On an Ordering of Convex Sublattices Preserving Planarity of the Lattice

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1 Introduction

For a lattice L, we shall denote by $CS(L)$ (respectively, $\mathrm{Sub}(L)$), the set of all convex sublattices (respectively, sublattices) of L, excluding the empty set. It is well known that both $CS(L) \cup \{\emptyset\}$ and $\mathrm{Sub}(L) \cup \{\emptyset\}$ form lattices, with respect to the inclusion order, and these lattices have been studied extensively by several authors (ref [1], [4], [5], [6], [9]). However, these lattices do not preserve much properties of L. In [7], we introduced a new partial order on $CS(L)$, which makes $CS(L)$ a lattice, and it appears to be more appropriate than the inclusion order. In [8], we extended this partial order to $\mathrm{Sub}(L)$.

In this paper we introduce a new partial order $\leq$ on $CS(L)$ which is an extension of the partial order defined earlier by us in [7] and prove some interesting new results.

The partial order $\leq$ on $CS(L)$ is defined as follows.

Definition 1.1. Define a binary relation $\leq'$ on $CS(L)$ by, for $A, B \in CS(L)$, $A \leq' B$ if and only if for each $a \in A$ there exists a $b \in B$ such that $a \leq b$. Then the binary relation $\leq$ on $CS(L)$ defined by, for $A, B \in CS(L)$, $A \leq B$ if and only if either $A \leq' B$ but $B \not\leq' A$

or

$A \leq' B$, $B \leq' A$ and $A \subseteq B$, is a partial order on $CS(L)$.

The poset $\langle CS(L), \leq \rangle$ will be simply denoted by $CS(L)$ in this paper.

In Section 2, we study some of the properties of the poset $CS(L)$. In Section 3, using a result of D. Kelly and I. Rival [3], we derive an interesting result that a finite lattice L is planar if and only if $CS(L)$ is planar.

For a subset A of a lattice L we denote by $(A]$ and $\langle A \rangle$ respectively, the ideal and the convex sublattice generated by A in L. For a poset p,

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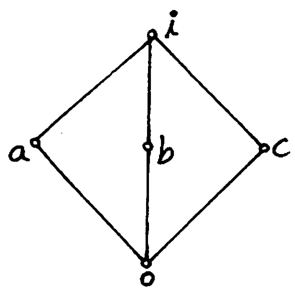
P^d denotes the dual of P. P is said to satisfy the Jordan-Dedekind chain condition if for every $a, b \in P$ with $a < b$ all maximal chains connecting a and b are of equal length. A meet semilattice S is said to be distributive [10], if for $w, a, b \in S$ with $w \geq a \wedge b$ there exist $x \geq a$, $y \geq b$ such that $x \wedge y = w$ and it is said to be modular [10] if for $w, a, b \in S$ with $w \geq a \wedge b$ there exist $x \geq a$, $y \geq b$ such that $x \wedge y = x \wedge w$.

For other undefined notions and notations used in this paper we refer to Gratzer [2].

2 On the poset $CS(L)$

Example 2.1. For the lattice L given in Figure 1, the poset $CS(L)$ is as shown in Figure 2.



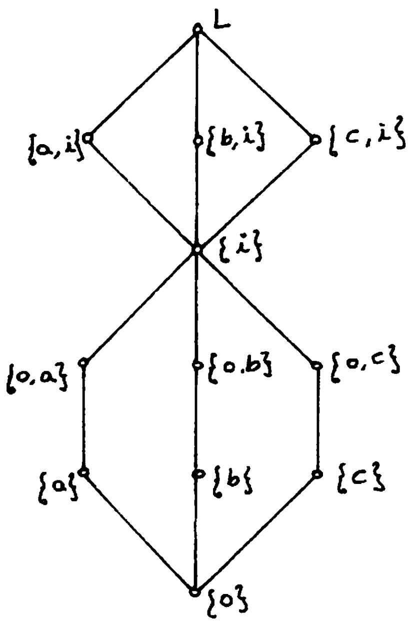


Figure 1 **Figure 2**

Remark 2.2. For any lattice L, $CS(L)$ is a meet semilattice. In fact, for $A, B \in CS(L)$ with $A \parallel B$,

$$A \wedge B = \begin{cases} A \cap B & \text{if } A \leq' B, B \leq' A \\ (\{a \wedge b \mid a \in A, b \in B\}] & \text{if } A \not\leq' B, B \not\leq' A \end{cases}$$

Remark 2.3. For any lattice L, the lattice of all ideals $I(L)$ of L with respect to $\subseteq$, is a meet subsemilattice of $CS(L)$. Moreover, $CS(L)$ is the ordinal sum [11] of the subposets $(CS(L) - D(L))$ and $D(L)$, where $D(L)$ is the lattice of dual ideals of L with respect to $\subseteq$.

Remark 2.4. Let L be a lattice satisfying the ascending and descending chain conditions. Then for $A, B \in CS(L)$,

(a) $A \prec B$ if and only if either $\max A = \max B$ and $\min B \prec \min A$, or $0 \in A$ (where 0 is the least element of L) with $\max A \prec \max B = \min B$.

(b) $A \parallel B$ if and only if either $\max A = \max B$ and $\min A \parallel \min B$, or $\max A \parallel \max B$.

(c) $A < B$ if and only if either $\max A = \max B$ and $\min A > \min B$, or $\max A < \max B$.

Theorem 2.5. *The following statements are equivalent in* $CS(L)$.

1) *L is a chain.*

2) *$CS(L)$ is a chain.*

3) *$CS(L)$ is a distributive meet semilattice.*

4) *$CS(L)$ is a modular meet semilattice.*

Proof: (1) $\Rightarrow$ (2): Let L be a chain and $A, B \in CS(L)$. If $A \not\leq' B$, then there exists an $a \in A$ such that $a \not\leq b$ for any $b \in B$. But then $a > b$ for every $b \in B$ so that $B \leq' A$ and hence $B \leq A$ holds. Similarly if $B \not\leq' A$, then $A \leq B$. Let $A \leq' B$ and $B \leq' A$. We assert that either $A \subseteq B$ or $B \subseteq A$. If $A \not\subseteq B$, then let $a_1 \in A$ and $a_1 \notin B$. Consider any $b \in B$. We have $b \leq a$ for some $a \in A$. If $a \not\leq b$ for any $a \in A$, then $b \leq a$ for every $a \in A$ and hence $b \leq a_1$ also, so that $a_1 \in B$, a contradiction. Therefore $b \geq a$ for some $a \in A$ which implies $b \in A$.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): Obvious.

(4) $\Rightarrow$ (1): Suppose L is not a chain, then L has a four element sublattice as given in Figure 3.

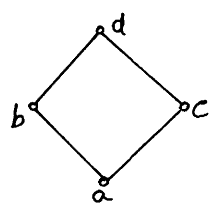


Figure 3

Consider $W, A, B \in CS(L)$ where $W = \{b\}$, $A = [a, b]$, $B = \{c\}$. We have $W \geq A \wedge B = (a]$. If there exist $X, Y \in CS(L)$ with $X \geq A$ and $Y \geq B$ such that $X \wedge Y = X \wedge W$, then since $X \wedge W = W$, $Y \geq W = \{b\}$. Therefore $Y \geq \{d\} \geq [a, b] = A$. But then $W < A \leq X \wedge Y$, a contradiction.

The following theorem characterizes lattices L for which $CS(L)$ is a lattice.

Theorem 2.6. *$CS(L)$ is a join semilattice if and only if L has no sublattice isomorphic with $C_2 \times C_\infty$. (Where C_∞ is the chain of positive integers with the usual ordering).*

Proof: ($\Rightarrow$): Suppose L has a sublattice isomorphic with $C_2 \times C_\infty$ as shown in Figure 4. Consider $A, B \in CS(L)$ where $A = \langle\{a_1, a_2, a_3, \ldots\}\rangle$ and $B = \{b_1\}$.

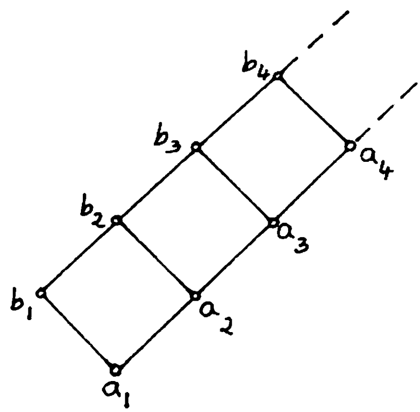


Figure 4

Clearly $\langle\{b_2, b_3, b_4, \ldots\}\rangle$ is an upper bound of $\{A, B\}$ in $CS(L)$. If $X = A \vee B$ exists, then $X \leq \langle\{b_2, b_3, b_4, \ldots\}\rangle$. But $A \leq' X$ and $B \leq' X$ implies that $b_1 \leq x_0$ for some $x_0 \in X$ and for each $i = 1, 2, \ldots$, $a_i \leq x_i$ for some $x_i \in X$. Hence $a_i \vee b_1 \leq x_0 \vee x_i \in X$. Therefore for each $j = 2, 3, \ldots$, $b_j \leq x_j$ for some $x_j \in X$ so that $\langle\{b_2, b_3, b_4, \ldots\}\rangle \leq' X$ and hence $X \subseteq \langle\{b_2, b_3, \ldots\}\rangle = [b_2, b_3] \cup [b_2, b_4] \cup \ldots$. Choose j such that $X \cap [b_2, b_j] \neq \emptyset$. Then $X \cap [b_{j+1}) \geq A, B$ and $X \cap [b_{j+1}) < X$, a contradiction.

($\Leftarrow$): Let $A, B \in CS(L)$ with $A \parallel B$. If $A \leq' B$ and $B \leq' A$, then clearly $A \vee B = \langle A \cup B\rangle$. Let $A \not\leq' B$, $B \not\leq' A$. If $\max\{a \vee b \mid a \in A, b \in B\}$ exists in L, then it equals $A \vee B$. Suppose $\max\{a \vee b \mid a \in A, b \in B\}$ does not exist in L for some $A, B \in CS(L)$ with $A \parallel B$, $A \not\leq' B$ and $B \not\leq' A$. Then there exist an $a_1 \in A$ such that $a_1 \not\leq b$ for any $b \in B$ and a $b_1 \in B$ such that $b_1 \not\leq a$ for any $a \in A$.

Since $\max\{a \vee b \mid a \in A, b \in B\}$ does not exist, either $\max\{a_1 \vee b \mid b \in B\}$ or $\max\{a \vee b_1 \mid a \in A\}$ does not exist. Without loss of generality it can be assumed that $\max\{a_1 \vee b \mid b \in B\}$ does not exist. Then we can choose $x_1, x_2, x_3, \cdots \in L$ and $b_1, b_2, \cdots \in B$ with $x_i = a_1 \vee b_i$, $x_i < x_j$ and $b_i < b_j$ whenever $i < j$ for $i = 1, 2, \ldots$. Clearly $b_j \parallel a_1 \vee b_{j-1}$ for $j = 2, 3, \ldots$.

Case (1): $\{a_1 \wedge b_i \mid i = 1, 2, \ldots\}$ has no maximum.

Then we can find an infinite chain $a_1 \wedge b_1 < a_1 \wedge b_{i_1} < a_1 \wedge b_{i_2} < \ldots$ where $i_1 < i_2 < \ldots$. But then L has a sublattice of the form $C_2 \times C_\infty$ as described in Figure 5.

Case (2): $\{a_1 \wedge b_i \mid i = 1, 2, \ldots\}$ has maximum, say $a_1 \wedge b_n$ for some n.

Then L has a sublattice of the form $C_2 \times C_\infty$ as described in Figure 6.

In either case we get a contradiction.

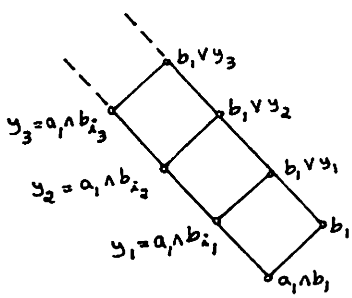


Figure 5

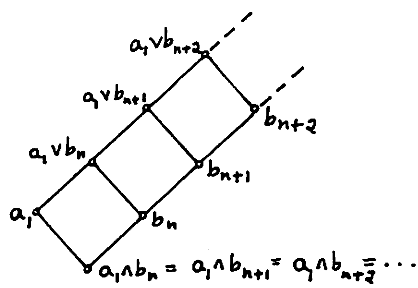


Figure 6

If L is a lattice satisfying the ascending chain condition, then we can define a congruence relation Θ on $CS(L)$ such that the quotient lattice $CS(L)/\Theta$ is isomorphic to L. In fact, we have the following.

Remark 2.7. Let L be a lattice satisfying the ascending chain condition. Then the binary relation Θ on $CS(L)$, defined by, for $X, Y \in CS(L)$, $X \equiv Y$ (Θ) if and only if $\max X = \max Y$ is a congruence relation on $CS(L)$. Also the mapping

$$f: CS(L)/\Theta \to L \text{ defined by, for } X \in CS(L),$$

$f([X]\,\Theta) = \max X$, is an isomorphism.

In the next theorem we derive an expression for the length of the lattice $CS(L)$, when L is a lattice of finite length.

Theorem 2.8. *Let L be a lattice of finite length. Then*

$$\ell(CS(L)) = \frac{\ell(L)\,(\ell(L)+1)}{2} + \ell(L)$$

Proof: By Remark 2.3, it suffices to show that $\ell([\{0\},\{i\}]) = \frac{\ell(L)(\ell(L)+1)}{2}$ in $CS(L)$. Let $\ell(L) = n$ and $0 = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_n = i$ be a maximal chain say C_1 in L of length n. By Remark 2.4 we have $\{0\} \prec \{x_1\} \prec [0,x_1] \prec \{x_2\} \prec [x_1,x_2] \prec [0,x_2] \prec \{x_3\} \prec \cdots \prec \{x_{n-1}\} \prec [x_{n-2},x_{n-1}] \prec \cdots \prec [0,x_{n-1}] \prec \{x_n\} = \{i\}$ is a maximal chain say, C_2, in $[\{0\},\{i\}]$ of $CS(L)$. For each r, $0 \le r \le n-1$, it is clear that there are precisely $r+1$ elements in C_2 with x_r as their maximum and there is one element in C_2 with x_n as its maximum. Therefore

$$\ell(C_2) = 1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

Let $\{0\} = A_0 \prec A_1 \prec \cdots \prec A_m = \{i\}$ be any maximal chain, say C_3, in $[\{0\},\{i\}]$ of $CS(L)$. Consider any j, $1 \le j \le m$. If $\max A_j \ne \max A_k$ for any $1 \le k \le m$, $k \ne j$, then $A_{j-1} < \{\max A_j\} \le A_j$. But since $A_{j-1} \prec A_j$, $\{\max A_j\} = A_j$ and also $j = m$ in this case. If $\max A_j = \max A_k = x$ say, for some $k \ne j$, $1 \le k \le m$, then let t be the least integer such that $\max A_t = x$. Now we have $A_{t-1} < \{\max A_t\} \le A_t$ which implies $A_t = \{\max A_t\}$. Therefore for every j, $\{y_j\} = \{\max A_j\}$ is an element in this chain. Let $y_{p_0}, y_{p_1}, \ldots, y_{p_r}$ be the distinct elements among y_j's, where $p_i < p_j$ if $i < j$. Then $0 = y_{p_0} \prec y_{p_1} \prec \cdots \prec y_{p_r} = i$. For each s, $1 \le s \le r$, if $A_k = \{y_{p_s}\}$, then $A_{k-1} = [0, y_{p_{s-1}}]$ and hence $\ell([\{y_{p_{s-1}}\},\{y_{p_s}\}])$ in $CS(L)$ is less than or equal to $\ell([0,y_{p_{s-1}}])$ in L. But $\ell([0,y_{p_r}]) = \ell([0,i]) = n$. Therefore $\ell([0,y_{p_{r-1}}]) \le n-1$, $\ell([0,y_{p_{r-2}}]) \le n-2$ and so on. Hence

$$\ell(C_3) \le n + (n-1) + \cdots + 1 = \frac{n(n+1)}{2}.$$

If $a_1 > a_2 > a_3 > \ldots$ is an infinite descending chain in a lattice L, then $\{a_1\} < [a_2,a_1] < [a_3,a_1] < \cdots < L$ is an infinite ascending chain in $CS(L)$. Hence we have the following remark.

Remark 2.9: For any lattice L, if $CS(L)$ is of locally finite length, then L is bounded below and hence $CS(L)$ will be of finite length.

Lemma 2.10. *Let L be a lattice with least element 0 and $A_0 \prec A_1 \prec \cdots \prec A_n$ be a maximal chain, say C, connecting A_0 and A_n in $CS(L)$, such that $\max A_i$ and $\min A_i$ exist for every i, $0 \le i \le n$. If $y_{p_0} \prec y_{p_1} \prec \cdots \prec y_{p_m}$ are the distinct elements among $y_i = \max A_i$ for $0 \le i \le n$, then*

$$\ell(C) \le \ell([0,\min A_0]) + \ell([\min A_n, \max A_n]) + \sum_{j=2}^{m} \ell([0,y_{p_j}]) \qquad (1)$$

Proof: Let $1 \le t \le n$ be the greatest integer such that $\max A_0 = \max A_t$ and $1 \le s \le n$ be the least integer such that
$maxA_n = \max A_s$. Let C_1 and C_2 be the subchains $A_0 \prec A_1 \prec \cdots \prec A_t$

and $A_s \prec A_{s+1} \prec \cdots \prec A_n$ of C respectively. Then by Remark 2.4, we have $\min A_t \prec \min A_{t-1} \prec \cdots \prec \min A_0$ where $\min A_t = 0$ and $\min A_n \prec \min A_{n-1} \prec \cdots \prec \min A_s$ where $\min A_s = \max A_n$. Now consider the remaining part $A_t \prec \cdots \prec A_s$, say C_3, of C where $A_{t+1} = \{y_{p_1}\}$ and $A_s = \{y_{p_m}\}$. We have for each j, $1 \leq j \leq m$, $\{y_{p_j}\}$ is an element in C_3 and for $1 \leq j \leq m$, $\ell[\{y_{p_j}\}, \{y_{p_{j+1}}\}] \leq \ell([0, y_{p_{j+1}}])$.

Therefore

$$\begin{aligned}\ell(C) &= \ell(C_1) + \ell(C_2) + \ell(C_3) \\ &\leq \ell([0, \min A_0]) + \ell([\min A_n, \max A_n] + \sum_{j=2}^{m} \ell([0, y_{p_j}]) + 1.\end{aligned}$$

(Note that addition of 1 is essential, since length of the subchain $A_t \prec A_{t+1}$ is not included in the summation).

Remark 2.11. For any lattice L, if L satisfies the Jordan-Dedekind chain condition, then equation (1) in the above theorem holds with equality.

Lemma 2.12. *Let L be a lattice satisfying the descending chain condition. If $A \prec B$ for some $A, B \in CS(L)$ and $\max B$ does not exist, then $\max A$ also does not exist and $\min B \prec \min A$. Also in this case $B \leq' A$.*

From Remark 2.11 and Lemma 2.12, we have the following theorem.

Theorem 2.13. *Let L be a lattice satisfying the descending chain condition. If L satisfies the Jordan-Dedekind chain condition, then $CS(L)$ also satisfies the Jordan-Dedekind chain condition.*

Note: If a lattice L does not satisfy the descending chain condition, then $CS(L)$ need not satisfy the Jordan-Dedekind chain condition even when L satisfies it.

For example, the lattice L in Figure 7 satisfies the Jordan–Dedekind chain condition. But

$$\begin{aligned}\{i, a_1, a_2, \ldots\} &\prec \{i, a_1, a_2, \ldots, b_1, b_2, \ldots\} \\ &\prec \{i, a_1, a_2, \ldots, b_1, b_2, \ldots, c_1, c_2, \ldots\} \prec L\end{aligned}$$

and

$$\{i, a_1, a_2, \ldots\} \prec \{i, d_1, d_2, \ldots, a_1, a_2, \ldots\} \prec L$$

are two maximal chains between $\{i, a_1, a_2, \ldots\}$ and L in $CS(L)$ of unequal lengths.

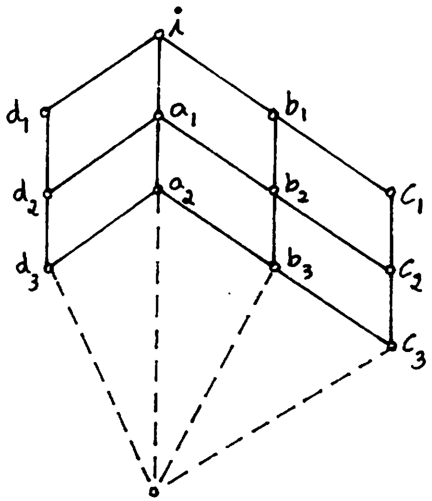


Figure 7

3 Equivalence of planarity for L and $CS(L)$

In this section, using a result of D. Kelly and I. Rival [2], we derive the main result of the paper which is obtained as a corollary of following theorem.

Theorem 3.1. *Let L be a finite lattice and $P = \{C_1, C_2, C_3, \ldots, C_n\}$, $n \geq 4$ be a subposet of $CS(L)$ having the following two properties (where $\propto$ denotes "is comparable to").*

1) *For each C_i, $i \geq 3$, there exist C_j, $C_k \in P$ with j, $k < i$ such that $C_i \propto C_j$, but $C_i \parallel C_k$.*

2) *For every pair (C_i, C_j), $i < j$, $j \geq 4$, there exists $C_k \in P$, $k < j$ such that $C_k \propto C_j$ but $C_k \parallel C_i$, or $C_k \propto C_i$ but $C_k \parallel C_j$.*

Then L has a subposet order isomorphic with P or P^d.

Proof: We separate the two cases $\max C_1 = \max C_2$ and $\max C_1 \neq \max C_2$.
Case (1): $\max C_1 = \max C_2$

From (1) we have $C_3 \propto C_1$ and $C_3 \parallel C_2$, or $C_3 \propto C_2$ and $C_3 \parallel C_1$. If $C_3 \propto C_1$, then $\max C_3 \propto \max C_1 = \max C_2$. But since $C_3 \parallel C_2$ we get $\max C_3 = \max C_2 = \max C_1$. Also, if $C_3 \propto C_2$, then $\max C_3 \propto \max C_2 = \max C_1$ and since $C_3 \parallel C_1$, $\max C_3 = \max C_2 = \max C_1$. In general, for any $j = 2, 3, \ldots, n-1$ if $\max C_i = \max C_1$, for every $i \leq j$, then since by (1), $C_{j+1} \propto C_{k_1}$ and $C_{j+1} \parallel C_{k_2}$ for some k_1, $k_2 \leq j$, it follows that $\max C_{j+1} = \max C_1$. Therefore $\max C_1 = \max C_2 = \cdots = \max C_n$.

Now, consider the subposet $L_1 = \{\min C_1, \min C_2, \dots, \min C_n\}$ of L. Then the map $f\colon L_1 \to p^d$ defined by $f(\min C_i) = C_i$ for $i = 1, 2, \dots, n$ is an order isomorphism.

Case 2: $\max C_1 \neq \max C_2$

We prove by induction on n, that $\max C_1 \neq \max C_j$ for any $1 \leq i, j \leq n$.

Let $n = 4$. Then $P = \{C_1, C_2, C_3, C_4\}$ is of one of the types as shown in Figure 8 or their duals.

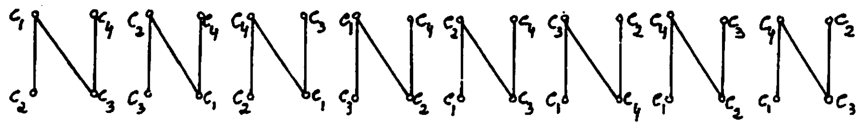


Figure 8

From the figures, using Remark 2.4, it follows that if $\max C_i = \max C_j$ for any $1 \leq i, j \leq 4$, then $\max C_i = \max C_j$ for every $1 \leq i, j \leq 4$. Assume that if $P = \{C_1, C_2, \dots, C_{n-1}\}$, $n > 4$, then $\max C_i \neq \max C_j$ for any $1 \leq i, j \leq n-1$.

Let $P = \{C_1, C_2, \dots, C_n\}$. By the assumption we have $\max C_i \neq \max C_j$ for any $1 \leq i, j \leq n-1$. Let $\max C_i = \max C_n$ for some $i < n$. By (2), there exists C_j, $j < n$ such that $C_j \propto C_i$ and $C_j \parallel C_n$, or $C_j \propto C_n$ and $C_j \parallel C_i$. In any case we get $\max C_i = \max C_j$, a contradiction. By induction it follows that $\max C_i \neq \max C_j$ for any $1 \leq i, j \leq n$. Consider the subposet $L_2 = \{\max C_1, \max C_2, \dots, \max C_n\}$ of L. Define the map $f\colon L_2 \to P$ by $f(\max C_i) = C_i$ for $i = 1, 2, \dots, n$. Then f is an order isomorphism.

Corollary 3.2. *A finite lattice L is planar if and only if $CS(L)$ is planar.*

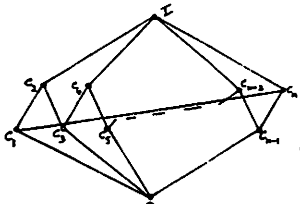
Proof: D. Kelly and I. Rival [3] prove that a finite lattice is planar if and only if it does not contain any lattice in M as a subposet where

$$M = \{A_n \mid n \geq 6\} \cup \{B, B^d, C, C^d, D, D^d\} \cup \{E_n, E_n^d \mid n \geq 4\}$$
$$\cup \{F_n \mid n \geq 4\} \cup \{G_n \mid n \geq 3\} \cup \{H_n \mid n \geq 6\}$$

A_n, B, C, D, E_n, F_n, G_n and H_n are as shown in the figures 9–16.

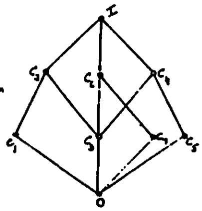
Let L be a planar finite lattice. Suppose $CS(L)$ is not planar, then it contains some lattice $X \in M$ as a subposet. But $X - \{0, I\}$ satisfies the conditions of the theorem with $C_1, C_2, \dots, C_n$ as marked in the figures. Therefore L contains a subposet say Y_1 order isomorphic to $X - \{0, I\}$ or $(X - \{0, I\})^d$. But then by adjoining the least element 0 and the greatest element i of L to Y_1 and mapping them to 0 and I of X respectively, Y_1 will be isomorphic to X or X^d, a contradiction to the planarity of L.

Converse part follows from Remark 2.3, by noting that L is isomorphic to $D(L)$.



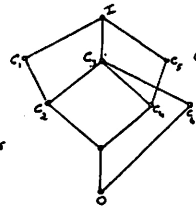
A_n $(n \geq 6)$

Fig. 9



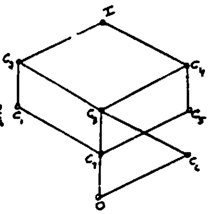
B

Fig. 10



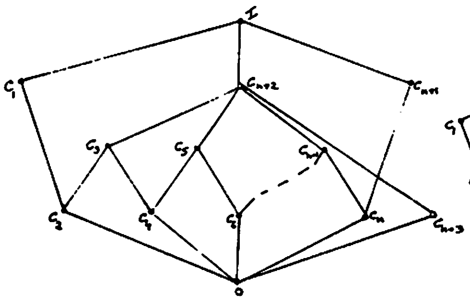
C

Fig. 11



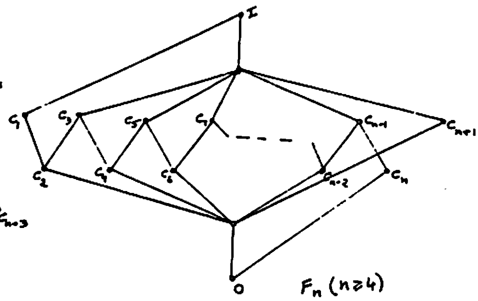
D

Fig. 12



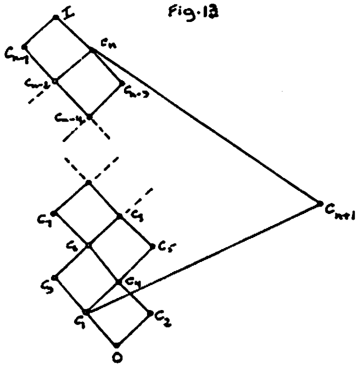
E_n $(n \geq 4)$

Fig. 13



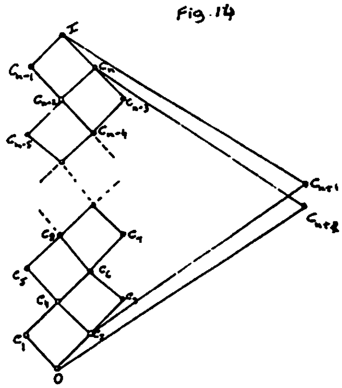
F_n $(n \geq 4)$

Fig. 14



G_n $(n \geq 3)$

Fig. 15



H_n $(n \geq 6)$

Fig. 16

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References

[1] C.C. Chen and K.M. Koh, On the lattice of convex sublattices of a finite lattice, *Nanta Math*, **5** (1972), 92–95.

[2] G. Gratzer, *General lattice theory*, Birkhauser Verlag, Basel, 1978.

[3] D. Kelly and I. Rival, Planar lattices, *Canad J. Math.*, **27** No. 3, (1975), 636–665.

[4] K.M. Koh, Lattices and their sublattice-lattices, *SEA Bull Math.* **10**, No. 2 (1986), 128–135.

[5] K.M. Koh, On the lattice of convex sublattices of a lattice, *Nanta Math*, **6** (1972), 18–37.

[6] K.M. Koh, On the complementation of the $CS(L)$ of a lattice L, *Tamkang J. Math*, **7** (1976), 145–150.

[7] S. Lavanya and S. Parameshwara Bhatta, A New approach to the lattice of convex sublattices of a lattice, to appear in *Algebra Universalis.*

[8] S. Lavanya and S. Parameshwara Bhatta, A New ordering on the set of all sublattices of a lattice, *Southeast Asia. Bull. Math.* **18**, No.2 (1994), 43–49.

[9] V.I. Marmazeev, The lattice of convex sublattices of a lattice (Russian), *Ordered sets and lattices*, **9** (1986), 50–58, 110–111, Saratov Gos Univ., Saratov.

[10] J.B. Rhodes, Modular and distributive semilattices, *Trans. Amer. Math. Soc.*, **201** (1975), 31–41.

[11] L. Skornjakov, *Elements of lattice theory*, 1977.