The Genus of a Minimal Connected Knight's Graph

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ABSTRACT. It has been proved that the smallest rectangular board on which a (p, q)-knight's graph is connected has sides $p + q$ by $2q$ when $p < q$. It has also been proved that these minimal connected knight's graphs are of genus 0 or 1, and that they are of genus 0 when p and q of the form $Md + 1$ and $(M + 1)d + 1$ with M a non-negative integer and d is a positive odd integer. It is proved in this paper that the minimal connected knight's graph is of genus 1 in all other cases.

1 Introduction

This paper has its roots in the study of the discrete metrics associated with knights' moves on an unrestricted chess board [1] and generalised knights' moves on unrestricted boards [2]. For applications to pattern recognition it is helpful to know the sizes of the restricted rectangular boards on which these discrete metrics are defined, i.e. on which the knights' graphs are connected. The sizes of the minimal connected rectangular (p, q)-knights' graphs were determined in [5]. In view of the complexity of the pattern of moves of a knight on a chess board a surprising result of that paper is that these minimal connected rectangular graphs are either planar or toroidal. The patterns of edges on the minimal boards are simpler than those on larger boards making it easier to study the related metrics, and the biperiodic patterns of the toroidal graphs are easier to study than the periodic patterns of the planar graphs. The aim of this paper is to determine which of the graphs are planar and which are toroidal.

The vertices of a knight's graph are the points with integer coordinates in the whole plane or in some rectangular region of the plane with sides parallel to the axes. An edge in a (p, q)-knight's graph joins two vertices when the difference of one of their coordinates is p and the difference of their other

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coordinate is q. In chess p and q are 1 and 2. A (p,q)-knight's graph on the whole plane is connected if and only if p and q are mutually prime and $p+q$ is odd [2, 3]. When p and q satisfy these conditions the (p,q)-knight's graph on a rectangular board of size X by Y is connected if and only if $\min\{X,Y\} \geq p+q$ and $\max\{X,Y\} \geq 2\max\{p,q\}$ [5]. It will be convenient to take $p < q$. Then the minimal connected rectangular (p,q)-knight's graph is of size $p+q$ by $2q$. It was proved in [5] that the genus of this graph is 0 or 1, and that it is 0 if $p = Md+1$ and $q = (M+1)d+1$ with M a non-negative integer and d a positive odd integer. It is shown here that the genus is 1 in all other cases, as conjectured in [5]. The proof is based on the theorem of Kuratowski that a graph is not planar if it contains a homeomorph of the complete bipartite graph $K_{3,3}$. In this case the vertices of the $K_{3,3}$ are six of the vertices in which three disjoint simple closed paths (x-cycles) meet two other disjoint simple closed paths (y-cycles). These cycles correspond to the two generators of a torus which supports the graph. Three cases have to be considered: $2 < 2p < q$, $1.5p < q < 2p-1$ and $p+1 < q < 1.5p$. The proof of the existence of the appropriate x-cycles is the same for all three cases, but the proof of the existence of the y-cycles becomes more delicate as the ratio q/p becomes smaller.

An algorithm for finding paths (though not necessarily shortest paths) between vertices in the unrestricted generalised knight's graphs is given in [3]. The problem of finding paths, particularly shortest paths, between vertices in restricted connected rectangular knights' graphs is much more difficult. It is hoped that the special simple closed paths studied in the course of this paper will help in an investigation of geodesics.

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2 Unwrapping minimal connected knight's graphs

The language and notation of [5] will be used throughout this paper. For p and q mutually prime with $p < q$ and $d = q-p$ odd the minimal connected (p,q)-knight's graph $\mathcal{G}$ is the (p,q)-knight's graph on the set of vertices (x,y) in $\mathbb{Z}^2$ such that $1 \leq x \leq p+q$ and $1 \leq y \leq 2q$. The set of points with first coordinate x will be called the *line* $\ell(x)$. The function

$$f(x) = \begin{cases} x+p & \text{if } 1 \leq x \leq q \\ x-q & \text{if } q+1 \leq x \leq p+q \end{cases}$$

i.e. $f(x) = x+p \pmod{p+q}$ which is a cyclic permutation of $[1, p+q]$, is used to order the lines $\ell(x)$. Two lines $\ell(x)$ and $\ell(x')$ are said to be adjacent if some vertex in $\ell(x)$ is adjacent to some vertex in $\ell(x')$. Each line $\ell(x)$ is adjacent to $\ell(f^{-1}(x))$ and $\ell(f(x))$ and to no other line.

It is convenient to think of the sequence of lines determined by the function f divided into blocks defined in the following way. If $p > 1$ then $Np < q < (N+1)p$ for some positive integer N. In this case when $1 \le x \le q - Np$, $f^{N+1}(x) = x + (N+1)p$ while $f^{N+2}(x) = x + (N+1)p - q$, and when $q - Np + 1 \le x \le p$, $f^N(x) = x + Np$ while $f^{N+1}(x) = x + Np - q$. Thus the sequence $1, f(1), f^2(1), \ldots, f^{p+q-1}(1)$ can be seen as $q - Np$ blocks of length $N+2$ and $(N+1)p - q$ blocks of length $N+1$ such that within each block successive values differ by p, while the difference between the value at the end of one block and the value at the beginning of the next block is $-q$. In each case there is at least one even block of lines and at least one odd block of lines. Note that if p is even and q is odd then the number of even blocks is odd, while if p is odd and q is even then the number of even blocks is even both for N even and for N odd. Thus in all cases q and the number ν of even blocks are both even or both odd.

It is shown in Section 5 of [5] that if $(f^j(x), y_j)$, $0 \le j \le p+q$, is a directed path in $\mathcal{G}$ then $y_0 + y_{p+q}$ is odd. Thus each line $\ell(x)$ splits into two half-lines, the vertices with even y-coordinates form the *even* half-line $\ell(x)$, and those with odd y-coordinates form the *odd* half-line $\ell(x)$. The two half lines will be said to be complementary to each other. The sequence of half-lines ordered by the function f can be extended by periodicity to an infinite sequence of half-lines of period $2(p+q)$.

The y-coordinates on each half-line are ordered using the function $g(n) = n + 2p \pmod{2q}$. This function divides the sequence of vertices on each half-line into groups. The difference between one y-coordinate and the next within a group is $-2d$ while the difference between the y-coordinate at the end of one group and that at the beginning of the next group is $2p$. When $2p < q$ each group contains one or two vertices, when $1.5p < q < 2p$ each group contains two or three vertices, while when $p + 1 < q < 1.5p$ and $p = Md + r$ each group contains $M+1$ or $M+2$ vertices.

Some of the adjacency properties of the unwrapped minimal knight's graphs are listed in the following two lemmas.

Lemma 2.1. *Suppose that $1 \le x \le q$. For each y there exists a unique y' such that (x, y) is adjacent to $(f(x), y')$, and then $(x, g^k(y))$ is adjacent to $(f(x), g^k(y'))$ for each integer k.*

Lemma 2.2. *Suppose that $q + 1 \le x \le p + q$.*

(i) *If (x, y) is adjacent to $(f(x), y')$ and $(f(x), y'')$ then y'' is $g(y')$ or $g^{-1}(y')$.*

(ii) *Suppose that both (x, y) and $(x, g(y))$ are adjacent to $(f(x), y')$. If (x, y) is adjacent to $(f(x), y'')$ then $y'' = g^{-1}(y')$. If $(x, g(y))$ is adjacent to $(f(x), y'')$ then $y'' = g(y')$.*

(iii) *If* (x,y) *is adjacent to* $(f(x),y')$ *and* $(x,g^k(y))$ *is adjacent to* $(f(x),y'')$ *then* y'' *is* $g^{k-1}(y')$ *or* $g^k(y')$ *or* $g^{k+1}(y')$.

It is convenient to display the unwrapped minimal knight's graphs with the half-lines running across the page. The y-coordinates of the vertices on each half-line are ordered from the left to the right of the page by the function g, while the x-coordinates of the half-lines are ordered from the top to the bottom of the page by the function f. If $f(x) = x + p$, so that $\ell(x)$ and $\ell(f(x))$ are in the same block, then each vertex (x,y) is adjacent to one of $(f(x),y+q)$ and $(f(x),y-q)$ in the following line. In this case the adjacent vertex $(f(x),y \pm q)$ is placed below (x,y) in the figures. Only the first and last lines of each block are included in the figures in this paper. If $f(x) = x - q$ and $p+1 \le y \le 2q-p$ then (x,y) is adjacent to both $(f(x),y+p)$ and $(f(x),y-p)$. In this case the latter vertices are placed astride (x,y) in the figures, while the other vertices on the half-lines $\ell(x)$ and $\ell(f(x))$ are placed as ordered by the function g.

The ordered pairs (p,q) for which it has been proved that the minimal connected (p,q)-knight's graph is planar are of four types. The first three types are ordered pairs $(1,2n)$, $(n,n+1)$ and $(2n,4n-1)$, for n a positive integer. In these cases $M = 0$, $d = 2n-1$ and $M = n-1$, $d = 1$ and $M = 1$, $d = 2n-1$, respectively. The fourth type is those ordered pairs $p = Md+1$ and $q = (M+1)d+1$ with $M > 1$ and $d > 1$. To describe this last type more notation is needed. If $d < p$ then $p = Md + r$ for some positive integers M and r with $r < d$. Then $d = Lr + s$ for some positive integer L and some non-negative integer s with $s < r$. Since p and d are mutually prime $s = 0$ if and only if $r = 1$. The fourth type of pairs (p,q) for which the (p,q)-knight's graph has been shown to be of genus 0 are those for which $p+1 < q < 1.5p$ and $r = 1$. Thus the pairs (p,q) for which it is conjectured in [5] that $\mathcal{G}$ has genus 1 fall into three classes: (i) those with $2 < 2p < q$, (ii) those with $1.5p < q < 2p-1$, (iii) those with $p+1 < q < 1.5p$ and $r > 1$.

3 x-cycles and y-cycles

In every minimal connected knight's graph there exist simple closed paths corresponding to one of the generators of a torus supporting the graph, directed by the action of the function f. These will be called x-cycles. When $p > 1$ and $d > 2$ there exist three disjoint x-cycles. When p and q are not of the form $Md+1$ and $(M+1)d+1$ there exist also two disjoint simple closed paths corresponding to the other generator, directed by the function g. These will be called y-cycles.

An *x-cycle* is a simple closed path in the minimal connected (p,q)-knight's graph $\mathcal{G}$ with $2p+2q$ edges and with vertices $(f^j(x), y_j)$, $0 \le j \le 2p+2q$, such that $y_0 = y_{2p+2q}$. It is directed by the function f. Note that as one moves along a path there is no choice in the change of the y-coordinate from one half-line to another within a block. In the x-cycles described below the changes of the y-coordinate between the last half-line of one block and the first half-line of the next alternate between p and $-p$: such x-cycles will be called will be called *alternating x-cycles*. In Figures 2, 3 and 4 they zig-zag down the page.

Proposition 3.1. *Let $q' = q+1$ if q is even and $q' = q+2$ if q is odd. If $p > 1$ and $d > 2$ then for each first half-line $\ell(x_0)$ of a block of half-lines there exist alternating x-cycles starting at $(x_0, 1)$, $(x_0, 3)$ and (x_0, q'). They are pairwise disjoint, and the ordering of their vertices is the same on each half-line.*

Proof: The notation δ_n will be used to denote 0 when n is even and 1 when n is odd. Let $\ell(x_0)$ be the first of a block of half-lines. For each number y_0 in $[1, d]$ or in $[q+1, q+d]$ define a directed path with vertices $(f^j(x_0), y_j)$, $0 \leq j \leq 2p+2q$, where $y_{j+1} = y_j + q \pmod{2q}$ if $\ell(f^{j+1}(x_0))$ and $\ell(f^j(x_0))$ are in the same block while $y_{j+1} = y_j - (-1)^m p$ if $\ell(f^j(x_0))$ is the last in the m-th block of half-lines following $\ell(x_0)$. In particular, if $\ell(f^j(x_0))$ is the last half-line of the block which starts with $\ell(x_0)$ then $y_{j+1} = y_j + p$. If the m blocks consist of n even blocks of half-lines and $m - n$ odd blocks of half-lines then $y_{j+1} = y_0 + \delta_m p + nq \pmod{2q}$. After $2(p+q)$ edges the path will have passed $2p$ blocks of half-lines of which 2ν are of even length, so that $(f^{2p+2q}(x_0), y_{2p+2q}) = (x_0, y_0)$. Thus the path is an alternating x-cycle. Now suppose $3 = g^\lambda(1) = 1+2\lambda p \pmod{2q}$ and $q' = g^\mu(1) = 1+2\mu p \pmod{2q}$, and consider the alternating x-cycles starting at $(x_0, 1)$, $(x_0, 3)$ and (x_0, q'). Their y-coordinates on the last line $\ell(f^j(x_0))$ of the m-th block of lines following $\ell(x_0)$, of which n are even blocks, are $1 + \delta_m p + nq \pmod{2q} = y'$ say, and $3 + \delta_m p + nq \pmod{2q} = g^\lambda(y')$ and $q' + \delta_m p + nq \pmod{2q} = g^\mu(y')$. On successive lines within each block the y-coordinates are the same as on the last line of the block, or all are increased by $q$or all are decreased by q. Thus on each half-line the y-coordinates of the three alternating x-cycles are related by the same powers of the function g, so that they are pairwise disjoint. □

A *y-cycle* based on an even half-line $\ell(x_0)$ or on an odd half-line $\ell(x_0)$ in $\mathcal{G}$ is a simple closed path which contains, in order, all the vertices (x_0, y) which are end points of groups of vertices on the half-line, and which does not meet both a half-line and its complementary half-line. It inherits its direction from the function g.

The last vertex (x, y) of a group in the first line $\ell(x)$ of a block can always be joined to the first vertex $(x, y+2p)$ of the next group by a path consisting of the two edges joining these vertices to the vertex $(x+q, y+p)$. It will be convenient to call such a path a *cap*. For certain half-lines $\ell(x_0)$

the first and last vertices of a group can be joined by special paths within the following half-lines $\ell(f^j(x_0))$ for $0 \le j < p+q$. It will be convenient to call such a path a *chain*. In the figures the caps stand above and the chains hang below the half-lines $\ell(x_0)$. The longer the groups of vertices the more complicated are the chains joining their ends. Nevertheless, in the figures each chain has a vertical axis of symmetry. The successions of caps and chains form disjoint y-cycles based on the even and odd half-lines $\ell(x_0)$.

The existence of y-cycles is proved in the following propositions. The existence of non-simple closed paths joining successive vertices on certain half-lines was proved in [5], but their patterns are complex and difficult to grasp. In contrast the patterns of the caps and chains joining the ends of groups of vertices are much more transparent so that it is possible to study their intersections with the x-cycles.

Case (i) $2 < 2p < q$

If $p > 1$ then $Np < q < (N+1)p$ for some positive integer N, and there are blocks of lines of length $N+1$ and blocks of length $N+2$. Thus in all cases there are blocks of even length. It is proved in the next proposition that when $2 < 2p < q$ there are y-cycles based on the even and odd half-lines at the beginning of an even block. Note that in this case the groups of vertices on each half-line contain one or two vertices, and runs of 1-groups are separated by single 2-groups. The lemma is based on Proposition 3 and Theorem 5 of [5].

Proposition 3.2. *Suppose that $2 < 2p < q$. If $\ell(x_0)$ is the first line of a block of $2k$ lines then $\mathcal{G}$ contains y-cycles based on the even and odd half-lines $\ell(x_0)$.*

Proof: If $1 \le y \le 2d$ then $g(y) = y+2p$ so that the vertex (x_0, y) is the last of one group while $(x_0, g(y))$ is the first of the next group. They are joined by a cap consisting of the two edges joining them to the vertex $(x_0+q, y+p)$. If $2d+1 \le y \le 2q$ then $g(y) = y - 2d$ so that (x, y) and $(x_0, g(y))$ are in the same group. They are joined by a chain whose vertices in order are $(f^j(x_0), y + jq \pmod{2q})$, $0 \le j \le 2k-1$, and $(f^{2k}(x_0), y+q+p)$ and $(f^{2k-1-j}(x_0), y+q+2p+jq \pmod{2q})$, $0 \le j \le 2k-1$. In Figure 2 the chain has a vertical axis of symmetry between (x_0, y) and $(x_0, y-2d)$ and through $(f^{2k}(x_0), y+q+p)$. The sequences of caps and chains based on the even and odd half-lines $\ell(x_0)$ are disjoint y-cycles. □

Case (ii) $1.5p < q < 2p-1$

When $1.5p < q < 2p-1$ the function g divides each half-line into groups of two or three vertices. Vertices with $2p+1 \le y \le 4d$ start 2-groups while vertices with $4d+1 \le y \le 2q$ start 3-groups. The first and last vertex in each group on the first half-lines of certain blocks can be connected by chains. The chains joining the ends of a 2-group are relatively short while the chains joining the ends of a 3-group are much longer. The existence of non-simple closed paths in this case is proved in Theorem 7 of [5].

Proposition 3.3. *If $1.5p < q < 2p - 1$ then $\mathcal{G}$ contains two successive 2-blocks which are separated by an even number of 3-blocks, and it contains y-cycles based on the even and odd half-lines of the first line $\ell(x_0)$ of the first of these 2-blocks.*

Proof: That there are successive 2-blocks separated by an even number of 3-blocks is proved in Lemma 5 of [5]. Let the first line of the first 2-block be $\ell(x_0)$, let the first line of the next 2-block be $\ell(x')$, and let there be $2j$ 3-blocks between them.

If (x_0, y) is the first vertex of a 2-group then $2p+1 \le y \le 4d$. Each of the vertices (x_0, y), $(x_0+p, y-q)$, $(x_0+p-q, y-q+p)$, $(x_0+p, y-q+2p)$, $(x_0, y-2q+2p)$ is adjacent to the next, all the y-coordinates being in $[1, 2q]$. Thus (x_0, y) is connected to $(x_0, y-2d)$, the last of the group, by a short chain. In Figure 3 the chain has a vertical axis of symmetry between (x_0, y) and $(x_0, y-2d)$ and through $(x_0+p-q, y-p+q)$.

If (x_0, y) is the first vertex of a 3-group then it is connected to vertices with y-coordinates $y-q$ in the first and last half-lines of the even 3-groups following $\ell(x_0)$ and to vertices with y-coordinates $y-q+p = y-d$ in the first and last half-lines in the odd 3-groups following $\ell(x_0)$ and also in the first half-line $\ell(x')$ of the next 2-block. Similarly, the last vertex of the group, $(x_0, y-4d)$, is connected to vertices with y-coordinates $y-4d+q$ in the first and last half-lines of the even 3-groups following $\ell(x_0)$ and to vertices with y-coordinates $y-4d+q-p = y-3d$ in the first and last half-lines in the odd 3-groups following $\ell(x_0)$ and also in the first half-line $\ell(x')$ of the next 2-block. The vertices $(x', y-d)$ and $(x', y-3d) = (x', g(y-d))$ are sometimes but not always the first and last vertices of a 2-group. Nevertheless, they are connected by a short chain in precisely the same way as detailed in the previous paragraph since in this case $4d+1 \le y \le 2q$ and as before the appropriate y-coordinates all lie in the interval $[1, 2q]$. Thus the end vertices of a 3-group are connected by a long chain. In Figure 3 the chain has a vertical axis of symmetry through the vertices with y-coordinates $y-q-2d$ in the first and last lines of the even 2-groups following $\ell(x_0)$.

Since two chains meet at most at a common end vertex, the sequences of caps and chains joining the ends of groups of vertices on the even and odd half-lines $\ell(x_0)$ are y-cycles. □

Case (iii) $p+1 < q < 1.5p$ and $r > 1$

If $p+1 < q < 1.5p$ then $p = Md + r$ and $q = (M+1)d + r$ with $M > 1$, and the cycle of y-coordinates contains groups of length $M+1$ and groups of length $M+2$. On certain half-lines the groups of vertices of length $M+1$ are joined by short chains while the groups of vertices of length $M+2$ are joined by long chains. The notion of monotone segments is introduced to clarify the description of these chains.

A directed path, or segment of a path, $(f^j(x), y_j)$ will be said to be *monotone increasing* if $y_{j+1} = y_j + p$ whenever $\ell(f^j(x))$ is the last line of a block, and *monotone decreasing* if $y_{j+1} = y_j - p$ whenever $\ell(f^j(x))$ is the last line of a block. Recall that connections between vertices on successive lines of a block are unique. In the figures the monotone increasing segments run from top left to bottom right, while monotone decreasing segments run from top right to bottom left.

Proposition 3.4. *If $p + 1 < q < 1.5p$ and $r > 1$ then $\mathcal{G}$ contains two successive runs of M 2-blocks which are separated by an odd number of runs of $M - 1$ 2-blocks, successive runs of 2-blocks being separated by a single 3-block, and it contains y-cycles based on the even and odd half-lines of the first line of the first of these 2-blocks.*

Proof: Let $\ell(x_0)$ be the first line of a run of M 2-blocks which is separated from the next run of M 2-blocks by an odd number of runs of $M - 1$ 2-blocks. The existence of such runs of lines is proved in Lemma 7 of [5].

If (x_0, y) is the first of an $(M+1)$-group of vertices then $2q-2d+1 \leq y \leq 2(M+1)d$. For the last vertex of the group $2q - 2(M+1)d + 1 \leq g^M(y) = y - 2Md \leq 2d$. In this case $y - (j-1)d - q$, $y - jd$, $g^M(y) + (j-1)d + q$ and $g^M(y) + jd$ lie in $[1, 2q]$ whenever $1 \leq j \leq M$. Thus (x_0, y) is joined to $(f^{2M}(x_0), y - Md)$ by a monotone increasing path and $(x_0, g^M(y))$ is joined to $(f^{2M}(x_0), g^M(y) + Md) = (f^{2M}(x_0), y - Md)$ by a decreasing path. Thus (x_0, y) is joined to $(x_0, g^M(y))$ by a chain in the half-lines from $\ell(x_0)$ to $\ell(f^{2M}(x_0))$. In Figure 4 the chain has a vertical axis of symmetry through the vertex $(f^{2M}(x_0), y - Md)$.

If (x_0, y) is the first of an $(M+2)$-group of vertices then $2(M+1)d+1 \leq y \leq 2q$. For the last vertex of the group $1 \leq g^{M+1}(y) \leq 2q - 2(M+1)d$. In this case (x_0, y) is joined to $(f^{2M}(x_0), y - Md)$ by a monotone increasing path and $(x_0, g^{M+1}(y))$ is joined to $(f^{2M}(x_0), g^{M+1}(y) + Md) = (f^{2M}(x_0), g(y - Md))$ by a monotone decreasing path. Now by the reflection principle described in the proof of Lemma 8 of [5], $(f^{2M}(x_0), y - Md)$ is connected to $(f^{4M+1}(x_0), y - q)$, and $(f^{2M}(x_0), g(y - Md))$ is connected to $(f^{4M+1}(x_0), g^{M+1}(y) + q)$, the paths being monotone deceasing and monotone increasing, respectively, in the first run of $M - 1$ 2-blocks. By repeated use of the reflection principle within the runs of $M - 1$ 2-blocks (x_0, y) is connected to $(f^{j(2M+1)-1}(x_0), y - q)$ if j is even and to $(f^{j(2M+1)-1}(x_0), y - Md)$ if j is odd, by a path which is alternately monotone increasing and monotone decreasing within the successive runs of $M - 1$ 2-blocks. In particular, if there are $2n - 1$ runs of $M - 1$ 2-blocks between the successive runs of M 2-blocks then taking $j = 2n$ the vertex (x_0, y) is joined to $(f^{2n(2M+1)-1}(x_0), y - q)$ in the first line of the 3-block preceding the next run of M 2-blocks, and so to $(x', y - d)$. Similarly, $(x_0, g^M(y))$ is joined to $(f^{j(2M+1)-1}(x_0), g^M(y) + q)$ if j is even and to $(f^{j(2M+1)-1}(x_0), g^M(y) + Md)$ if j is odd. In particular, $(x_0, g^M(y)$ is joined to $(f^{2n(2M+1)-1}(x_0), g^M(y) + q)$ and so to $(x', g^M(y) + d) = (x', g^{M-1}(y - d))$. In this case $(x', y - d)$ is the first of an $(M+1)$-group so it is connected to $(x', g^{M-1}(y-d))$ by a short chain in the following $2M$ half-lines via the vertex $(f^{2M}(x'), y - (M + 1)d)$, so that (x_0, y) and $(x_0, g^{M+1}(y))$ are joined by a long chain in the half-lines $\ell(x_0)$ to $\ell(f^{2M}(x'))$. In Figure 4 the chain has a vertical axis of symmetry through $(f^{2M}(x'), y - (M + 1)d)$.

Since two chains meet at most at a common end vertex, the sequences of caps and chains joining the ends of groups of vertices on the even and odd half-lines $\ell(x_0)$ are y-cycles. □

4 Non-planar knight's graphs

The proof that the minimal connected knight's graph is of genus 1 when p and q are not of the form $Md+1$ and $(M+1)d+1$ is based on the existence of subgraphs which are homeomorphic to $K_{3,3}$. The vertices of the $K_{3,3}$ are six of the common vertices of three alternating x-cycles with two disjoint y-cycles. The basic situation is described in the following proposition. Recall that $q' = q + 1$ if q is even and $q' = q + 2$ if q is odd.

Proposition 4.1. *Suppose that for some x_0 the graph $\mathcal{G}$ contains two y-cycles γ_1 and γ_2 based on the odd and even half-lines $\ell(x_0)$. Let α_1, α_2 and α_3 be the segments of the alternating x-cycles through $(x_0, 1)$, $(x_0, 3)$ and (x_0, q') starting at the last common vertex $b(\alpha_i)$ with γ_1 and ending at the first common vertex $e(\alpha_i)$ with γ_2 in the direction of the function f, and let β be the segment of the alternating x-cycle through $(x_0, 1)$ starting at the last common vertex $b(\beta)$ with γ_1 and ending at the first common vertex $e(\beta)$ with γ_1 in the direction of the function f^{-1}. If the vertices $b(\alpha_1)$, $b(\beta)$, $b(\alpha_2)$, $b(\alpha_3)$ lie on γ_1 in that order, and the vertices $e(\beta)$, $e(\alpha_1)$, $e(\alpha_2)$, $e(\alpha_3)$ lie on γ_2 in that order, relative to the ordering induced from the function g, or if for both sets of vertices the roles of α_2 and α_3 are reversed, then $\mathcal{G}$ is not planar.*

Proof: Note that by definition the y-cycles γ_1 and γ_2 are disjoint. If the roles of α_2 and α_3 are as given in the first reading of the proposition then the conditions ensure that the two triples of vertices $b(\alpha_1)$, $b(\alpha_2)$, $e(\alpha_3)$ and $b(\beta)$, $b(\alpha_3)$, $e(\alpha_2)$ are the vertices of a homeomorph of a $K_{3,3}$. Its edges are the four segments of the loop γ_1 between the four vertices $b(\alpha_1)$, $b(\alpha_2)$, $b(\alpha_2)$ and $b(\beta)$, the paths α_2 and α_3, a path from $b(\alpha_1)$ to $e(\alpha_2)$ via $e(\alpha_1)$ along α_1 and γ_2, and a path from $b(\beta)$ to $e(\alpha_3)$ via $e(\beta)$ along β and γ_2. Then by a theorem of Kuratowski [4] the graph is not planar. In the other reading the roles of α_2 and α_3 are reversed. The first reading is illustrated in Figure 1. □

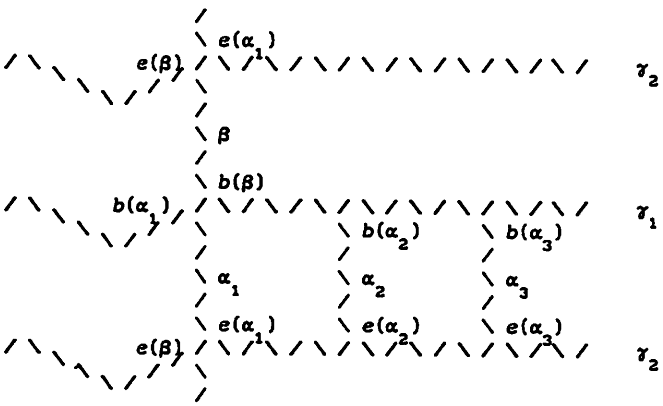

Figure 1.
Construction of a $K_{3,3}$ on a minimal connected knight's graph

Note that $p > 1$ and $d > 2$ in all the three cases considered in the following theorem so that by Proposition 3.1 there exist disjoint alternating x-cycles through the vertices $(x_0, 1)$, $(x_0, 3)$ and (x_0, q') whenever $\ell(x_0)$ is the first of a block of lines. Propositions 3.2, 3.3 and 3.4 show that in each case there exist y-cycles γ_1 and γ_2. It will be necessary to investigate the orderings of $b(\alpha_1)$ and $b(\beta)$ on γ_1 and of $e(\alpha_1)$ and $e(\beta)$ on γ_2.

Theorem 4.1. *Let p and q be mutually prime with $p+q$ odd. Then the minimal connected knight's graph $\mathcal{G}$ is of genus 0 when $p = Md+1$ and $q = (M+1)d+1$ for some non-negative integer M and positive odd integer d, and is of genus 1 otherwise.*

Proof: The graph $\mathcal{G}$ is proved to be of genus 0 or 1 in Proposition 4 of [5]. It is proved to be of genus 0 when $p = Md+1$ and $q = (M+1)d+1$ in Theorem 10 of [5]. Thus it remains to be proved that it has genus 1 for Case (i) $2 < 2p < q$, Case (ii), $1.5p < q < 2p-1$ and Case (iii) $p+1 < q < 1.5p$ with $r > 1$.

In each of the three cases $p > 1$ and $d > 2$, so that by Proposition 3.1 whenever $\ell(x_0)$ is the first of a block of half-lines there exist pairwise disjoint alternating x-cycles through $(x_0, 1)$, $(x_0, 3)$ and (x_0, q'). Moreover, by Propositions 3.2, 3.3 and 3.4 there exist y-cycles γ_1 and γ_2 based on the even and the odd half-lines $\ell(x_0)$ described in those propositions. In each case the vertex $(x_0, 1)$ is the last of a group of vertices and $(x_0, g(1))$ is the first of the next group. They are joined by a cap whose middle vertex is $(f^{-1}(x_0), 1+p)$. The vertex $(x_0, 1)$ is joined by a chain to the first vertex (x_0, y') of its group where in Case (i) $(x_0, y') = (x_0, g^{-1}(1))$, in

Case (ii) $(x_0, y') = (x_0, g^{-2}(1))$ and in Case (iii) $(x_0, y') = (x_0, g^{-M-1}(1))$. The alternating x-cycle through $(x_0, 1)$ meets the y-cycle γ_1 at $(x_0, 1)$ and $(f^{-1}(x_0), 1+p)$ and some other vertices in the chain joining $(x_0, 1)$ to (x_0, y'), but in no other chain or cap. Thus, in the notation of Proposition 4.1, in each case $b(\beta) = (f^{-1}(x_0), 1+p)$ and $b(\alpha_1)$ lies in the chain joining $(x_0, 1)$ to (x_0, y'), and these vertices lie on γ_1 in the order (x_0, y'), $b(\alpha_1)$, $b(\beta)$, $(x_0, 1)$.

If p is odd then ν is even and the alternating x-cycle through $(x_0, 1)$ meets the even half-line $\ell(x_0)$ at the vertex $(x_0, 1+p)$, the previous vertex on the x-cycle being $(f^{-1}(x_0), 1)$. In Case (i) $(x_0, 1+p)$ is the last vertex of a 2-group while in Case (ii) it is the last vertex of a 2-group or a 3-group so that $e(\alpha_1) = (x_0, 1+p)$ and $e(\beta)$ lies in the chain joining $(x_0, 1+p)$ to the first vertex of the group. In Case (iii), if $M = 2n-1$ then $(x_0, 1+p)$ is the right vertex of the centre pair of an $(M+1)$-group whose first and last vertices are $(x_0, g^{-n}(1+p))$ and $(x_0, g^{n-1}(1+p))$, while if $M = 2n$ then $(x_0, 1+p)$ is the right vertex of the centre pair of an $(M+2)$-group whose first and last vertices are $(x_0, g^{-n-1}(1+p))$ and $(x_0, g^{n}(1+p))$. Thus in the blocks following the even half-line $\ell(x_0)$ the alternating x-cycle through $(x_0, 1)$ has vertices alternately on and to the right of the line of symmetry of the chain joining the ends of the group. Thus $e(\alpha_1)$ and $e(\beta)$ lie in this chain and $e(\beta)$ precedes $e(\alpha_1)$.

If p is even then q is odd and the alternating x-cycle through $(x_0, 1)$ meets the even half-line $\ell(x_0)$ at the vertex $(x_0, 1+q)$, the previous vertex on the x-cycle being $(f^{-1}(x_0), 1+q+p)$. In Case (i) the vertex $(x_0, 1+q)$ is preceded and followed by caps in γ_2 so that $e(\alpha_1) = (x_0, 1+q+p)$ and $e(\beta) = (x_0, 1+q)$ and they lie on γ_2 in the order $(x_0, g^{-1}(1+q))$, $e(\beta)$, $e(\alpha_1)$, $(x_0, g(1+q))$. In Case (ii) the vertex $(x_0, 1+q)$ is the central vertex of a 3-group starting at $(x_0, g^{-1}(1+q))$ and ending at $(x_0, g(1+q))$. In Case (iii) if $M = 2n$ or if $M = 2n-1$ then $(x_0, 1+q)$ is the central vertex of a group starting at $(x_0, g^{-n}(1+q))$ and ending at $(x_0, g^{n}(1+q))$, these vertices being joined by a short chain when $M = 2n$ and by a long chain when $M = 2n-1$. Now in the blocks following the even half-line $\ell(x_0)$ the alternating x-cycle through $(x_0, 1)$ has vertices alternately to the right and on the line of symmetry of the chain joining the ends of the group. Thus $e(\alpha_1)$ and $e(\beta)$ lie in this chain and $e(\beta)$ precedes $e(\alpha)1$.

Since the relative positions of the vertices of the three alternating x-cycles are the same on each half-line, it follows that in all three cases the two y-cycles and the three alternating x-cycles satisfy the conditions of Proposition 4.1 so that the graph is of genus 1. □

The two y-cycles γ_1 and γ_2, and the four paths α_1, α_2, α_3 and β are illustrated in the following three figures. Case (i) with p even is illustrated in Figure 2. Cases (ii) and (iii) with p even are illustrated in Figures 3 and 4. Only the first and last lines of each block of lines are included in the figures. The edges in the y-cycles are marked in bold print, the edges in the α and β paths are marked with solid lines and the other edges in the graph are marked in dotted lines. The α paths start at vertices marked in large print while the path β starts at a vertex marked in bold print. They end at corresponding underlined vertices.

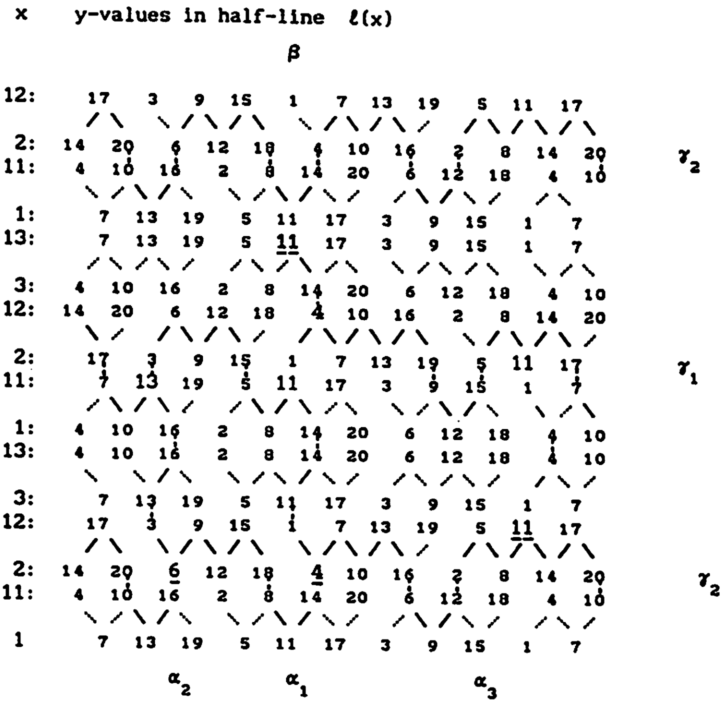

Figure 2.
The unwrapped (3, 10)-knight'sgraph with $x_0 = 2$.

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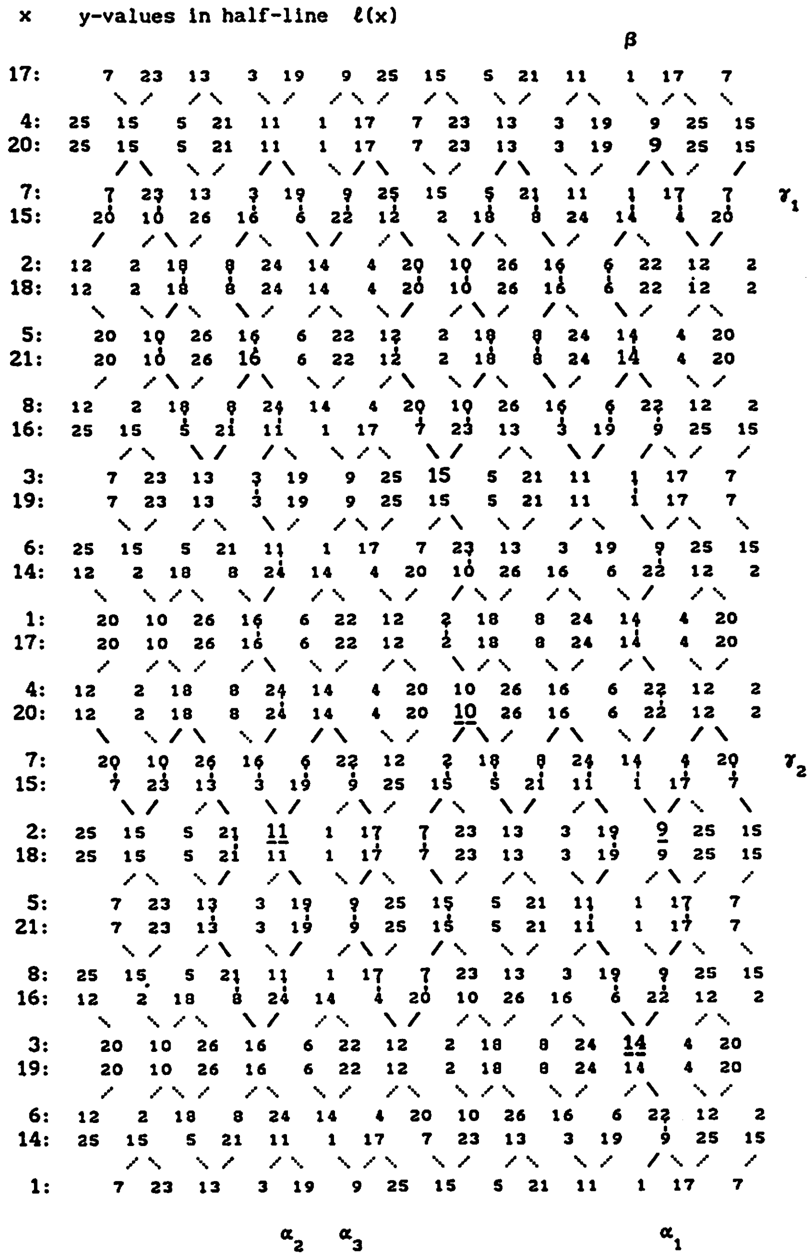

Figure 3.
The unwrapped (8, 13)-knight's graph with $x_0 = 7$ and $x' = 8$.

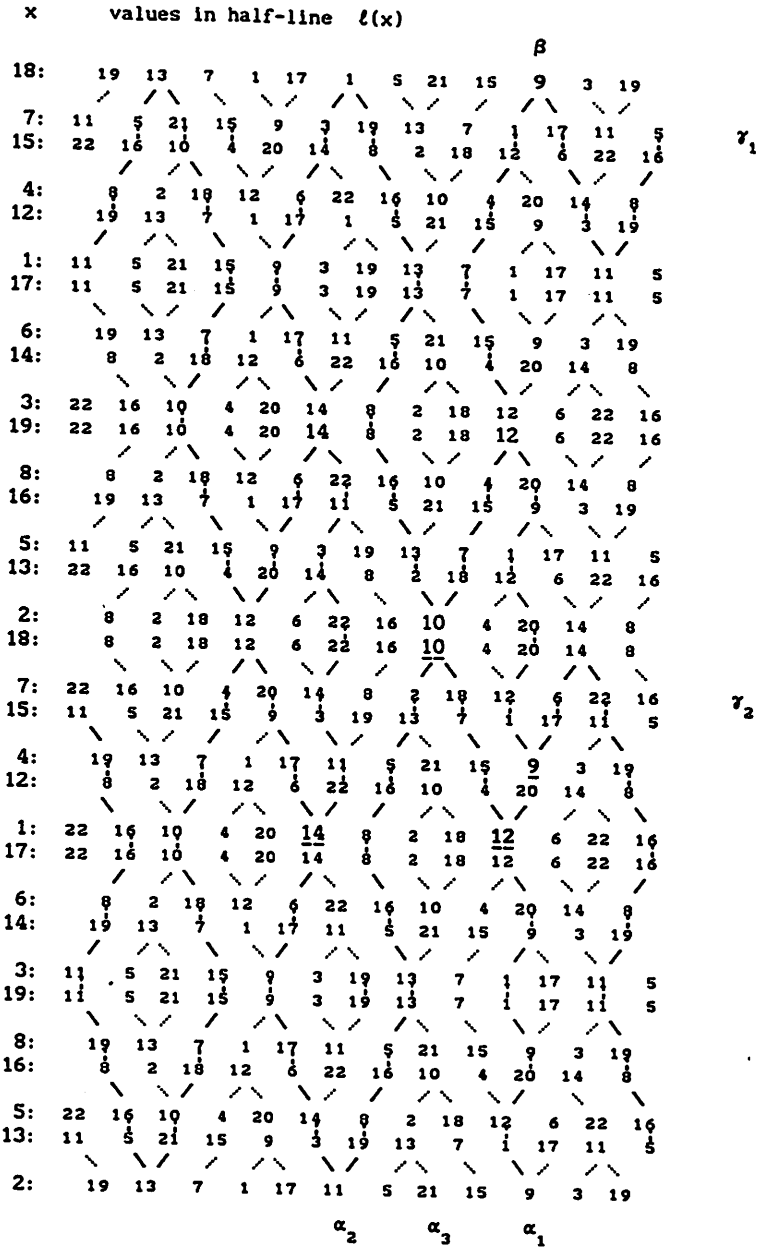

Figure 4.
The unwrapped (8, 11)-knight's graph with $x_0 = 7$ and $x' = 8$.