-(22,10,48)	6	D_{12}, F

New design		The existence follows from		
Parameters	n	Initial design	Upper bound on p	Proof
3-(26,5,n)	3	3-(26,6,1)	2	D_{10}
	43	3-(26,7,35)	4	D_{20}, F
	100	3-(26,6,1)	2	$D_{10} \cup D_{21}$, F
	103	3-(26,6,1)	2	D_{21}, F
	105	3-(26,5,1)	2	D_{11}
	106	3-(26,5,1)	2	$D_{00} \cup D_{11}$
3-(26,6,n)	42	3-(26,5,1)	2	D_{01}
	120	3-(26,6,1)	2	D_{11}
	121	3-(26,6,1)	2	$D_{00} \cup D_{11}$
	140	3-(26,7,35)	4	D_{10}
	840	3-(26,6,7)	3	D_{11}
3-(26,7,n35)	21	3-(26,5,1)	2	D_{02}
	57	3-(26,6,1)	2	D_{12}
	57	3-(26,6,1)	2	$D_{01} \cup D_{12}$
	119	3-(26,7,35)	4	$D_{00} \cup D_{11}$, F
	120	3-(26,7,35)	4	D_{11}, F
3-(26,8,n7)	76	3-(26,6,1)	2	D_{02}
3-(26,9,n21)	228	3-(26,6,1)	2	D_{03}
3-(28,9,n28)	55	3-(28,10,220)	7	D_{10}
	495	3-(28,11,495)	8	D_{20}
3-(28,10,n20)	198	3-(28,11,495)	8	D_{10}
	1683	3-(28,12,935)	9	D_{20}
	1980	3-(28,10,220)	7	D_{11}
	1991	3-(28,10,220)	7	$D_{00} \cup D_{11}$
3-(28,11,n495)	11	3-(28,10,220)	7	D_{01}
	17	3-(28,12,935)	9	D_{10}
	187	3-(28,11,495)	8	D_{11}
	188	3-(28,11,495)	8	$D_{00} \cup D_{11}$
3-(28,12,n55)	204	3-(28,11,495)	8	D_{01}
	1122	3-(28,10,220)	7	D_{02}
	1568	3-(28,13,8624)	11	D_{10}
	3264	3-(28,12,935)	9	D_{11}
	3281	3-(28,12,935)	9	$D_{00} \cup D_{11}$
3-(28,13,n22)	884	3-(28,12,935)	9	D_{01}
	5096	3-(28,12,5390)	10	D_{01}
	5304	3-(28,11,495)	8	D_{02}
	53300	3-(28,16,143500)	14	D_{10}, E
3-(28,14,n6)	27440	3-(28,13,8624)	11	D_{01}
	30940	3-(28,12,935)	9	D_{02}
	287000	3-(28,15,143500)	13	D_{10}

4 Concluding remarks

We have investigated the maximal intersection number of some known designs. The results, by applying the theorem of Driessen and the corollaries, lead to the proof of the existence of the 151 new designs from the table above. The table does not contain a design which is derived or residual of a design from the same table. So, the designs found above lead to many other designs. Some other well-known theorems can be also used to obtaining new designs from the ones obtained so far. The total of more than 500 new designs are generated this way [5]. A complete list can be found in [6].

Note that some of the designs from the table have the smallest known λ, for example, the 3-$(26, 8, 532)$, 3-$(26, 9, 4788)$, 4-$(27, 8, 270)$, 5-$(24, 10, 180)$, 5-$(24, 11, 336)$, 5-$(28, 8, 56)$ designs (cf.[2]). Some of the designs have even the smallest possible λ, for example, the 5-(30,11,1540) design. Some of the sets are the only known sets of parameters for fixed t, v, k, for example, the 4-$(27, 9, 4788)$, 5-$(28, 9, 1260)$ designs. At this point, it is obvious that the study of designs with small maximal intersection number p or, moreover, DMDB's, could be a source for finding many new designs.

5 References

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