On Bisectability of Trees

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ABSTRACT. The present paper studies bisectable trees, i.e., trees whose edges can be colored by two colors so that the induced monochromatic subgraphs are isomorphic. It is proved that the number of edges that have to be removed from a tree with maximum degree three to make it bisectable can be bounded by an absolute constant.

1 Introduction

A graph G is *bisectable* (or *even*), if its edges can be colored by two colors such that the monochromatic subgraphs of G induced by the coloring are isomorphic. The problem of deciding whether a graph is bisectable appears difficult even for trees (see Graham and Robinson [2]). Harary and Robinson [3] conjectured that the problem is easy for trees T with maximum degree $\Delta(T) \leq 3$. More precisely, they conjectured that with the exception of two trees, every tree with $\Delta(T) \leq 3$ and with an even number of edges is bisectable (of course, no graphs with an odd number of edges can be bisectable). This conjecture was proven false by Heinrich and Horak [4], who constructed an infinite class of non-bisectable trees with an even number of edges and with maximum degree equal to three. This opened the question of characterization of the class of non-bisectable maximum-degree-three trees.

Within this context, one may want to investigate how far a tree with $\Delta(T) = 3$ can stray from bisectability. The smallest number $R(G)$ of edges whose removal from G leaves a bisectable graph can serve as a measure of this. For general trees, Alon, Caro and Krasikov [1] proved that $R(T) \leq O(v/(\log\log v))$, where v denotes the number of vertices of T. They also constructed trees with $R(T) \geq \Omega(\log v)$. Horak and Zhu [5] proved that for trees with $\Delta(T) \leq 3$, $R(T) \leq O(\log v')$, where v' is the number of vertices of degree three in T. In the present paper we improve their result by showing that for trees with $\Delta(T) \leq 3$, $R(T)$ can be bounded by a constant. In fact, our proof was motivated by their paper, and can be viewed as a refinement of the proof of Theorem 3 from there.

Before we proceed with the paper, we will introduce a few definitions. For a tree T with $\Delta(T) = 3$, we denote by cub(T) the tree obtained from T by suppressing the vertices of degree two (that is, cub(T) is topologically isomorphic to T, but has only vertices of degrees one and three). A tree is *rooted*, if one of its vertices, called the *root*, is designated as special. We say that two rooted trees are isomorphic, if they admit an isomorphism preserving roots.

2 The Result

Lemma 1. *Let T be a tree with $\Delta(T) \leq 3$, and with at most three vertices of degree three. Then $R(T) \leq 7$.*

Proof: It is readily seen that if P is a path, then $R(P) \leq 1$. If T has at most three vertices of degree three, then it can be disconnected into a collection of at most four paths by a removal of at most three edges. Consequently, $R(T) \leq 3 + 4 = 7$. □

Lemma 2. *Let T be a tree with $\Delta(T) \leq 3$, and with at least four vertices of degree three. Then there is a vertex u of degree three in T, and rooted trees B, B', and B'' with the common root u (which has degree one in each of them), and with $T = B \cup B' \cup B''$, such that B and B' have at most three vertices of degree three each, while $B \cup B'$ has at least three degree three vertices.*

Proof: Let u_0 be a vertex of degree one in T, and let u_1 be the vertex with degree three in T which is closest to u_0. T may then be written as $T = B^1 \cup (B')^1 \cup (B'')^1$, where u_1 is the common root of B^1, $(B')^1$, and $(B'')^1$ having degree one in each of them, and where $(B'')^1$ is the path u_0u_1. Since T has at least four vertices of degree three, either the rooted trees B^1, $(B')^1$, and $(B'')^1$ are as required, or one of B^1 and $(B')^1$, say B^1, has at least four vertices of degree three. In the latter case, we let u_2 be the vertex of degree three in B^1 nearest to u_1, and let $T = B^2 \cup (B')^2 \cup (B'')^2$, where B^2, $(B')^2$, and $(B'')^2$ are all rooted in u_2, u_2 has degree one in each of them, and $(B'')^2$ contains the path u_0u_1 (or, equivalently, $B^2 \cup (B')^2 \subseteq B^1$). Again, either B^2, $(B')^2$, and $(B'')^2$ are as desired, or one of B^2 and $(B')^2$ has at least four vertices of degree three. Continuing this way, we construct B^k, $(B')^k$, and $(B'')^k$, $k = 3, 4, 5, \ldots$, and eventually reach the point where neither of B^k and $(B')^k$ has more than three vertices of degree three. The resulting trees will be as desired. □

Lemma 3. *If T is a rooted tree with root u whose degree in T is equal to one, and if cub(T) is isomorphic (as a rooted tree) to one of $B_1 - B_5$ from Figure 4, then either T is isomorphic to one of $H_1 - H_8$ from Figure 4, or it can be written as $T_1 \cup T_2$, where $T_1 \cap T_2 = \{v\}$ for some vertex v, where T_1 is a rooted tree with root v, isomorphic to one of $G_1 - G_{18}$ of Figure 2, where T_1 doesn't contain u, where v has degree one in T_2, and where T_2 is a tree rooted at u (see Figure 1).*

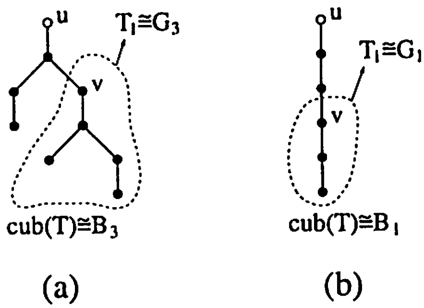

Figure 1

Proof: The proof consists of a straightforward, but fairly tedious checking, whose details are omitted, and whose outline is as follows: If cub(T) $\cong$ B_1 (the isomorphism being an isomorphism of rooted trees), and if T is isomorphic neither to H_1 nor to H_2, then it can be seen that the required decomposition $T_1 \cup T_2$ can be achieved with $T_1 \cong G_1$, (see Figure 1(b)). Similarly, if cub(T) $\cong B_2$, but T is not isomorphic to any of $H_3 - H_5$, then one of $G_1 - G_4$ can be used as T_1.

It requires somewhat more patience to check that if cub(T) $\cong B_3$, but $T \not\cong H_6$, then one of $G_1 - G_{10}$ will work as T_1, that if cub(T) $\cong B_4$, but T is isomorphic to neither H_7 nor H_8, then one of $G_1 - G_4$, or of $G_{13} - G_{18}$ will do, and, finally, that if cub(T) $\cong B_5$, then T_1 will take the form of one of $G_1 - G_{12}$. □

Lemma 4. *Let T be a tree with $\Delta(T) = 3$, and with at least four vertices of degree three. Then T can be written as $T = T_1 \cup T_2$, where*

(i) *T_1 and T_2 have exactly one vertex in common,*

(ii) *T_1 (as a tree rooted at $T_1 \cap T_2$) is isomorphic to one of the graphs $G_1 - G_{37}$ of Figures 2 and 3, and*

(iii) *the vertex $T_1 \cap T_2$ has degree one in T_2.*

Proof: Let u, B, B', and B'' be as in Lemma 2. It is easily seen that the rooted tree cub(B) (and, similarly, the rooted tree cub(B')) has to be isomorphic to one of $B_1 - B_5$. The result follows by Lemma 3 when at least one of B and B' is not isomorphic to any of $H_1 - H_8$. However, if B and B' are each isomorphic to one of $H_1 - H_8$, then it follows from the fact that $B \cup B'$ has at least three vertices of degree three that $T_1 = B \cup B'$ must be isomorphic to one of $G_{19} - G_{37}$. □

Lemma 5. *Let T be a tree with maximum degree at most three. Then there is a sequence $T_0, T_1, \ldots, T_n$ of rooted trees such that:*

(i) *T_0 has at most three vertices of degree three,*

(ii) *for each i, $1 \le i \le n$, $T_i \cap \{T_0 \cup T_1 \cup \cdots \cup T_{i-1}\}$ consists of a single vertex, which is also the root of T_i, and which has degree one in $\{T_0 \cup T_1 \cup \cdots \cup T_{i-1}\}$,*

(iii) *each of $T_1, T_2, \ldots, T_n$ is isomorphic to one of $G_1 - G_{37}$, and*

(iv) *$T = T_0 \cup T_1 \cup \cdots \cup T_n$.*

Proof: This is proved by a repeated use of Lemma 4 (the trees T_i are constructed in the descending order of i). □

Figures 2 and 3 give a list of edge-colorings of each of the 37 graphs $G_1 - G_{37}$, the number of colorings of G_i ($1 \le i \le 37$) in this list will be denoted by n_i. Thus, for example, $n_4 = 4$ (although not all four colorings of G_4 in Figure 2 are distinct). For $1 \le i \le 37$, the colorings of G_i from Figures 2 and 3 will be denoted, as in those figures, by G_i^k, $1 \le k \le n_i$. We will use these colorings in the proof of our main result. More precisely, we will use their representations in colors 0 and 1. Since each of these colorings gives rise to two such representations, we will us a prefix to distinguish between them. Thus, for $\epsilon \in \{0, 1\}$, $\epsilon - G_i^k$ will denote the coloring of G_i, corresponding to G_i^k, in which the edges incident with the root are assigned color ϵ, or, equivalently, $\epsilon - G_i^k$ is the coloring obtained from G_i^k by coloring solid edges in Figure 2 (or 3) with color ϵ, and coloring dashed edges with color $1 - \epsilon$.

We will now describe an algorithm for coloring the edges of a maximum-degree-three tree T. To this end, set $s_{\epsilon,1} = s_{\epsilon,2} = \cdots = s_{\epsilon,37} = 0$ for $\epsilon = 0, 1$, and let $T_0, T_1, \ldots, T_n$ be the decomposition of T described by Lemma 5. The algorithm then goes on as follows:

- First, color the edges of T_0 with colors 0 and 1 so that no more than seven edges need to be removed to make the monochromatic subgraphs isomorphic (Lemma 1 guarantees that this is possible). Then repeat the following three steps for $i = 1, 2, \ldots, 37$.

- (Trees $T_0, T_1, \ldots, T_{i-1}$ are assumed to have already been colored). Let ϵ be the color by which the edge of $T_0 \cup T_1 \cup \cdots \cup T_{i-1}$ incident to the root of T_i is colored.
- Let j be such that T_i is isomorphic to G_j, use coloring $(1-\epsilon) - G_j^{s_{1-\epsilon,j}+1}$ to color T_i, and increase $s_{1-\epsilon,j}$ by one.
- Finally, if $s_{1-\epsilon,j} = n_j$, set $s_{1-\epsilon,j} = 0$.

We now prove the following

Theorem 6. *The coloring described by the above algorithm is such that no more than 3559 edges need to be removed from T to make its monochromatic components isomorphic.*

Proof: First, note that the coloring algorithm is such that the monochromatic components of each T_i are also monochromatic components in T (this fact will be tacitly assume throughout the proof).

We will now partition $\{T_1, T_2, \ldots, T_n\}$ into 74 classes $\mathcal{H}_{\epsilon,j}$, $\epsilon \in \{0,1\}, 1 \leq j \leq 37$, so that the class $\mathcal{H}_{\epsilon,j}$ contains those T_i which are isomorphic to G_j, and which are colored by a coloring of the form $\epsilon - G_j^k$ for some k (or, equivalently, in which the edges incident with the root are colored by color ϵ).

Consider now a fixed $\mathcal{H}_{\epsilon,j} = \{T_{i_1}, T_{i_2}, \ldots, T_{i_m}\}$, where $i_1 < i_2 < \cdots < i_m$. In particular, $m \equiv s_{\epsilon,j} \pmod{n_j}$, where the value of $s_{\epsilon,j}$ is its terminal value from the algorithm. Therefore, $m = rn_j + s_{\epsilon,j}$ for some r. Also, the algorithm colored graphs $T_{i_1}, T_{i_2}, \ldots, T_{i_{n_j}}, T_{i_{n_j+1}}, \ldots, T_{i_m}$ with $\epsilon - G_j^1, \epsilon - G_j^2, \ldots, \epsilon - G_j^{n_j}, \epsilon - G_j^1, \ldots, \epsilon - G_j^{s_{\epsilon,j}}$, respectively.

We will now show that the removal of $T_{rn_j+1}, T_{rn_j+2}, \ldots, T_{rn_j+s_{\epsilon,j}} = T_m$ from $\mathcal{H}_{\epsilon,j}$ results in a collection $\mathcal{H}'_{\epsilon,j}$ such that the induced monochromatic subgraphs of the union of trees in $\mathcal{H}'_{\epsilon,j}$ are isomorphic.

To this end, let $\mathcal{G}_{\epsilon,j}$ be the collection $\{\epsilon - G_j^k \colon 1 \leq k \leq n_j\}$. One may check in Figures 2 and 3 that, for every j, the induced monochromatic subgraphs of $\bigcup\{T \colon T \in \mathcal{G}_{\epsilon,j}\}$ are isomorphic. Since $\mathcal{H}'_{\epsilon,j}$ is essentially just a collection of copies of $\mathcal{G}_{\epsilon,j}$, the monochromatic subgraphs of $\bigcup\{T \colon T \in \mathcal{H}'_{\epsilon,j}\}$ must be isomorphic too. Also, since for all j, $s_{\epsilon,j} < n_j \leq 4$, and the number of edges in G_j is at most 16, the total number of edges of the graphs removed from $\mathcal{H}_{\epsilon,j}$ is at most $3 \cdot 16 = 48$.

If we do this for every $\epsilon \in \{0,1\}$, and for every j, $1 \leq j \leq 37$, and then remove at most seven more edges so that the monochromatic subgraphs of (what will be left of) T_0 are isomorphic, then the resulting graph will have isomorphic induced monochromatic subgraphs, and we will have removed at most $7 + 2 \cdot 37 \cdot 48 = 3559$ edges. □

Corollary 7. *Let T be a tree with $\Delta(T) \leq 3$. Then $R(T) \leq 3559$.* □

Remark The estimate in Theorem 6 is far from best possible. It can be proved (using, e.g., Theorem 9 of [4]) that in Lemma 1, two can be used instead of seven. More importantly, a more detailed version of the argument of Theorem 6 may be used to further reduce the estimated number of edges that need to be removed. In fact, it can be shown that the number 3559 in Theorem 6 can be replaced by 23. Although this may, at the first sight, appear to be vastly superior to the estimate provided by the proof of Theorem 6, the author feels that the tight constant upper bound on $R(T)$ is more likely to be much smaller, maybe as small as two. In light of this, it was felt that the contribution of our result to the estimate of $R(T)$ was in bounding it by a constant, and that unless this constant could be brought down to at least a number smaller than, say, ten, the exact value of the constant was of little significance. Consequently, for simplicity's sake, we decided not to exhibit the strongest — and most complicated — version of the method of Theorem 6.

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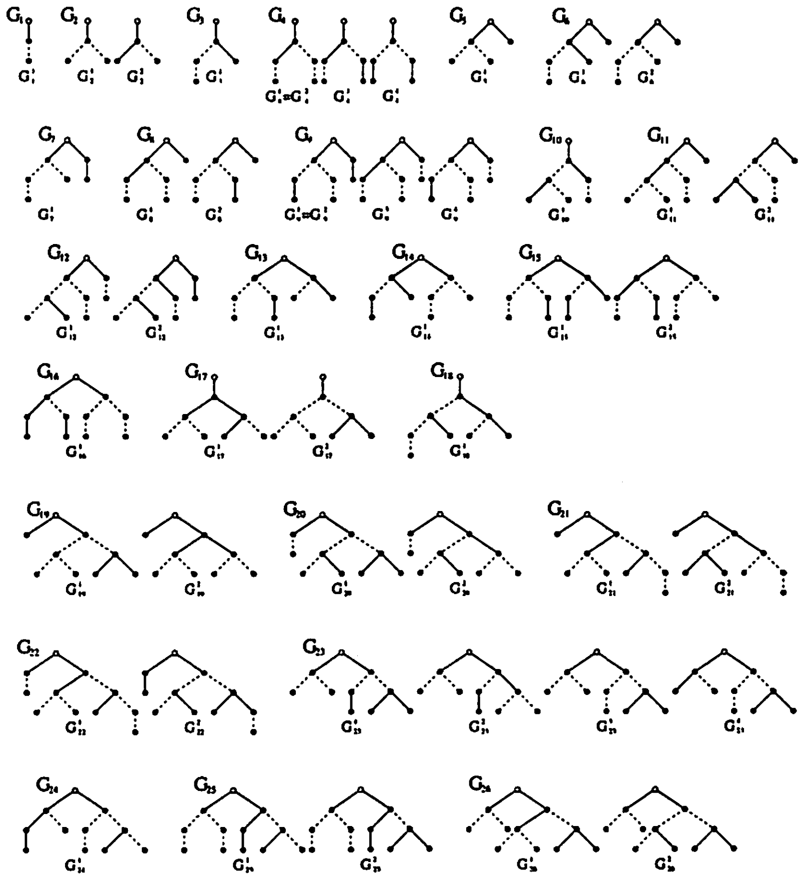

Figure 2

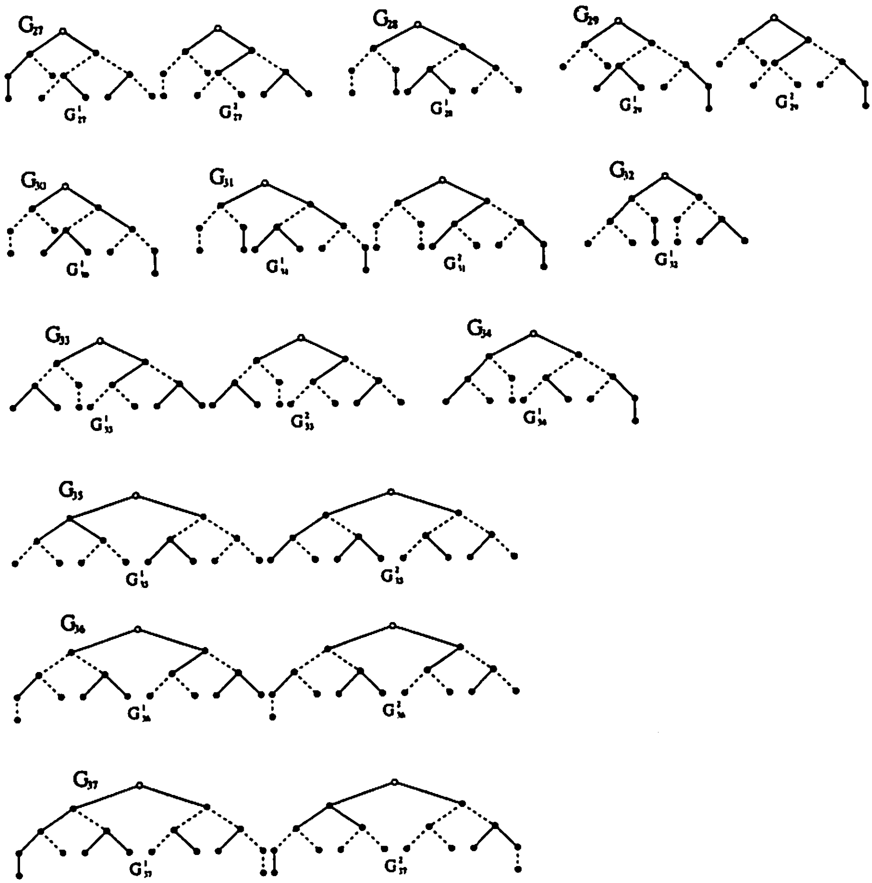

Figure 3

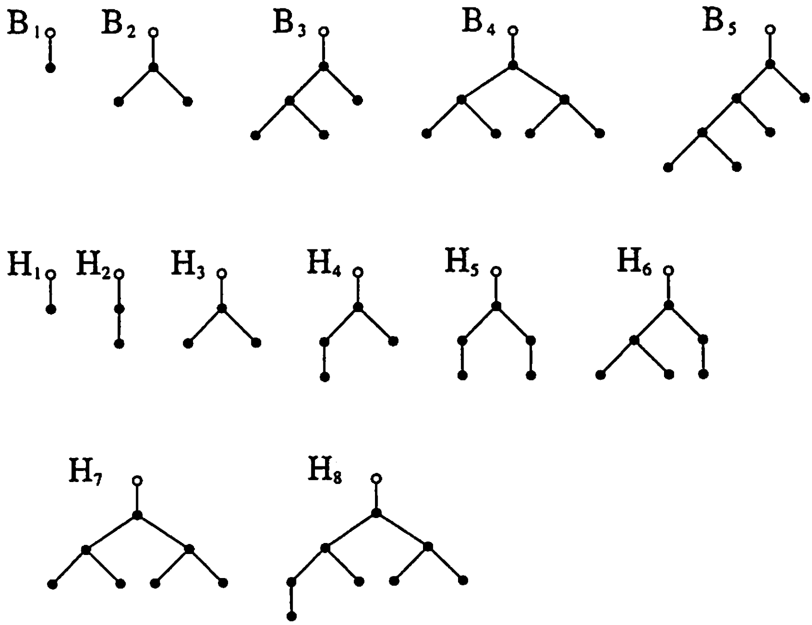

Figure 4