Beyond Chordal Graphs

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ABSTRACT. Much of chordal graph theory and its applications is based on chordal graphs being the intersection graphs of subtrees of trees. This suggests also looking at intersection graphs of subgraphs of chordal graphs, and so on, with appropriate conditions imposed on the subgraphs. This paper investigates such a hierarchy of generalizations of "chordal-type" graphs, emphasizing the so-called "ekachordal graphs" — those next in line beyond chordal graphs. Parts of the theory of chordal graphs do carry over to chordal-type graphs, including a recursive, elimination characterization for ekachordal graphs.

1 Introduction

Chordal graphs — those that contain no induced cycles of length four or more — form one of the most structured, most studied and most applied families of graphs. Among their many characterizations, perhaps the most intrinsic is that they are precisely the intersection graphs of subtrees of trees; see [3, Chapter 4] or [2] for more information. Indeed, several of the richest applications of graph theory (especially to statistics and matrices; see [6]) exploit chordal graphs being intersection graphs.

This paper introduces a hierarchy of nested, increasingly larger families of "chordal-type" graphs. The chordal-type-0 graphs are precisely the forests. For positive t, the chordal-type-t graphs are the intersection graphs of families of chordal-type-$(t-1)$ graphs that satisfy conditions given in the next section. (This means that chordal-type-t graphs are also chordal-type-t' for

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all $t' > t$.) These conditions virtually disappear when $t = 1$, making the chordal-type-1 graphs precisely the chordal graphs.

There are, predictably, many different ways to generalize chordal graphs. The approach taken here is to put appropriate conditions onto this natural generalization of chordal so as to retain as much as possible of the structure and results that proved important in [6]. Section 2 discusses these conditions and the resulting chordal-type graphs. Section 3 focuses on chordal-type-2 graphs — those that stand next in line in the sequence $\langle forest, chordal, \ldots \rangle$. These "ekachordal graphs" have a recursive characterization that extends the traditional perfect elimination orderings for chordal graphs. Section 4 surveys a variety of open questions and suggestions for future work.

2 Chordal-Type Graphs

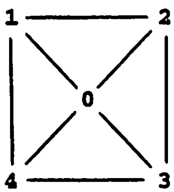
Define a graph to be chordal-type-0 if it is a forest. For each positive integer t, define a graph G to be chordal-type-t if it is the intersection graph of a family $\{H_v : v \text{ a vertex of } G\}$ of induced subgraphs of a *host* chordal-type-$(t-1)$ graph H such that the following three conditions hold on the complete subgraphs of the *guest* graph G and the host graph H.

- **Clique intersection condition:** If Q is any complete subgraph of G, then $H_Q =_{def} \cap\{H_v : v \in Q\}$ is nonempty, connected and chordal-type-$(t-1)$.
- **Clique union condition:** If Q is any complete subgraph of G, then the subgraph induced by $\cup\{H_v : v \in Q\}$ is chordal-type-$(t-1)$.
- **Clique cover condition:** Every complete subgraph of H is contained in some H_v.

A *chordal-type* graph is one that is chordal-type-t for some $t \geq 0$. Notice in particular that each H_v must be connected and chordal-type. The clique intersection condition can be viewed as a strengthened Helly condition since it can be rephrased as follows: If $v_1, \ldots, v_k$ are vertices of G such that $1 \leq i \leq j \leq k$ implies $H_{v_i} \cap H_{v_j} \neq \emptyset$, then the induced subgraph $\bigcap_{i \leq k} H_{v_i}$ of H is nonempty, connected and chordal-type-$(t-1)$; weakening the requirement just to $\bigcap_{i \leq k} H_{v_i} \neq \emptyset$ would produce the traditional Helly condition. (The choice of these three conditions is discussed further in §4.1.)

Call the vertices of a chordal-type-$(t-1)$ host graph H *nodes* to prevent confusion with the vertices of the guest graph G. Each node of H corresponds to the complete subgraph of G induced by those vertices v of G for which H_v contains that node; this allows us to say that a vertex or complete subgraph of G is *contained in a node*. The *maxcliques* of a graph are its inclusion-maximal complete subgraphs.

Figure 1 shows that the 4-spoked wheel $G = W_5$ is chordal-type-2 by exhibiting a chordal-type-1 (= chordal) host H. The node 012 of H corresponds to the maxclique $\langle 0, 1, 2\rangle$ of G and so *contains* the vertices $0, 1, 2$ of G. Thus H_1 is the path induced by the two nodes of H that contain 1, $H_{\langle 0,1\rangle} = H_1$, H_0 is all of H, etc. (The matrix B will be explained in the proof of Theorem 2.)



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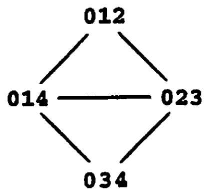


H

1	5	-8	4
4	12	-12	4
-5	-9	4	0
2	2	0	0

B

Figure 1

Lemma 1 *The chordal-type-t graphs are closed under edge contraction.*

Proof. Suppose G is chordal-type-t and note that the $t = 0$ case (G a forest) is trivial. Suppose $t \geq 1$, H is a chordal-type-$(t-1)$ host for G, and G/uv is G with edge uv contracted into the vertex v (u disappearing). Let H^- be H with each occurrence of u contained in a node replaced by v. Since H^- has not changed as a graph, it is still chordal-type-$(t-1)$. Since H_v^- is induced by $H_u \cup H_v$ and $x \notin \{u, v\} \Rightarrow H_x^- = H_x$, the clique union condition ensures G/uv is a guest graph with host H^- with respect to subgraphs $\{H_x : x \in G/uv\}$. Thus G/uv is chordal-type-t. □

The parallels between the theory of chordal-type-t graphs (for positive t) and the theory of chordal graphs as laid out in [6] begin with the following basic result.

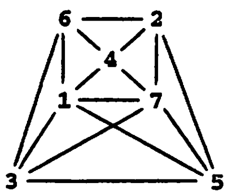
Theorem 1 *The nodes of a host chordal-type-$(t-1)$ graph for a guest chordal-type-t graph G can always be taken to be (precisely, without repetition) the maxcliques of G.*

Proof. Suppose H is any chordal-type-$(t-1)$ host with a guest chordal-type-t graph G. Argue by induction on t, with the basis $t = 1$ case (G chordal) well-known [3, Theorem 4.8]. Suppose $t \geq 2$. If Q is any maxclique of G, then the clique intersection condition guarantees that $H_Q \neq \emptyset$ and so Q is contained in some node of H; indeed Q must actually be a node, since any other vertex of G contained in that node would have to be adjacent to every vertex of Q and so would be in Q. Thus each maxclique of G occurs as a node in H. All we need to show is that there exists a host H for which no node (viewed as a set of vertices of G) contains another node.

Suppose H has minimum possible order among all possible chordal-type-$(t-1)$ hosts for G and suppose (arguing toward a contradiction) H contains two distinct nodes Q and Q' with $Q \subseteq Q'$; since H_Q is connected and $|H_Q| \geq 2$, we can choose Q' so that QQ' is an edge of H. By Lemma 1, contracting edge QQ' into the node Q' (Q disappearing) would produce a chordal-type-$(t-1)$ host for G of smaller order than H. This contradicts the assumed minimality of H and so completes the proof. □

Call a chordal-type-$(t-1)$ graph H and a family $\{H_v : v \in G\}$ of subgraphs of H with guest chordal-type-t graph G a *clique* $(t-1)$-*host* for G whenever the nodes of H correspond (precisely, without repetition) to the maxcliques of G. The graph in Figure 1 is a clique 1-host for $G = W_5$. Observe that the clique cover condition (ensuring that the "diagonal" edge of H has some vertex v contained in both its end nodes) prevents C_4 from having a clique 1-host obtained from H by removing 0 from each node. Indeed, when $n \geq 4$, C_n is not even chordal-type: a clique $(t-1)$-host would have to have n nodes, the clique intersection condition would require n edges forming a cycle, and the clique cover condition would prevent all other edges — i.e., C_n would be the only possible host of itself. All wheels are chordal-type-2, as are also any join of P_k with ℓK_1. (The *join* of two graphs is their union plus new edges joining each vertex of one with each vertex of the other.) As illustrated by C_4 being an induced subgraph of W_5, the chordal-type-t graphs are not closed under induced subgraphs when $t > 1$.

The complement of P_7 is a simple example of a chordal-type-3 graph that is not chordal-type-2; Figure 2 shows it has a natural clique 2-host based on W_7. (Showing that there is no clique 1-host is instructive practice with the three conditions.)



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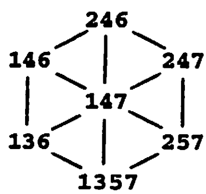

H

1	7	-15	10	-1
7	22	-24	10	-1
-12	-21	9	0	0
6	6	0	0	0

B

Figure 2

By the clique cover condition, each complete subgraph of a clique $(t-1)$-host H can be associated with a complete subgraph of the host G induced by the intersection of the host nodes. For instance in Figure 1, each node of H is associated with a triangle of G, each "perimeter" edge of H with a "spoke" edge of G, the "diagonal" edge of H with the vertex 0 of G, and both the triangles of H with the vertex 0 of G. For each $i \geq 1$, let $H(i)$ consist of all the (guest) complete subgraphs of G that are associated with (host) complete subgraphs of order i of H. So $H(1)$ is the set of maxcliques of G. Since different complete subgraphs of H of orders greater than one may correspond to the same complete subgraph of G, it is important to interpret $H(2), H(3), \ldots$ as **multisets** (i.e., sets with repeated elements). For instance in Figure 1, $H(2)$ ={01, 02, 03, 04, 0} and $H(3) = \{0, 0\}$; in Figure 2, $H(2)$ contains nine edges and three vertices and $H(3) = \{1, 1, 4, 4, 7, 7\}$.

For any graph G and positive integer i, define $k_i(G)$ to be the number of complete subgraphs of order i in G; thus $k_1(G)$, $k_2(G)$ and $k_3(G)$ count, respectively, the numbers of vertices, edges and triangles in G. Define $k_0(G) = 1$. Let $\chi(G) = k_1(G) - k_2(G) + k_3(G) - k_4(G) + \cdots$ be the (Euler) *characteristic* of G. Theorem 2 generalizes [6, equality (5)].

Theorem 2 *Every connected chordal-type graph has characteristic one.*

Proof. Suppose G is a connected chordal-type-t graph, noting that the result is immediate when $t = 0$ (G is a tree). Suppose H is a clique $(t-1)$-host for G. Argue by induction on $t \geq 1$, noting that the basis $t = 1$ case (G chordal) is known from [6]. Define a matrix A with entries

$$a_{ij} = (-1)^{i+j} \sum_{Q \in H(i)} \binom{|Q|}{j};$$

i.e., $|a_{ij}|$ counts the number of occurrences of (guest) K_j's of G inside (host) K_i's of H, with entries negated in the usual checkerboard pattern. Define matrix B from A by augmenting a set-off 0th column of row sums and 0th row of column sums; $b_{00} = \sum_j b_{0j} = \sum_i b_{i0}$ is thus the sum of all the entries of A. See Figures 1 and 2.

For each $i \geq 1$, each host complete subgraph $Q \in H(i)$ has (as a complete subgraph of G) characteristic one. Thus each $Q \in H(i)$ corresponds to an equality $1 = k_1(Q) - k_2(Q) + \cdots$ and adding these (if i odd, or their negations if i even) for every $Q \in H(i)$ gives $|H(i)| = |b_{i1} + b_{i2} + \cdots| = |b_{i0}|$. (In the example of Figure 1 with $i = 2$, this means adding four $(1 = 2 - 1 + 0)$'s [from the "perimeter" edges of H] and one $(1 = 1 - 0 + 0)$ [from the "diagonal" edge of H] to get $5 = 9 - 4 + 0$.) Hence $|H(1)| - |H(2)| + \cdots = \chi(H) = b_{00}$ and so $b_{00} = 1$ by the inductive hypothesis.

For each guest complete subgraph Q of G, H_Q is chordal-type-$(t-1)$ and so $\chi(H_Q) = 1$ by the induction hypothesis. For every $j \geq 1$, each order-j guest Q corresponds to an equality $1 = k_1(H_Q) - k_2(H_Q) + \cdots$ and adding these (if j odd, or their negations if j even) for every such Q gives $|k_j(G)| = |b_{1j} + b_{2j} + \cdots| = |b_{0j}|$. (In the example of Figure 1 with $j = 2$, this means adding four $(1 = 2 - 1 + 0)$'s [from the "spoke" edges of G] and four $(1 = 1 - 0 + 0)$'s [from the "felly" edges of G] to get $8 = 12 - 4 + 0$.) Hence $k_1(G) - k_2(G) + \cdots = \chi(G) = b_{00} = 1$. □

As a corollary, cycles C_n with $n \geq 4$, $K_{3,3}$ and the octahedron $K_{2,2,2}$ (having characteristics 0, −3 and 2 respectively) are not chordal-type. The converse of Theorem 2 fails, as the family of graphs of characteristic one is not closed under edge contraction as would be required by Lemma 1. Starting with the octahedron $K_{2,2,2}$ and introducing a seventh vertex adjacent to two nonadjacent vertices produces a graph that has characteristic one but that is not chordal-type (having an edge contraction to a graph of characteristic zero). On the other hand, starting with the octahedron $K_{2,2,2}$ and introducing a seventh vertex adjacent to all the vertices of the octahedron produces a chordal-type-2 graph (with host isomorphic to K_8) that does have characteristic one.

Theorem 3 generalizes the basic "arboreal equalities" from [6].

Theorem 3 *Suppose H is a clique $(t-1)$-host for a guest chordal-type-t graph G. Then, for all $j \geq 0$,*

$$k_j(G) = \sum_{Q \in H(1)} k_j(Q) - \sum_{Q \in H(2)} k_j(Q) + \sum_{Q \in H(3)} k_j(Q) - \cdots . \qquad (1)$$

Proof. Suppose H is a clique $(t-1)$-host with guest G. The $j = 0$ case follows from Theorem 2 since (1) becomes $1 = \sum(-1)^i |H(i)| = \chi(H)$. The $j \geq 1$ cases follow by summing all the identities $1 = \chi(H_Q)$ over all order-i complete subgraphs Q of G (as in the proof of Theorem 2). □

3 Ekachordal (= Chordal-Type-2) Graphs

The definition of chordal-type-t simplifies considerably when $t = 2$: the clique intersection condition merely requires that each H_Q be nonempty and connected, since induced subgraphs of chordal-type-1 (i.e., chordal) graphs are chordal; the clique union condition is automatically true; and the clique cover condition remains unchanged (and, while it is not traditionally required in chordal graph theory, it is harmless).

Because the chordal-type-2 graphs seem to form a particularly natural family standing next in order beyond chordal graphs, and because there is no "simple" characterization (paralleling "no induced cycles" for chordal) yet known, we call chordal-type-2 graphs *ekachordal graphs*. (The prefix *eka-* is from chemistry, where it is used "to denote provisionally a predicted element that should stand next in order to a given element.") Thus wheels are examples of ekachordal graphs, but the complement of P_7 in Figure 2 is not. (Appropriating another Sanskrit-based chemical prefix, that graph would be "dvichordal," i.e., in the next family beyond ekachordal graphs.)

The next theorem shows how to recognize whether a given H with family $\{H_v : v \in G\}$ is a host for G, and so whether G is ekachordal, without directly verifying the clique intersection and clique cover conditions on all the guest and host complete subgraphs. (This parallels [6, Theorem 3], except there only $j = 1$ was needed.)

Theorem 4 *Suppose H is a chordal graph whose nodes are precisely the maxcliques of a connected graph G and each member of an $H(i)$ (defined as above) is a nonempty complete subgraph of G. Then H and $\{H_v : v \in G\}$ form a clique 1-host for G (and so G is an ekachordal graph with clique 1-host H) if and only if equality* (1) *holds for all $j \geq 1$.*

Proof. Suppose H and G are as in the theorem. The "only if" direction is from Theorem 3. For the converse, suppose (1) holds for all $j \geq 1$. Vertices u and v are adjacent in G if and only if they are in a common complete subgraph of G, and that is equivalent (since the nodes of H are the maxcliques of G) to $H_u \cap H_v \neq \emptyset$; thus G is the intersection graph of the family $\{H_v : v \in G\}$. The clique cover condition holds since each $H(i) \neq \emptyset$. All that needs to be checked is the clique intersection condition: that H_Q is connected whenever Q is a complete subgraph of G. Since each H_Q is chordal, Theorem 2 implies that $\chi(H_Q)$ is the number of components of H_Q; i.e., $1 \leq \chi(H_Q)$ with equality if and only if H_Q is connected. Comparing the sum of the inequalities $1 \leq \chi(H_Q)$ over all order-j complete subgraphs of G with the equality (1) shows that each $\chi(H_Q) = 1$ and so each H_Q must be connected. □

The following recursive approach to ekachordal graphs, while regrettably complicated when the parameter s is larger than one, does lead to a useful characterization. For $s \geq 1$, define the *s-elements* of a graph to be all its complete subgraphs of orders less than or equal to s. Use the symbols $\cap^s$ and $\cup^s$ instead of the usual set symbols $\cap$ and $\cup$ to signify working with respect to all s-elements, rather than just vertices (and rather than a graph-theoretic operation that employs set-theoretic notation). Define an *s-clique elimination ordering* (abbreviated *s-CEO*) to be an ordering $Q_1, \ldots, Q_c$ of all the maxcliques of G (each viewed as a set of s-elements) with, for each $i \in \{1, \ldots, c-1\}$, a number $s_i \in \{1, \ldots, s\}$ and indices $f(i,1), \ldots, f(i,s_i)$ such that $i < f(i,1) < \cdots < f(i,s_i) \leq c$ such that the following three conditions hold:

(i) If $1 \leq j_1 < j_2 \leq s_i$, then for some k, $f(i,j_2) = f(f(i,j_1),k)$.

(ii) $Q_i \cap^s (\bigcup^s_{j>i} Q_j) \subseteq Q_{f(i,1)} \cup^s \cdots \cup^s Q_{f(i,s_i)}$.

(iii) $Q_i \cap (Q_{f(i,1)} \cap \cdots \cap Q_{f(i,s_i)}) \neq \emptyset$.

When $s = 1$, s-elements are just vertices and each $s_i = 1$. Condition (i) is vacuous. Condition (ii) means that each $f(i,1) > i$ is an index of some maxclique of G that contains $Q_i \cap (Q_{i+1} \cup \cdots \cup Q_c)$, where $Q_i \backslash (\cup_{j>i} Q_j)$ is a maximal set of pairwise-adjacent *simplicial vertices* (vertices in a unique maxclique) of the induced subgraph $\langle Q_i, \ldots, Q_c \rangle$. (Notice that, since each $i \in \{1, \ldots, c-1\}$ determines an index $f(i,1) \in \{i+1, \ldots, c\}$, it must be that $f(1,1) = 2$, $f(f(1,1),1) = 3$, etc.) Thus the 1-CEO is a conventional *perfect elimination ordering* (an ordering $v_1, \ldots, v_n$ of the vertices of G such that each v_i is simplicial in the induced subgraph $\langle v_i, \ldots, v_n \rangle$ of G); by the traditional theory of chordal graphs (see [3] or [2]), this is equivalent to G being chordal. Condition (iii) means that G is connected, and so having a 1-CEO corresponds to being a connected chordal graph. (Such clique elimination orderings are called "running intersection orderings" in [1] and [2] and "creation orderings" in [5]; they are also used in the statistics literature cited in [6].) Think of the vertices as being stripped off so as to destroy the maxcliques one at a time, with each Q_i linked to the remaining maxcliques by one maxclique $Q_{f(i,1)}$.

In a 2-CEO, the vertices and edges (the 2-elements) are again stripped off to destroy maxcliques one at a time, but now with each Q_i linked to the remaining maxcliques either by one maxclique, $Q_{f(i,1)}$, or by two maxcliques, $Q_{f(i,1)}$ and $Q_{f(i,2)}$, and if by two, then the first will be linked to the second later on (i.e., $f(f(i,1),k) = f(i,2)$ for some $k \in \{1,2\}$) by condition (i). As an example of a 2-CEO for G in Figure 1, take $Q_1 = \{0, 1, 2, 01, 02, 12\}$ to be all 2-elements of $\langle 0,1,2 \rangle$, Q_2 to be all 2-elements of $\langle 0,2,3 \rangle$, Q_3 to be all 2-elements of $\langle 0,3,4 \rangle$, Q_4 to be all 2-elements of $\langle 0,1,4 \rangle$, $s_1 = s_2 = 2$, and $s_3 = 1$. Put $f(1,1) = 2$, $f(1,2) = 4$, $f(2,1) = 3$, $f(2,2) = 4$ and $f(3,1) = 4$. For condition (i), note that $f(1,2) = 4 = f(2,2) = f(f(1,1),2)$ (so for $i = 1$, $j_1 = 1$ & $j_2 = 2 \Rightarrow k = 2$) and $f(2,2) = 4 = f(3,1) = f(f(2,1),1)$ (so for $i = 2$, $j_1 = 1$ & $j_2 = 2 \Rightarrow k = 1$). For condition (ii), note that $Q_1 \cap^2 (Q_2 \cup^2 Q_3 \cup^2 Q_4) = \{0, 1, 2, 01, 02\} \subseteq Q_{f(1,1)} \cup^2 Q_{f(1,2)} = Q_2 \cup^2 Q_4 = \{0, 1, 2, 3, 4, 01, 02, 03, 04, 14, 23\}$ and $Q_2 \cap^2 (Q_3 \cup^2 Q_4) = \{0, 3, 03\} \subseteq Q_{f(2,1)} \cup^2 Q_{f(2,2)} = Q_3 \cup^2 Q_4 = \{0, 1, 3, 4, 01, 03, 04, 14, 34\}$ and $Q_3 \cap^2 (Q_4) = \{0, 4, 04\} \subseteq Q_{f(3,1)} = Q_4 = \{0, 1, 4, 01, 04, 14\}$. For condition (iii), each expression contains 0 and so is nonempty. The join of P_4 with $3K_1$ is an example of an ekachordal graph that has a 3-CEO, but no 2-CEO.

Theorem 5 *A connected graph is ekachordal if and only if it has an s-CEO for some value s.*

Proof. First suppose H is a clique 1-host for a connected ekachordal graph G, and (since H is chordal) that $Q_1, \ldots, Q_c$ is a perfect elimination ordering for H. Let s be the order of a largest maxclique of H. For each i, let $f(i,1) < \cdots < f(i,s_i) \leq s$ be the subscripts of the neighbors of Q_i in $\langle Q_i, \ldots, Q_c \rangle$ inside H. Condition (i) of the definition of s-CEO follows from the definition of perfect elimination ordering, since every two neighbors Q_{j_1}, Q_{j_2} of Q_i will be adjacent in $\langle Q_i, \ldots, Q_c \rangle$, and so either Q_{j_2} will be a neighbor of Q_{j_1} in $\langle Q_{j_1}, \ldots, Q_c \rangle$ inside H or similarly with j_1, j_2 reversed. Condition (ii) follows from the clique intersection condition, since $Q \subseteq Q_i \cap^s Q_j$ $(j > i)$ implies H_Q contains a Q_i-to-Q_j path beginning with some edge $Q_i Q_{f(i,k)}$, showing $Q \subseteq Q_{f(i,k)}$. Condition (iii) follows from G being connected and the clique cover condition, since $Q_i \cap (Q_{f(i,1)} \cap \cdots \cap Q_{f(i,s_i)})$ is a complete subgraph of H, so it is contained in some H_v and so it will contain this v.

Conversely, suppose $Q_1, \ldots, Q_c$ is an s-CEO for G. Define H on the nodes $Q_1, \ldots, Q_c$ such that Q_i is adjacent to Q_j $(i < j)$ if and only if there exists a k such that $f(i,k) = j$. Then G is the intersection graph of the H_v's since the nodes for H are the maxcliques of G (arguing as in the proof of Theorem 4). That H is a clique 1-host for G can be argued by induction on c, noting that the basis case $c = 1$ (G is complete) is trivial. In showing that H is chordal, note that Q_1 will be simplicial in H by condition (i) of the definition of s-CEO, and so the induction hypothesis can be used. In showing the clique intersection condition, suppose Q is any complete subgraph of G and $Q \subseteq Q_1 \cap^s Q_j$ where $1 < j$; by condition (ii) there will be an edge $Q_1 Q_{f(i,k)}$ in H_Q, and so the induction hypothesis can be used. To show the clique cover condition, any complete subgraph of H that contains Q_1 will have to be contained in $Q_1 \cap (Q_{11} \cap \cdots \cap Q_{f(1,s_1)}) \neq \emptyset$ by condition (iii). □

A chordal graph whose largest clique has order at most $s+1$ has been called an *s-chordal graph*. Thus the 1-chordal graphs are the forests. Define a graph to be an ***s-ekachordal graph*** if it is ekachordal with an s-chordal clique 1-host, so that the 1-ekachordal graphs are precisely the chordal graphs. The 2-ekachordal graphs form a very conservative expansion of the class of chordal graphs (but still include, for instance, all wheels, but not the join of P_4 with $3K_1$).

Theorem 6 *A connected graph is s-ekachordal if and only if it has an s-CEO.*

Proof. This follows as a refinement of the proof of Theorem 5. □

4 Directions for Further Work

4.1 Conditions: Is the definition of chordal-type-t in Section 1 the "right one"? Conditions are certainly needed to produce interesting families of graphs, conditions that somehow reflect those aspects of trees that underlie chordal graph theory, but are these the best choices? When $t = 1, 2$ the conditions coalesce into simple, natural ones, but the choice becomes less certain beyond $t = 2$ — i.e., beyond ekachordal graphs.

In particular: Is the clique union condition necessary? Could somewhat weaker conditions give the same results? Would somewhat stronger conditions give "nicer" results?

4.2 Characterizations: Is there a "simple" characterization of chordal-type graphs? Or of ekachordal, s-ekachordal or 2-ekachordal graphs? (Equivalently, what is a more descriptive name for ekachordal graphs?) Can more be made of the role of the characteristic χ? (Compare: A connected graph is chordal if and only if it and all its connected induced subgraphs have characteristic one.) Can the notion of s -CEO be modified for chordal-type-3 graphs?

4.3 Representation Procedures: One of the most important aspects of chordal graph theory as surveyed in [6] (or from a somewhat different viewpoint in [2]) is the use of a greedy algorithm to construct clique 0-hosts (clique trees) for chordal graphs. How can clique 1-hosts be constructed for ekachordal graphs (even knowing all their maxcliques)? Such a procedure could also be used on an arbitrary graph, and then Theorem 4 used to test for being ekachordal. How can 2-chordal clique 1-hosts be constructed for 2-ekachordal graphs?

4.4 Identification of $H(i)$'s: For a chordal graph with various clique 0-hosts H, $H(2)$ is uniquely determined and its members (and their multiplicities) can be identified graph-theoretically; see [6]. But for an ekachordal graph, the $H(i)$'s are not even uniquely determined. There appears to be hope, however, that if $H(4) = \emptyset$, then the terms in (1) after cancellation are uniquely determined by G (and so the corresponding members of $H(2)$ and $H(3)$ should be identifiable graph-theoretically).

4.5 Within Chordal-Type Graphs: The s-chordal and s-ekachordal graphs can be viewed as the first two rows of a "periodic table" of chordal-type graphs. Define *s-chordal-type-t graphs* inductively: s-chordal-type-1 if s-chordal; s-chordal-type-t if the intersection graph of subgraphs (satisfying the clique intersection, union and cover conditions) of an s-chordal-type-$(t-1)$ graph. The s-chordal and s-ekachordal graphs are, respectively, those of s-chordal-type-1 and -2. The chordal-type-t graphs are those of s-chordal-type-t for some s or (equivalently!) those of 1-chordal-type-$(t+1)$. Is there a "simple" characterization of (or a representation procedure for) s-chordal-type-t graphs?

Notice that we could mimic the above using any other hierarchy of chordal graphs instead of being s-chordal; e.g., having leafage at most s as in [5], or having rank at most s as in [4].

4.6 Applications: Since the emphasis of [6, Section 4] was on how chordal graphs are applied to matrices and statistics, will some vestigial forms of these applications extend to chordal-type graphs? Or to ekachordal graphs? Or at least to 2-ekachordal graphs? Is there a way to generalize the Decomposition Theorem of [6] to these broader contexts?

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