HPMDs of type 2^n3^1 with Block Size Four and Related HCOLSs

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ABSTRACT. A holey perfect Mendelsohn design of type $h_1^{n_1}h_2^{n_2}\cdots h_k^{n_k}$ (HPMD($h_1^{n_1}h_2^{n_2}\cdots h_k^{n_k}$)) with block size four is equivalent to a frame idempotent quasigroup satisfying Stein's third law with the same type, where a frame quasigroup of type $h_1^{n_1}h_2^{n_2}\cdots h_k^{n_k}$ means a quasigroup of order n with n_i missing subquasigroups (holes) of order $h_i, 1 \le i \le k$, which are disjoint and spanning, that is $\sum_{1\le i\le k} n_ih_i = n$. In this paper, we investigate the existence of HPMD(2^n3^1) and show that an HPMD(2^n3^1) exists if and only if $n \ge 4$. As an application, we readily obtain HSOLS(2^n3^1) and establish the existence of (2,3,1) [or (3,1,2)]-HCOLS(2^n3^1) for all $n \ge 4$.

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1 Introduction

A *quasigroup* is an ordered pair $(Q, \cdot)$, where Q is a set and $(\cdot)$ is a binary operation on Q such that the equations

$$a \cdot x = b \text{ and } y \cdot a = b \tag{1}$$

are uniquely solvable for every pair of elements $a, b \in Q$. A quasigroup is called *idempotent* if the identity

$$x \cdot x = x \tag{2}$$

is satisfied for all x in Q. If the identity

$$(y \cdot x) \cdot (x \cdot y) = x \tag{3}$$

holds for all $x, y \in Q$, then we say that the quasigroup satisfies *Stein's third law*. An idempotent quasigroup satisfying the Stein's third law is denoted by *IQST*. The *order* of the quasigroup is $|Q|$.

Let $(Q, \cdot)$ be a quasigroup where the multiplication table of $(\cdot)$ forms a Latin square indexed by Q. The (i, j, k)-conjugate of $(Q, \cdot)$ is $(Q, *_{i,j,k})$, where (i, j, k) is a permutation of $(1, 2, 3)$ and $x_i *_{i,j,k} x_j = x_k$ if and only if $x_1 \cdot x_2 = x_3$. Following the convention (see [5]), we also call Q a Latin square. A Latin square is said to be (i, j, k)-*conjugate-orthogonal* ((i, j, k)-COLS for short) if $x \cdot y = z \cdot w$ and $x *_{i,j,k} y = z *_{i,j,k} w$ imply $x = z$ and $y = w$. We will use (i,j,k)–HCOLS$(h_1^{n_1} \cdots h_k^{n_k})$ to denote the type of holey (i, j, k)-COLS of order $\sum_{i=1}^{k} h_i n_i$, that have n_i holes of size h_i, $1 \le i \le k$, and all the holes are assumed to be mutually disjoint, and each of them corresponds to a missing sub-Latin square. It is well-known that there does not exist any $(1, 2, 3)$-HCOLS$(h_1^{n_1} \cdots h_k^{n_k})$ for order $n > 1$; a $(3, 2, 1)$-HCOLS$(h_1^{n_1} \cdots h_k^{n_k})$ exists if and only if a $(1, 3, 2)$-HCOLS$(h_1^{n_1} \cdots h_k^{n_k})$ exists; a $(2, 3, 1)$-HCOLS$(h_1^{n_1} \cdots h_k^{n_k})$ exists if and only if a $(3, 1, 2)$-HCOLS$(h_1^{n_1} \cdots h_k^{n_k})$ exists. A $(2, 1, 3)$-HCOLS$(h_1^{n_1} \cdots h_k^{n_k})$ is also called a holey self-orthogonal Latin square and denoted by HSOLS$(h_1^{n_1} \cdots h_k^{n_k})$.

We shall discuss the existence of some kinds of HCOLSs in Section 4. Now we turn our attention to Mendelsohn designs.

The existence of IQST is equivalent to the existence of perfect Mendelsohn design with block size four (see for example [5]).

Let v, k be positive integers. A *(v, k, 1)–Mendelsohn design*, briefly $(v, k, 1)$–MD, is a pair $(X, \mathcal{B})$, where X is a v–set (of *points*) and $\mathcal{B}$ is a collection of cyclically ordered k–subsets of X (called *blocks*) such that every ordered pair of points of X are consecutive in exactly one block of $\mathcal{B}$, where a cyclically ordered block $(a_1, a_2, \cdots, a_k)$ means $a_1 \le a_2 \le \cdots \le a_k \le a_1$. If for all $t = 1, 2, \cdots, k-1$, every ordered pair of points of X are t–apart in exactly one block of $\mathcal{B}$, then the $(v,k,1)$–MD is called *perfect* and is denoted by $(v,k,1)$–PMD.

Suppose $(Q,\cdot)$ is an IQST. Let $\mathcal{B}=\{(x,x\cdot y,y,y\cdot x):x,y\in Q,x\neq y\}$. Then $(Q,\mathcal{B})$ is a perfect Mendelsohn design of block size four. For the existence of PMDs of block size four, we have the following theorem from [4, 11, 7].

Theorem 1.1 *A $(v,4,1)-PMD$ exists if and only if $v\equiv 0,1$ $(mod\ 4)$ and $v\neq 4,8$.*

Let Q be a set and $\mathcal{H}=\{S_1,S_2,\cdots,S_k\}$ be a set of subsets of Q. A *holey IQST* having hole set $\mathcal{H}$ is a triple $(Q,\mathcal{H},\cdot)$, which satisfies the following properties:

1. $(\cdot)$ is a binary operation defined on Q. However, when both points a and b belong to the same set S_i, there is no definition for a·b,
2. the equations (1) hold when a and b are not contained in the same set $S_i, 1\leq i\leq k$,
3. the identity (2) holds for any $x\notin\cup_{1\leq i\leq k}S_i$,
4. the identity (3) holds when x and y are not contained in the same set $S_i, 1\leq i\leq k$.

We denote holey IQST by HIQST$(n;s_1,s_2,\cdots,s_k)$, where $n=|Q|$ is the *order* and $s_i=|S_i|, 1\leq i\leq k$. Each S_i is called a *hole*. When $\mathcal{H}=\emptyset$, we obtain an IQST, and denote it by IQST(n).

From the definition of HIQST, we can obtain the definition of frame IQST as follows. If $\mathcal{H}=\{S_1,S_2,\cdots,S_k\}$ is a partition of Q, then a holey IQST is called *frame IQST*. The *type* of the frame IQST is defined to be the multiset $\{|S_i|:1\leq i\leq k\}$. We shall use an "exponential" notation $s_1^{n_1}s_2^{n_2}cdotss_t^{n_t}$ to describe the type of n_i occurrences of $s_i, 1\leq i\leq t$ in the multiset. We briefly denote a frame IQST of type $s_1^{n_1}s_2^{n_2}\cdots s_t^{n_t}$ by FIQST$(s_1^{n_1}s_2^{n_2}\cdots s_t^{n_t})$.

Corresponding to FIQST$(s_1^{n_1}s_2^{n_2}\cdots s_t^{n_t})$, we can define holey perfect Mendelsohn design, denoted by $HPMD(s_1^{n_1}s_2^{n_2}\cdots s_t^{n_t})$ as follows. A *holey perfect Mendelsohn design* is a triple (X,$\mathcal{H}$, B) which satisfies the following properties:

1. $\mathcal{H}$ is a partition of X into subsets called *holes*,
2. $\mathcal{B}$ is a family of cyclically ordered k-subsets of X (called *blocks*) such that a hole and a block contain at most one common point,

3. every ordered pair of points from distinct holes are $t-$ apart in exactly one block for $t = 1, 2, \cdots, k-1$.

The *type* of the HPMD is the multiset $\{|H| : H \in \mathcal{H}\}$ and it is also described by an exponential notation.

In graph theoretic terminology, an HPMD is a decomposition of a complete multipartite directed graph $DK_{n_1,n_2,\cdots,n_h}$ into $k-$circuits such that for any two vertices x and y from different components, there is one circuit along which the directed distance from x to y is t, where $1 \le t \le k-1$.

Another class of design related to HPMD is GDD. A GDD is a triple (X, $\mathcal{G}, \mathcal{B}$) which satisfied the following properties:

1. $\mathcal{G}$ is a partition of X into subsets called *groups*,

2. $\mathcal{B}$ is a family of subsets of X (called *blocks*) such that a group and a block contain at most one common point,

3. every pair of points from distinct groups occurs in exactly λ blocks.

The *type* of the GDD is the multiset $\{ |G| : G \in \mathcal{G}\}$. We also use the notation GD(K, M; λ) to denote the GDD when its set of block sizes is K and set of group sizes is M.

If M $=\{1\}$, then the GDD becomes a *PBD*. If K $= \{k\}$, M $= \{n\}$ and the type n^k, then the GDD becomes TD(k, n). It is well known that the existence of a TD(k, n) is equivalent to the existence of $k-2$ MOLS(n).

It is easy to see that if we ignore the cyclic order of elements in the blocks, the HPMD becomes a GDD with block size four and $\lambda = 3$. But the converse may be not true. It is proved in [6] that such a GDD of type h^u exists if and only if $h^2u(u-1) \equiv 0 \pmod 4$. However, for the existence of HPMDs of block size four with equal - sized holes, [3] gives the following complete solution.

Theorem 1.2 *An HPMD(h^u) exists if and only if $h^2u(u-1) \equiv 0 \pmod 4$ with the exception of types $2^4, 1^8$ and h^4 for odd h.*

In this paper, we consider the existence of HPMDs of type 2^n3^1 with block size four. We shall use HPMD to denote HPMD of block size four throughout this paper. The main result of this paper is the following theorem.

Theorem 1.3 *An HPMD(2^n3^1) exists if and only if $n \ge 4$.*

2 Constructions

In this section, we display some direct and recursive constructions of HPMDs. To construct HPMDs directly, we usually use *starter blocks.* Suppose the block set $\mathcal{B}$ of an HPMD is closed under the action of some Abelian group G, then we need to list only part of the blocks (starter or base blocks) which determines the structure of the HPMD. We can also attach some infinite points to an Abelian group G. When the group acts on the blocks, the infinite points remain fixed. For HPMD(2^n3^1), we use $Z_{2n} \cup \{x, y, z\}$. In the following example x, y, z are infinite points.

Example 2.1 *An HPMD*(2^83^1):

points: $Z_{16} \cup \{x, y, z\}$

holes: $\{\{i, i+8\} : 0 \leq i \leq 7\} \cup \{x, y, z\}$

starter blocks: $(x, 0, 7, 6), (y, 0, 13, 1), (z, 0, 14, 9), (0, 2, 12, 15), (0, 9, 5, 11)$.

In this example, the whole set of blocks is developed from the starter blocks by Z_{16}, i.e., adding 1 (mod 16) to each point of the starter blocks. But when n is odd, it is impossible to use Z_{2n} to develop the blocks. So for odd n, we use a different method as stated in the following lemma. We refer this as the "+2 method".

Lemma 2.2 *Suppose there exist blocks* $A = (w, a_1, a_2, a_3)$ *and* $B = (b_1, b_2, b_3, b_4)$, *where* $w \in W$ *is the infinite point,* $a_i, b_j \in Z_{2n}$, $A \in \mathcal{A}$ *and* $B \in \mathcal{B}$. *Let* $D^{(-1)} = \{a_t - a_{t+1}, b_s - b_{s+1} : t = 1, 2, \ s = 1, 2, 3, 4 \ (mod\ 4), \ A \in \mathcal{A}, B \in \mathcal{B}\}$ *and* $D^{(-2)} = \{a_t - a_{t+2}, b_s - b_{s+2} : t = 1, 3 \ (mod\ 4), s = 1, 2, 3, 4 \ (mod\ 4), A \in \mathcal{A}, \ B \in \mathcal{B}\}$. *If*

(1) *every element of* $Z_{2n} \setminus \{0, n\}$ *appears twice both in* $D^{(-1)}$ *and* $D^{(-2)}$;

(2) $c_m - c_n, c_p - c_q \in D^{(-i)}$, $c_m - c_n \equiv c_p - c_q \ (mod\ 2n)$ *implies* $c_m \not\equiv c_p (mod\ 2)$, *where* i = *1 or 2;*

(3) *each infinite point* w *is contained in two blocks* $A^{(1)}, A^{(2)}$ *such that* $a_j^{(1)} \not\equiv a_j^{(2)} (mod\ 2), j = 1, 2, 3,$

then there exists an HPMD($2^n u^1$), *where* $u = |W|$.

Proof Develop the blocks $A \in \mathcal{A}$ and $B \in \mathcal{B}$ by adding 2 to every element modulo $2n$. It is readily checked that these blocks form an HPMD($2^n u^1$) on set $Z_{2n} \cup W$. □

The following HPMD is constructed by the +2 method.

Example 2.3 *An HPMD*(2^73^1)

points: $Z_{14} \cup \{x, y, z\}$

holes: $\{\{i, i+7\} : 0 \leq i \leq 6\} \cup \{x, y, z\}$

starter blocks: $(x, 0, 9, 10), (x, 1, 10, 13), (y, 0, 1, 11), (y, 1, 12, 10), (z, 0, 2, 8), (z, 1, 3, 7), (0, 4, 3, 6), (0, 5, 13, 4), (0, 11, 9, 1)$.

In general, if we cannot find any starter blocks of the HPMD, we often list the FIQST's multiplication table which corresponds to the HPMD. The Table 2.1 is an FIQST($2^5 1^1$) which corresponds to an HPMD($2^5 1^1$).

Next, we state several recursive constructions of HPMDs, which are commonly used in other block designs. The following construction comes from the weighting construction of GDDs [9].

Construction 2.4 (Weighting) *Suppose $(X, \mathcal{H}, B)$ is a GDD with $\lambda = 1$ and let w: $X \mapsto Z^+ \cup \{0\}$. Suppose there exist HPMDs of type $\{w(x)$: $x \in B\}$ for every $B \in \mathcal{B}$. Then there exists an HPMD of type $\{\sum_{x \in H} w(x) : H \in \mathcal{H}\}$.*

The next construction, called "filling in holes", is quite standard in constructing designs.

*	0	1	2	3	4	5	6	7	8	9	x
0		7	3	9	8		2	x	4	1	6
1	4		8	x	2	7		9	0	5	3
2	9	x		4	3	1	0		6	8	5
3	6	9	1		7	x	4	5		2	0
4	1	3	6	5		2	x	0	7		8
5		8	9	1	x		7	3	2	6	4
6	8		x	2	5	3		4	9	0	7
7	3	0		6	1	4	8		5	x	9
8	x	4	0		6	9	5	1		7	2
9	7	2	5	0		8	3	6	x		1
x	2	5	4	7	0	6	9	8	1	3	

Table 2.1 An FIQST($2^5 1^1$)

Construction 2.5 *Suppose there exist an HPMD of type $\{ s_i : 1 \le i \le k\}$ and HPMDs of type $\{ h_{i_j} : 1 \le j \le n_i\} \cup \{a\}$, where $\sum_{j=1}^{n_i} h_{i_j} = s_i$ and $1 \le i \le k-1$, then there exists an HPMD of type $\{ h_{i_j} : 1 \le j \le n_i, 1 \le i \le k-1\} \cup \{s_k + a\}$.*

Construction 2.6 *If there exist an HPMD($2^n t^1$) and an HPMD($2^m u^1$), where $2m + u = t$, then there exists an HPMD($2^{n+m} u^1$).*

To use the previous constructions, we need some small HPMDs which are provided in the following lemma.

Lemma 2.7 *There exist HPMDs of type* $2^5, 2^6, 2^5 1^1, 2^5 3^1, 2^5 4^1$.

Proof The HPMDs of type $2^5, 2^6$ come from Theorem 1.1. An HPMD($2^5 1^1$) is displayed in Table 2.1. HPMDs of type $2^5 3^1, 2^5 4^1$ are constructed in the Appendix. □

Now we are in a position to state the following main recursive construction of this paper.

Construction 2.8 *If there exists a TD(6,m), then there exists an HPMD* ($2^{5m} u^1$), *where* $0 \leq u \leq 4m$.

Proof Give weight 2 to each point of the first five groups of a TD(6,m) and give weight 0, 1, 2, 3 or 4 to the points of the sixth group such that the total weights of all points of this group is u. Since there exist HPMDs of type $2^5, 2^5 1^1, 2^6, 2^5 3^1, 2^5 4^1$ by Lemma 2.7, we obtain an HPMD($(2m)^5 u^1$). Filling in holes of size 2m with HPMD(2^m) which exist from Theorem 1.2, we obtained the desired HPMD. □

To use Construction 2.8, we need some results of the existence of TD(6,m). The following results about $TD(6, m)$ come from the existence of 4 MOLSs (see [1]).

Lemma 2.9 *For* $m \geq 5$ *and* $m \notin \{6, 10, 14, 18, 22\}$, *there exists a* $TD(6, m)$.

We can also give weight to the points of an HPMD to construct a new HPMD. The following construction is obtained in this way.

Construction 2.10 *If there exists an HPMD*($h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k}$), *then there exists an HPMD*($(mh_1)^{n_1} (mh_2)^{n_2} \cdots (mh_k)^{n_k}$), *where* $m \neq 2$, 6.

Proof Give weight m to each point of the HPMD($h_1^{n_1} h_2^{n_2} \cdots h_k^{n_k}$). For every block B of this HPMD, construct a TD(4, m) such that the blocks in the TD are cyclically ordered in the same way as B. The resulting design is an HPMD($(mh_1)^{n_1} (mh_2)^{n_2} \cdots (mh_k)^{n_k}$). □

3 Existence of HPMD($2^n 3^1$)

In this section, we shall prove our main result. First we construct some HPMDs of type $2^n 3^1$ for small values of n. Most of these designs are found by computer using the direct construction methods provided in Section 1.

Lemma 3.1 *There exist HPMDs of types* $2^n 3^1$ *for* $4 \leq n \leq 15$.

Proof The types $2^7 3^1$ and $2^8 3^1$ are constructed in Section 2. For other types see the Appendix. □

Combining Constructions 2.6, 2.8 and Lemma 3.1, we can handle most cases of the existence of HPMD($2^n 3^1$).

Lemma 3.2 *There exists an HPMD*(2^n3^1) *for* $n \geq 25$ *and* $n \neq 26, 27, 28, 36, 37, 38$.

Proof Let m = 5 and u = 3, 11, 13, 15, 17, 19 in Construction 2.8 to obtain HPMDs of type $2^{25}u^1$. As there exist HPMDs of type 2^s3^1 for $s = 4, 5, 6, 7, 8$ by Lemma 3.1, we then obtain HPMDs of type 2^n3^1 for $n = 25, 29, 30, 31, 32, 33$ by Construction 2.6. Similarly, we can let m = 7 and u = $3, 11, 13, \cdots, 27$ in Construction 2.8 to obtain HPMDs of type 2^n3^1 for $n = 35, 39, 40, \cdots, 47$. For $n \geq 49$, we let m$\geq$ 9, m odd and u = $11, 13, \cdots, 33$ in Construction 2.8 to obtain the desired HPMDs. The existence of TD(6,m) comes from Lemma 2.9. □

Lemma 3.3 *There exists an HPMD*(2^n3^1) *for* $n \equiv 0$ (*mod* 4) *and* $n \geq 16$.

Proof There exist HPMDs of type 8^n8^1 for $n \geq 3$ from Theorem 1.2. Now let $a = 3$ in Construction 2.5. Since an HPMD(2^43^1) exists from Lemma 3.1, we obtain HPMD($2^{4(n+1)}3^1$) for $n \geq 3$. □

Lemma 3.4 *There exists an HPMD*(2^n3^1) *for* $n = 22, 26, 38$.

Proof From an HPMD(2^43^1) we obtain an HPMD(8^412^1) by Construction 2.10. Let a = 3 in Construction 2.5, we obtain an HPMD($2^{22}3^1$), since HPMDs of types $2^43^1, 2^63^1$ exist. Similarly, from the HPMD(2^53^1) we get an HPMD(8^512^1) and then add 3 points to get an HPMD($2^{26}3^1$). Finally, from an HPMD(2^83^1) We get an HPMD(8^812^1) and then obtain an HPMD($2^{38}3^1$) by adding 3 new points. □

Lemma 3.5 *There exists an HPMD*(2^n3^1) *for* $n = 17, 18, 19, 21, 23, 27, 37$.

Proof For $n = 19$, add one point to an HPMD(10^4) to obtain an HPMD ($2^{15}11^1$) by Construction 2.5, then use Construction 2.6 to obtain HPMD ($2^{19}3^1$). For $n = 27$, from an HPMD(2^43^1) we obtain an HPMD(10^415^1) by Construction 2.10. Let a = 2 in Construction 2.5 and fill in holes with HPMDs of type $2^52^1, 2^73^1$ to obtain an HPMD($2^{27}3^1$). For $n = 37$, first delete 5 points from a block of a TD(6,7) to obtain a {5, 6}–GDD of type 6^57^1. Then give weight 2 to each point of that GDD to obtain an HPMD(12^514^1). An HPMD($2^{37}3^1$) can be constructed by adding 3 points to this HPMD by Construction 2.5. HPMDs of types $2^{13}11^1$, $2^{14}11^1, 2^{16}13^1, 2^{18}13^1$ are constructed in the Appendix, so we can obtain HPMDs of type 2^n3^1 for $n = 17, 18, 21, 23$ by Construction 2.6. □

Theorem 3.6 *There exist HPMDs of type* 2^n3^1 *if and only if* $n \geq 4$.

Proof The conclusion comes from Lemmas 3.1–3.5 directly. □

4 Application to HCOLSs

As we indicated in Section 1, the multiplication table of a quasigroup defines a Latin square. So a holey quasigroup defines a holey Latin square. From an HPMD(2^n3^1) one can obtain a holey Latin square of the same type which is self-orthogonal ((2, 1, 3)-HCOLS) from its correspodence with Stein's third law. A (2, 1, 3)-HCOLS is also denoted as HSOLS or FSOLS. In [8], an almost complete solution of the existence of HSOLS(2^n3^1) is given. And [10] gives a complete solution. As a corollary of Theorem 3.6, we also have the following result about the existence of (2, 1, 3)-HCOLS.

Theorem 4.1 *A (2, 1, 3)-HCOLS(2^n3^1) exists if and only if $n \geq 4$.*

On the other hand, Stein's third law $(y \cdot x) \cdot (x \cdot y) = x$ is conjugate-equivalent to the identity $(x \cdot (y \cdot x)) \cdot y = x$, using (1, 3, 2) - conjugate operation. This means that the (1, 3, 2) - conjugate of an FIQST satisfies the identity $(x \cdot (y \cdot x)) \cdot y = x$. It is not difficult to check that an idempotent quasigroup satisfying $(x \cdot (y \cdot x)) \cdot y = x$ is orthogonal to its (2, 3, 1) - conjugate. So Theorem 3.6 implies the following new result.

Theorem 4.2 *A (2, 3, 1)[or (3, 1, 2)]-HCOLS(2^n3^1) exists if and only if $n \geq 4$.*

We wish to indicate that (3, 2, 1) [or (1, 3, 2)]-HCOLS(2^n3^1) also has been investigated and the problem of existence has been settled with six possible exceptions of n, namely, n = 17, 18, 19, 21, 22, 23. The interested readers are referred to [2].

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Appendix

Here we list the starter blocks of some HPMDs which are used in the previous sections. Most of them are obtained by computer. In the following list, the point set of an $HPMD(2^nu^1)$ are Z_{2n} and u infinitive points which are denoted by alphabet.

Some HPMDs of type 2^n3^1

n = 6 (+1 mod 12):
(0,2,5,4), (a,0,1,11), (b,0,5,9), (c,0,7,4).

n = 9 (+2 mod 18):
(a,0,10,7), (a,1,5,6), (b,0,4,2), (b,1,7,3), (c,0,2,3),
(c,1,3,0), (0,3,10,4),(0,5,4,12),(0,7,15,13),(0,11,5,15)
(0,13,6,5).

n = 10 (+1 mod 20):
(0,2,16,19), (0,15,7,13), (0,17,8,16), (a,0,9,14),
(b,0,18,11), (3,0,19,15).

n = 11 (+2 mod 22):
(a,0,14,18), (a,1,13,11), (b,0,3,6), (b,1,16,9), (c,0,2,3),
(c,1,3,0), (0,5,15,21), (0,6,5,1), (0,7,2,10), (0,9,14,2),
(0,13,21,15),(0,16,13,4), (0,17,9,13).

n = 12 (+1 mod 24):
(0,2,20,23), (0,10,8,17), (0,13,10,18), (0,19,6,20),
(a,0,15,11), (b,0,16,15), (c,0,17,22).

n = 13 (+2 mod 26):
(a,0,7,5), (a,1,0,6), (b,0,15,6), (b,1,4,9), (c,0,2,3),
(c,1,3,0), (0,3,22,12), (0,4,25,6), (0,8,7,16),(0,9,19,25),
(0,11,16,14),(0,17,11,15),(0,18,15,4),(0,19,9,5),(1,9,21,13).

n = 14 (+1 mod 28):
(0,2,24,27), (0,10,22,21), (0,13,9,20), (0,17,5,9),
(0,18,27,5), (a,0,15,12), (b,0,20,18), (c,0,21,26).

n = 15 (+2 mod 30):
(a,0,4,28), (a,1,19,25), (b,0,6,26), (b,1,17,15), (c,0,2,3),
(c,1,3,26), (0,3,11,5), (0,5,16,13),(0,7,6,14), (0,9,21,12),
(0,10,8,27),(0,11,25,21),(0,12,7,11),(0,13,18,17),(0,14,10,3),
(0,17,9,29),(0,21,1,8).

Some HPMDs of type $2^n u^1$

n = 5, u = 4 (+1 mod 10):
(a,0,1,3), (b,0,3,9), (c,0,4,2), (d,0,7,6).

n = 13, u = 11: (+2 mod 26):
(a,0,5,24), (a,1,22,25), (b,0,17,22), (b,1,8,23),
(c,0,10,21), (c,1,23,22), (d,0,8,20), (d,1,9,21),
(e,0,9,19), (e,1,16,20), (f,0,24,18), (f,1,21,19),
(g,0,25,17), (g,1,4,18), (h,0,19,16), (h,1,18,15),
(i,0,6,1), (i,1,15,16), (j,0,16,12), (j,1,5,11),
(k,0,2,3), (k,1,3,12), (0,7,23,8).

n = 14, u = 11 (+1 mod 28):
(0,2,24,27), (a,0,9,17), (b,0,10,6), (c,0,11,18), (d,0,13,8),
(e,0,15,21), (f,0,16,13),(g,0,17,1), (h,0,18,16), (i,0,19,23),
(j,0,20,19), (k,0,21,26).

n = 16, u = 13 (+1 mod 32):
(0,1,31,30), (a,0,7,13), (b,0,9,22), (c,0,12,23), (d,0,14,24),
(e,0,15,2), (f,0,17,25), (g,0,18,12),(h,0,20,15), (i,0,21,26),
(j,0,22,18),(k,0,23,27), (l,0,24,21),(m,0,25,28).

n = 18, u = 13 (+1 mod 36):
(0,1,27,33), (0,5,35,34), (a,0,9,13), (b,0,11,21), (c,0,13,2),
(d,0,14,26), (e,0,15,3), (f,0,16,24),(g,0,17,8), (h,0,19,11),
(i,0,20,16), (j,0,21,14), (k,0,22,17),(l,0,23,30),(m,0,34,31).

FIQST of types $2^4 3^1$ and $2^5 3^1$

*	0	1	2	3	4	5	6	7	a	b	c
0		c	3	2		7	a	b	5	6	1
1	3		4	a	b		c	2	0	7	6
2	a	b		c	5	4		1	7	0	3
3	c	4	5		6	a	b		2	1	0
4		3	a	b		c	7	6	1	2	5
5	b		c	6	7		0	a	4	3	2
6	1	0		5	a	b		c	3	4	7
7	2	a	b		c	0	1		6	5	4
a	7	6	1	0	3	2	5	4			
b	5	2	7	4	1	6	3	0			
c	6	7	0	1	2	3	4	5			

*	0	1	2	3	4	5	6	7	8	9	a	b	c
0		2	3	6	b		c	8	7	a	4	9	1
1	a		9	c	3	7		5	4	b	0	8	2
2	b	5		9	c	a	3		6	8	1	4	0
3	9	0	b		a	c	7	1		5	6	2	4
4	6	c	1	7		b	2	a	0		8	5	3
5		3	8	4	2		b	c	a	1	9	7	6
6	2		a	b	8	4		9	c	3	5	0	7
7	4	b		5	0	9	a		1	c	3	6	8
8	c	a	4		7	6	5	b		0	2	1	9
9	1	8	c	a		2	0	6	b		7	3	5
a	7	9	0	2	1	3	8	4	5	6			
b	8	7	5	0	6	1	4	3	9	2			
c	3	4	6	1	5	8	9	0	2	7			

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