The intersection problem for Semi-symmetric Latin Squares

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ABSTRACT. Let $J[v]$ denote the set of numbers k so that there exist two semi- symmetric latin squares (SSLS) of order v which have k entries in common. In this paper, we show that
$J[3] = \{0, 9\}$, $J[4] = \{0, 1, 3, 4, 9, 12, 16\}$,
$J[5] = \{0, 1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 18, 21, 25\}$,
$J[6] = \{0, 1, 2, \ldots, 23, 24, 26, 27, 28, 29, 32, 36\}$, and
$J[v] = \{0, 1, 2, \ldots, v^2\} \setminus \{v^2 - 1, v^2 - 2, v^2 - 3, v^2 - 5, v^2 - 6\}$
for each $v \geq 7$.

1 Introduction

A quasigroup $(V, \cdot)$ satisfying the identity $y \cdot (x \cdot y) = x$, for each x, y in V, is called a semi-symmetric quasigroup (SSQG). That is if $a \cdot b = c$ then $b \cdot c = a$ and $c \cdot a = b$.

A complete directed graph D_n on n vertices is a directed graph such that each pair of vertices x, y is joined by two arcs (x, y) and (y, x). Let D_n^+ denote D_n together with a loop at each vertex. A graph on two vertices consisting of exactly two arcs is called an edge. A graph on two vertices consisting of exactly one loop and exactly one edge is called a lollipop. As usual, K_n is the complete graph on n vertices.

Lemma 1. *A SSQG(v) is equivalent to a partition of the arcs of D_n^+ into directed triangles, lollipops and loops.*

Proof: Suppose $a \cdot b = c$. If $a \neq b \neq c \neq a$, they correspond to a directed triangle (a, b, c), which consists of arcs (a, b), (b, c), and (c, a), in D_n^+. If

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$a \neq b = c$, they correspond to a lollipop on a, b with the loop at b. If $a = b = c$, they correspond to a loop at a. □

The embedding problem has been solved by Hoffman, see [5]. In this paper we deal with the intersection of two SSLSs. For clearness and completeness, we do some embeddings of SSQG(v) in the following to help us get the answer.

2 Embedding of SSQG of order v

A set F of edges of a graph is a 1-factor if every vertex of the graph is an end of exactly one edge in F. A spanning cycle of a graph is a cycle which contains all the vertices of the graph.

Throughout the sequel, v, w, and t are positive integers with $t = w - v$.

Lemma 2.. *A SSQG(v) can be embedded in a SSQG(w) if there is a partition of the arcs of D_t^+ into directed triangles, lollipops and loops and v 1-factors.*

Proof.: Given a $1-1$ correspondence between the 1-factors and the elements of V, if 1-factor F corresponds to the element a, then an edge in F on vertices b, c corresponds to two directed triangles (a, b, c) and (a, c, b). □

Lemma 3. *Let v be even. A SSQG(v) can be embedded in a SSQG(w) if there is a partition of the arcs of D_t^+ into directed triangles, lollipops and loops and $v/2$ spanning cycles.*

Proof.: If a spanning cycle $S = \{c_1, cc_2, \ldots, c_t\}$ in D_t^+ corresponds to two elements a, b of V, then two directed cycles $(c_1, c_2, \ldots, c_t)$ and $(c_t, c_{t-1}, \ldots, c_1)$ of S correspond to directed triangles $(a, c_1, c_2), (a, c_2, c_3), \ldots, (a, c_t, c_1)$, $(b, c_t, c_{t-1}), (b, c_{t-1}, c_{t-2}), \ldots,$ and (b, c_1, c_t). □

Lemma 4. [2] *If n is even then K_n can be decomposed into $n-1$ 1-factors, and if n is odd then K_n can be decomposed into $(n-1)/2$ edge-disjoint spanning cycles.*

Lemma 5. *A SSQG(v) can be embedded in a SSQG(2v).*

Proof:

(i) If v is even, then we can partition D_t^+ into $v-1$ 1-factors and v loops. Pick any $v - 1$ elements from V corresponding to those 1-factors to form directed triangles as Lemma 2. Each loop corresponds to a lollipop on the vth element of V and b with its loop at b, for each b in D_t^+.

(ii) If v is odd, then D_t^+ can be decomposed into $(v-1)/2$ spanning cycles and v loops. Each two vertices of V correspond to one spanning cycle to form directed triangles as Lemma 3. The vth element of V and each loop in D_t^+ form lollipops. □

Lemma 6. *A SSQG(v) can be embedded in a SSQG($2v+1$).*

Proof: If v is even, then by Lemma 4 D_t^+ can be partitioned into $v/2$ spanning cycles and $v+1$ loops. If v is odd, D_t^+ can be partitioned into v 1-factors and $v+1$ loops. By Lemma 2 and Lemma 3, we conclude the proof. □

Lemma 7. *A SSQG(v) can be embedded in a SSQG($2v+3$).*

Proof: If v is even, then D_t^+ can be partitioned into $(v/2)+1$ spanning cycles and $v+3$ loops. One spanning cycle with $v+3$ loops can be decomposed into $v+3$ lollipops. If v is odd, D_t^+ can be partitioned into $v+2$ 1-factors and $v+3$ loops. Since two 1-factors form a union of cycles, it, with $v+3$ loops, can be partitioned into $v+3$ lollipops. By Lemma 2 and Lemma 3, a SSQG(v) can be embedded in a SSQG($2v+3$). □

3 Main results

We define two semi-symmetric latin squares $L=[l_{i,j}]$ and $M=[m_{i,j}]$ have k entries in common if there are k cells (i,j) such that $l_{i,j}=m_{i,j}$. Let $J[v]=\{k\colon$ there exist two SSLSs of order v which have k entries in common$\}$. For convenience, we define $I[v]=\{0,1,2,\ldots,v^2-8,v^2-7,v^2-4,v^2\}$. It is trivial that $J[1]=\{1\}$ and $J[2]=\{0,4\}$. Using the computer, we can get the following results:

Lemma 8. $J[3]=\{0,9\}$, $J[4]=\{0,1,3,4,9,12,16\}$, $J[5]=\{0,1,3,4,6,7,9,10,12,13,15,18,21,25\}$.

Proof: Using the computer, we obtain that there are three SSLS(3), 18 SSLS(4), and 120 SSLS(5) which are listed in Appendix A. Compare each pair of them in each order, we have the results. □

Lemma 9. $J[v]\subseteq I[v]$, *for each positive integer v.*

Proof: It is easy to see that there do not exist two SSLSs of order v which have v^2-1, v^2-2, v^2-3, or v^2-5 entries in common [1]. It suffices to show that v^2-6 not in $J[v]$. If $v^2-6\in J[v]$, then we should find two SSLS(v) such that they have 6 corresponding entries filled different symbols. There are only two cases where this situation can arise:

$$\begin{array}{ccc} a & b & c \\ b & c & a \end{array} \quad \begin{array}{ccc} b & c & a \\ a & b & c \end{array} \quad \text{and} \quad \begin{array}{ccc} a & b & \\ b & & a \\ & a & b \end{array} \quad \begin{array}{ccc} b & a & \\ a & & b \\ & b & a \end{array}$$

But these two cases can not be contained in two SSLS(v) respectively, such that other corresponding entries are filled the same symbols. □

Lemma 10. $J[6] = I[6] \setminus \{25\} = \{0, 1, 2, \ldots, 23, 24, 26, 27, 28, 29, 32, 36\}$.

Proof: If $25 \in J[6]$ then there exist two SSLSs of order 6 such that they have 11 corresponding entries are filled different symbols. Since corresponding to two SSLSs of order 6 having only 11 different entries, there are two arc-labellings of D_6 such that there are three lollipops and two loops labelled different. From Figure 1, we can see that it is impossible to complete them to a latin square. Therefore $25 \notin J[6]$. For the other results are listed in Appendix B. □

	a	b	c	d	e	f
a	a	b				
b	b	a				
c			c	d	e	
d			d	e		
e			e		c	
f						

	a	b	c	d	e	f
a	b	a				
b	a	b				
c			d	e	c	
d			e	d		
e			c		e	
f						

Figure 1

Lemma 11. $J[7] = I[7]$.

Proof: According to Appendix C, we have $J[7] = I[7]$. □

Next we use the recurrsive construction to find $J[v]$.

Lemma 12. *If* $J[v] = I[v]$ *and* v *is an integer* ≥ 7 *then* $J[2v] = I[2v]$.

Proof: By Lemma 5, we can embed a SSLS(v) in a SSLS($2v$). By replacing a SSLS(v) and interchanging any two vertices of V corresponding to different 1-factors or spanning cycles to form different directed triangle or lollipops, we obtain that $J[2v] \supseteq J[v] + \{0, 3v, 6v, \ldots, 3v(v-2), 3v^2\}$. If $v \geq 7$, and $J[v] = I[v]$ then $v^2 - 7 > 6v$ and $J[2v] \supseteq I[v] + \{0, 3v, 6v, \ldots, 3v(v-2), 3v^2\}$. Therefore $J[2v] \supseteq I[2v]$. This implies $J[2v] = I[2v]$. □

Lemma 13. *If* $J[v] = I[v]$ *and* $v \geq 8$, *then* $J[2v+1] = I[2v+1]$.

Proof: By Lemma 6, we can embed a SSLS(v) in a SSLS($2v+1$). By replacing a SSLS(v) and interchanging any two vertices of V corresponding to distinct 1-factors or spanning cycles to form distinct directed triangles, we obtain $J[2v+1] \supseteq J[v] + v + 1 + \{0, 3(v+1), 6(v+1), \ldots, 3(v-2)(v+1), 3v(v+1)\}$. If $v > 7$ and $J[v] = I[v]$, then $J[2v+1] \supseteq I[2v+1] \setminus \{0, 1, \ldots, v\}$. (Since we cannot do anything about the loops in D_t^+). Next, we use Lemma 7 to embed a SSLS(u) in a SSLS($2u+3$). We can construct two SSLS($2u+3$) such that all entries which are not in SSLS(u) are different by permuting all the vertices of V which correspond to 1-factor or spanning cycles and changing the loops of lollipops by another end of edges. Thus we can get $J[2u+3] \supseteq J[u]+\{0\}$. By the assumption and Lemma 11, we have $J[u] \supseteq \{0,1,2,\ldots,u+1\}$ for $u \geq 7$, it implies $J[2u+3] \supseteq \{0,1,2,\ldots,u+1\}$ for $u \geq 7$. Combine Lemma 10, we get $\{0,1,\ldots,v\} \subset J[2v+1]$ for $v \geq 8$. Therefore $J[2v+1] = I[2v+1]$. □

Since we use the recurrsive construction, we need to fill in some gaps to obtain that $J[v] = I[v]$ for $v \geq 7$.

Lemma 14. $J[8] = I[8]$, *and* $J[9] = I[9]$.

Proof: By Appendices D and E, we obtain that $J[8]=I[8]$ and $J[9]=I[9]$. □

Lemma 15. $J[10] = I[10]$.

Proof: From Lemma 5, we have $J[10] \supseteq J[5] + \{0,15,30,45,75\}$. As Lemma 4, we can embed a SSLS(4) in a SSLS(10). D_6^+ can be partitioned into five 1-factors and six loops. Therefore we can get $J[10] \supseteq J[4]+\{0,18,36,72\}+\{0,4,8,12\}$. Combine the results in Appendix F, we conclude that $J[10] = I[10]$. □

Lemma 16. $J[11] = I[11]$.

Proof: From Lemma 6, we have $J[11] \supseteq J[5]+6+\{0,18,36,54,90\}$ and $J[11] \supseteq J[4]+\{0,21,42,63,105\}$. Combine the results in Appendix G, we have $J[11] = I[11]$. □

Lemma 17. $J[12] = I[12]$.

Proof: From Lemma 5, we get $J[12] \supseteq J[6]+\{0,18,36,54,72,108\} = \{0,1,2,\ldots,135,136,137,140,144\}\setminus\{97,102,103,105,106,107,133\}$. In Figure 2, A and B are two SSLSs of order 6 and $|A \cap B| \in J[6]$. Thus $|L_1 \cap L_2| \in J[6]+\{90\} = \{90,91,\ldots,114,116,117,118,119,122,126\}$. And $|L_3 \cap L_4| = 133$. Therefore $J[12] = I[12]$. □

Lemma 18. $J[13] = I[13]$.

Proof: From Lemma 6 and Lemma 7, we have $J[13] \supseteq J[6]+7+\{0,21,42,63,84,126\} = I[13] \setminus \{0,1,2,3,4,5,6,116,121,122,124,125,126,128,129,130,131,132,158\}$ and $J[13] \supseteq J[5]+\{0,24,48,72,120\}+\{0,24\} = \{0,1,3,4,6,7,9,10,12,13,15,18,21,25\}+\{0,24,48,72,96,120,144\}$. Thus $J[13] \supseteq I[13]\setminus\{2,5,116,122,125,128,131,158\}$. By Appendix H, we conclude that $J[13] = I[13]$. □

L_1:

						8	11	7	9	12	10
						9	7	10	12	8	11
	A					10	12	11	7	9	8
						11	10	12	8	7	9
						12	9	8	11	10	7
						7	8	9	10	11	12
9	8	10	11	12	7	6	1	2	3	4	5
7	11	12	10	9	8	2	6	5	4	1	3
10	7	11	12	8	9	1	5	6	2	3	4
12	9	7	8	11	10	3	4	1	6	5	2
8	12	9	7	10	11	4	2	3	5	6	1
11	10	8	9	7	12	5	3	4	1	2	6

L_2:

						8	7	10	9	12	11
						9	11	7	12	8	10
	B					10	12	11	7	9	8
						11	10	12	8	7	9
						12	9	8	11	10	7
						7	8	9	10	11	12
8	9	10	11	12	7	6	1	2	3	4	5
7	11	12	10	9	8	1	6	5	4	2	3
10	7	11	12	8	9	2	5	6	1	3	4
9	12	7	8	11	10	3	4	1	6	5	2
12	8	9	7	10	11	4	2	3	5	6	1
11	10	8	9	7	12	5	3	4	2	1	6

L_3:

2	1	3	4	9	10	11	12	5	6	7	8
1	2	4	3	10	9	12	11	6	5	9	7
3	4	1	2	11	12	9	10	7	8	5	6
4	3	2	1	12	11	10	9	8	7	6	5
9	10	11	12	6	5	7	8	1	2	3	4
10	9	12	11	5	6	8	7	2	1	4	3
11	12	9	10	7	8	5	6	3	4	1	2
12	11	10	9	8	7	6	5	4	3	2	1
5	6	7	8	1	2	3	4	9	10	11	12
6	5	8	7	2	1	4	3	10	9	12	11
7	8	5	6	3	4	1	2	11	12	9	10
8	7	6	5	4	3	2	1	12	11	10	9

L_4:

1	2	3	4	9	10	11	12	5	6	7	8
2	1	4	3	10	9	12	11	6	5	8	7
3	4	1	2	11	12	9	10	7	8	5	6
4	3	2	1	12	11	10	9	8	7	6	5
9	10	11	12	8	6	7	5	1	2	3	4
10	9	12	11	6	5	8	7	2	1	4	3
11	12	9	10	7	8	5	6	3	4	1	2
12	11	10	9	5	7	6	8	4	3	2	1
5	6	7	8	1	2	3	4	9	10	11	12
6	5	8	7	2	1	4	3	10	9	12	11
7	8	5	6	3	4	1	2	11	12	9	10
8	7	6	5	4	3	2	1	12	11	10	9

Lemma 19. $J[15] = I[15]$.

Proof: By Lemma 6 and Lemma 7, we can get that $J[15] \supseteq J[7] + 8 + \{0, 24, 48, 72, 96, 120, 168\} = I[15] \setminus \{0, 1, 2, \ldots, 7, 171, 172, 174, 175\}$ and $J[15] \supseteq J[6] + \{0, 27, 54, 81, 108, 162\} + \{0, 27\} = \{0, 1, \ldots, 24, 26, 27, 28, 29, 32, 36\} + \{0, 27, 54, 81, 108, 135, 189\}$. Thus we have $J[15] \supseteq I[15] \setminus \{172, 174, 175\}$.

Since D_{15}^{+} can be decomposed into three D_5^{+} and one tripartite graph $B_{5,5,5}$. For the tripartite graph $B_{5,5,5}$, we can decompose it into edge-disjoint triangles. Hence $J[15] \supseteq J[5] + J[5] + J[5] + \{150\} + \{172, 174, 175\}$. Therefore $J[15] = I[15]$. □

Combine the above Lemmas 11 to 19, we can get the following result:

Theorem 20. $J[v] = I[v] = \{0, 1, 2, \ldots, v^2\} \setminus \{v^2 - 1, v^2 - 2, v^2 - 3, v^2 - 5, v^2 - 6\}$ *for each* $v \geq 7$.

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Appendix A

(1) SSLS(3)

1	3	2		2	1	3		3	2	1
3	2	1		1	3	2		2	1	3
2	1	3		3	2	1		1	3	2

(2) SSLS(4)

L1	1234	2143	3412	4321
L2	1243	2134	4321	3412
L3	1324	3412	2143	4231
L4	1342	4213	2431	3124
L5	1423	3241	4132	2314
L6	1432	4321	3214	2143
L7	2134	1243	3412	4321
L8	2143	1234	4321	3412
L9	2143	1324	4231	3412
L10	2143	1432	4321	3214
L11	3214	2143	1432	4321
L12	3412	4231	1324	2143
L13	3412	4321	1234	2143
L14	3412	4321	1243	2134
L15	4231	2143	3412	1324
L16	4321	3214	2143	1432
L17	4321	3412	2134	1243
L18	4321	3412	2143	1234

(3) SSLS(5)

L1	12345	21453	35124	43512	4231
L2	12345	21534	34152	45213	53421
L3	12453	21345	53214	34521	45132
L4	12453	21534	54312	35241	43125
L5	12534	21345	43251	54123	35412
L6	12534	21453	45321	53142	34215
L7	13254	34125	21543	52431	45312
L8	13254	35142	21435	54321	42513
L9	13425	42153	25314	31542	54231
L10	13425	45132	24513	31254	52341
L11	13452	52134	24315	35241	41523
L12	13452	52134	24513	35241	41325
L13	13452	52143	25314	34521	41235
L14	13452	54123	25314	32541	41235
L15	13524	42135	24351	51243	35412
L16	13524	42153	25341	51432	34215
L17	13524	42153	25431	51342	34215
L18	13524	45132	24351	51243	32415

L6	21534	12453	45321	53142	34215
L62	21534	13245	42351	54123	35412
L63	21534	14325	43251	52143	35412
L64	21534	15342	43251	54123	32415
L65	21543	13254	52431	45312	34125
L66	21543	15432	54321	43215	32154
L67	32145	21453	15324	43512	54231
L68	32145	21534	14352	45213	53421
L69	32154	21435	14523	53241	45312
L70	32154	21543	15432	54321	43215
L71	34125	42513	15432	21354	53241
L72	34125	45312	13254	21543	52431
L73	34152	52314	13245	25431	41523
L74	34152	52413	15324	23541	41235
L75	34152	53214	12345	25431	41523
L76	34152	53214	12435	25341	41523
L77	34152	53214	12543	25431	41325
L78	35124	42351	13245	51432	24513

L19	13542	52134	24351	45213	31425
L20	13542	54123	25431	42315	31254
L21	14235	32514	41352	25143	53421
L22	14235	35412	41523	23154	52341
L23	14253	32415	51324	23541	45132
L24	14253	32514	51342	25431	43125
L25	14253	32514	51432	25341	43125
L26	14253	35412	51324	23541	42135
L27	14352	52413	35124	23541	41235
L28	14352	53214	32145	25431	41523
L29	14523	43215	52431	21354	35142
L30	14523	45312	53241	21435	32154
L31	14532	52314	43251	25143	31425
L32	14532	52413	45321	23145	31254
L33	14532	52413	45321	23154	31245
L34	14532	53214	42351	25143	31425
L35	15234	32451	41325	53142	24513
L36	15234	32451	41523	53142	24315
L37	15234	32541	41352	54123	23415
L38	15234	34521	41352	52143	23415
L39	15243	32451	51324	43512	24135
L40	15243	34521	51432	42315	23154
L41	15324	42531	34152	51243	23415
L42	15324	43251	32145	51432	24513
L43	15423	42351	53214	31542	24135
L44	15423	42531	54312	31245	23154
L45	15423	42531	54312	31254	23145
L46	15423	43251	52314	31542	24135
L47	15432	53241	42513	34125	21354
L48	15432	54321	43215	32154	21543
L49	21345	12453	35124	43512	54231
L50	21345	12534	34152	45213	53421
L51	21354	14523	35142	52431	43215
L52	21354	15432	34125	53241	42513
L53	21435	13254	42513	35142	54321
L54	21435	14523	45312	32154	53241
L55	21453	12345	53214	34521	45132
L56	21453	12534	54312	35241	43125
L57	21453	13245	52314	34521	45132
L58	21453	14325	53214	32541	45132
L59	21453	15342	53214	34521	42135
L60	21534	12345	43251	54123	35412

L79	35124	42531	14352	51243	23415
L80	35124	43251	12345	51432	24513
L81	35124	43251	12435	51342	24513
L82	35124	43251	12543	51432	24315
L83	35142	52431	14523	43215	21354
L84	35142	54321	13254	42513	21435
L85	42315	21453	35124	13542	54231
L86	42315	21534	34152	152435	3421
L87	42513	21354	53241	15432	34125
L88	42513	21435	54321	13254	35142
L89	43215	32154	21543	15432	54321
L90	43215	35142	21354	14523	52431
L91	43512	52134	24351	15243	31425
L92	43512	52143	25431	14325	31254
L93	43512	54123	25341	12435	31254
L94	43512	54123	25431	12345	31254
L95	43512	54123	25431	12354	31245
L96	45213	32451	51324	13542	24135
L97	45213	32541	51432	14325	23154
L98	45213	34521	51342	12435	23154
L99	45213	34521	51432	12345	23154
L100	45213	34521	51432	12354	23145
L101	45312	52431	34125	13254	21543
L102	45312	53241	32154	14523	21435
L103	52341	21453	35124	43512	14235
L104	52341	21534	34152	45213	13425
L105	52431	21354	43215	35142	14523
L106	52431	21543	45312	34125	13254
L107	53241	32154	21435	45312	14523
L108	53241	34125	21354	42513	15432
L109	53421	42135	24513	31254	15342
L110	53421	42153	25314	31542	14235
L111	53421	45132	24315	31254	12543
L112	53421	45132	24513	31245	12354
L113	53421	45132	24513	31254	12345
L114	54231	32415	41523	23154	15342
L115	54231	32514	41352	25143	13425
L116	54231	35412	41325	23154	12543
L117	54231	35412	41523	23145	12354
L118	54231	35412	41523	23154	12345
L119	54321	42513	35142	21435	13254
L120	54321	43215	32154	21543	15432

Appendix B

L1	124563	213645	632154	356412	465231	541326
L2	213456	132564	321645	465123	546312	654231
L3	124563	215436	653124	346215	432651	561342
L4	132456	321564	213645	465123	546312	654231
L5	124536	215463	563124	346215	431652	652341
L6	124365	215634	456123	361452	632541	543216
L7	123564	216345	341652	652413	465231	534126
L8	124365	215634	453126	361542	632451	546213
L9	124365	215634	453126	361452	632541	546213
L10	124563	216345	645132	352614	463251	531426
L11	134562	621345	243156	352614	465231	516423
L12	124563	216345	643152	352416	465231	531624
L13	132564	321645	214356	653412	465231	546123
L14	132564	21645	215436	654312	463251	546123

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	0													
3	19	0												
4	1	27	2											
5	13	6	24	8										
6	15	3	16	4	10									
7	21	6	10	4	7	9								
8	13	3	18	5	12	28	7							
9	14	3	17	5	11	32	8	32						
10	23	0	17	1	11	11	23	11	10					
11	19	0	11	6	8	7	19	11	11	22				
12	26	0	17	2	11	11	26	11	12	28	23			
13	20	0	7	8	4	8	20	10	11	13	23	17		
14	16	0	8	8	5	7	16	11	10	17	20	13	29	

$$J[6] = \{0, 1, 2, \ldots, 23, 24, 26, 27, 28, 29, 36\}$$

Appendix C

L1	2156734	1267345	6574123	7643512	3725461	4312657	5431276
L2	7326451	3215674	2164735	5743216	6471523	4537162	1652347
L3	1327456	3215674	2136745	5764231	6471523	7543162	4652317
L4	2176453	1257346	7534261	5643712	6321574	4762135	3415627
L5	2176453	1257346	7543261	5634712	6321574	4762135	3415627
L6	1763452	7256341	4537216	5614723	6321574	3472165	2145637
L7	1275436	2163754	6437521	5724163	4651372	7316245	3542617
L8	1345672	7412536	2631754	3274165	4567213	5726341	6153427
L9	1563472	7234156	4327615	5412763	2671534	3756241	6145327
L10	1356472	7214536	2637145	5462713	3571264	4723651	6145327
L11	1725634	3264571	6137425	7452163	4573216	5316742	2641357
L12	1276453	2157346	7564231	5643712	6321574	4732165	3415627
L13	1725634	3462751	6137425	7256143	4673512	5314267	2541376
L14	1725436	3264751	6137524	5472163	4651372	7316245	2543617
L15	2137456	1325674	3216745	5764231	6471523	7543162	4652317
L16	2175436	1263754	6457321	5726143	4631572	7314265	3542617
L17	1345672	7412536	2631754	3276145	4567213	5724361	6153427
L18	1347652	7412536	2631745	3265471	6574213	5723164	4156327
L19	1725634	3462751	6137425	7254163	4673512	5316247	2541376
L20	1725634	3264751	6137425	7452163	4673512	5316247	2541376

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2	7																			
3	1	34																		
4	19	15	10																	
5	16	12	10	45																
6	7	10	13	25	25															
7	6	7	13	11	11	13														
8	0	7	10	5	5	7	10													
9	1	9	10	12	12	22	14	14												
10	10	14	16	15	14	16	9	21	27											
11	10	11	13	3	2	10	18	15	15	16										
12	15	18	10	41	38	25	14	5	12	14	2									
13	12	8	10	2	1	10	20	11	11	8	32	3								
14	7	8	13	9	8	16	36	12	18	13	28	8	26							
15	3	27	40	11	12	10	11	5	8	10	7	9	5	7						
16	10	10	13	15	15	13	37	4	12	8	14	12	23	29	15					
17	0	8	10	5	4	7	6	45	11	21	12	6	15	9	5	8				
18	0	14	22	8	7	10	3	30	11	24	12	9	12	3	17	2	31			
19	12	7	10	2	2	10	24	15	14	8	35	2	45	29	5	19	11	11		
20	13	8	10	3	2	10	23	11	18	12	39	2	42	33	4	20	8	8	45	

$$J[7] = \{0, 1, \ldots, 41, 42, 45, 49\}.$$

Appendix D

L1	1243678521348567432158763412765886573142578624136875423175681324
L2	2134678512438567341258764321765886573142578624136875423175681324
L3	1324678534128567214358764231765886573142578624136875423175681324
L4	2143678513248567427158363412765886573142578624136835427175681324
L5	2143678513248567425138763412765886375142578624136875423175681324
L6	1234678521438567341258764321765886573142578624136875423175681324
L7	2143685713248576423157683412768576583142587624318765132465874213
L8	1324785634125678214385674231678575682413867532415786132468574132
L9	1243567821346785432178563412856758761234658741237658341287652341
L10	1843627563254781427158363517264884523167518674237634851227681354
L11	1574683287651324764352184831267524583167532674813217854661824753
L12	1278345621874365564378126534872178126534872156433456128743652178
L13	1245678321834567843176523524187646725138576823146857324173168425
L14	1245678321834567845136723527184646327158576823146874523173168425
L15	2145678312834567846173523524187646728135573826146857324173165428
L16	1423678532418567413278562314567886754132578624136857324175681324
L17	1326784532178456213857647685412387543612487265315461237865431287
L18	2136784513278456321857647685412387543612487265315461237865431287
L19	1326784532178456213547687658312487436512487256315461238765841273
L20	1325678432184567213478568543167246725138578624136857324174618325
L21	2145678312834567843176523524187646728135576823146857324173165428
L22	2143658712854763483176523514287664728135576813248657324173265418
L23	2156784312873456854317627634812537286514487256315461238763154278
L24	1256784321873456853417627643812537285614487265315461237863154287
L25	1326784532178456218547637658312487435612487265315461237865341287
L26	1345678285172346268174533768152442765138582346176452387171348265
L27	2143658712583764457128363816745263278145578416238635427174625318
L28	1345876282175346263174583768152445762831782346156452318751846273
L29	1345876282175346268174533764182545782631782365146452318751364278
L30	1345876282175346268174533764182545782631782365146452317851364287

$\|L4 \cap L8\| = 0$	$\|L8 \cap L9\| = 1$	$\|L17 \cap L13\| = 2$	$\|L2 \cap L8\| = 3$
$\|L6 \cap L8\| = 4$	$\|L27 \cap L28\| = 5$	$\|L2 \cap L12\| = 6$	$\|L1 \cap L19\| = 7$
$\|L24 \cap L28\| = 8$	$\|L7 \cap L9\| = 9$	$\|L1 \cap L11\| = 10$	$\|L23 \cap L28\| = 11$
$\|L2 \cap L11\| = 12$	$\|L1 \cap L26\| = 13$	$\|L19 \cap L29\| = 14$	$\|L4 \cap L9\| = 15$
$\|L3 \cap L11\| = 16$	$\|L20 \cap L29\| = 17$	$\|L7 \cap L11\| = 18$	$\|L3 \cap L13\| = 19$
$\|L13 \cap L30\| = 20$	$\|L2 \cap L10\| = 21$	$\|L3 \cap L10\| = 22$	$\|L20 \cap L22\| = 23$
$\|L2 \cap L14\| = 24$	$\|L2 \cap L15\| = 25$	$\|L13 \cap L26\| = 26$	$\|L1 \cap L7\| = 27$
$\|L1 \cap L10\| = 28$	$\|L15 \cap L16\| = 29$	$\|L5 \cap L7\| = 30$	$\|L17 \cap L23\| = 31$
$\|L22 \cap L27\| = 32$	$\|L4 \cap L7\| = 33$	$\|L3 \cap L20\| = 34$	$\|L20 \cap L21\| = 35$
$\|L18 \cap L19\| = 36$	$\|L18 \cap L24\| = 37$	$\|L19 \cap L23\| = 38$	$\|L4 \cap L16\| = 39$
$\|L1 \cap L16\| = 40$	$\|L26 \cap L30\| = 41$	$\|L14 \cap L15\| = 42$	$\|L18 \cap L25\| = 43$
$\|L26 \cap L28\| = 44$	$\|L3 \cap L4\| = 45$	$\|L21 \cap L22\| = 46$	$\|L28 \cap L30\| = 47$
$\|L1 \cap L2\| = 48$	$\|L1 \cap L3\| = 49$	$\|L28 \cap L29\| = 50$	$\|L2 \cap L3\| = 51$
$\|L1 \cap L6\| = 52$	$\|L19 \cap L25\| = 53$	$\|L1 \cap L4\| = 54$	$\|L17 \cap L18\| = 55$
$\|L13 \cap L21\| = 56$	$\|L4 \cap L5\| = 57$	$\|L2 \cap L6\| = 60$	

Appendix E

L1	213567894132945678321789456957418362468251937579826143684392715795634281846173529
L2	132547698321965874213789456597418362468152937759826143684391725975634281846273519
L3	213547698132965874321789456597418362468152937759826143684391725975634281846273519
L4	124567893219345678946183752352719486468251937573928164687492315795836241831674529
L5	184365927843297615432159876321674598695731482579418263968542731217986354756823149
L6	132547698381674529217986354569412873478159236746293185653821947925738461894365712
L7	145237698486173529563912874219786453371859246732694185658421937927548361894365712
L8	132547698321684957213965874569712483486193725745238169698471532957826341874359216
L9	132987654321879546213798465987465321879654213798546132654321798546213987465132879
L10	145237698426183957563912874219478563381759426732896145698541732957624381874365219
L11	159624873528719436983547621675431298214358967497186352846293715732965184361872549
L12	159624873528719436983547621675831249214376598497165382846253917732498165361982754
L13	159624873528719436983547621675931284214365798497158362846273519732896145361482957
L14	159624873528719436986543721675431298214397685493178562847265319732986154361852947
L15	159624873532716498928547631675931284214358967467182359846293715793865142381479526
L16	351624987528719436189547623675431298214378569497186352946253871832965714763892145
L17	159624873568712439983547621675431298214379586427198365846253917732986154391865742
L18	745236198486153927563912874219487653351829746632791485198674532927548361874365219
L19	159624873528719436987546321675831249214357698496173582843265917732498165361982754
L20	159624873528719436986543721675931284214357698493178562847265319732896145361482957
L21	759624183528719436984357621673891245215938764497186352146273598832465917361542879
L22	759624183583719426932857641678491235215938764497186352146273598824365917361542879
L23	159624873563712498932857641678491235215938764427189356846273519794365182381546927
L24	351624879526713498165832947678491235213978564432189756849257613794365182987546321
L25	159624873532716498925837641678951234213548967467182359846293715794365182381479526
L26	136587294721698435283971546978412653497856312349765128615234879562349781854123967
L27	136587294721698435283971546978412653497865312349756128615234987562349871854123769
L28	136587294721698435283971546978412653497856312349765128615234789562349871854123967
L29	136587294791368452248971536972815643487692315359746128615234879563429781824153967
L30	136587294791368452248971536972815643487692315359746128615234789563429871824153967
L31	136587294791368452243951876972415683485632917359746128618294735567829341824173569
L32	136587294791368452248971536972815643487652319359746128615234987563429871824193765
L33	136587294791368452249871563972415638487692315358749126615234987564923871823156749
L34	136587294791358462248971536972815643457629318389746125615234879563492781824163957
L35	136587294751328469248971536972815643427659318389746125615234987563492871894163752
L36	136587294791368452243971568972815643487692315358749126615234879569423781824156937

$\lvert L3 \cap L12\rvert = 0$	$\lvert L5 \cap L6\rvert = 1$	$\lvert L6 \cap L14\rvert = 2$	$\lvert L5 \cap L9\rvert = 3$	$\lvert L1 \cap L14\rvert = 4$
$\lvert L6 \cap L14\rvert = 5$	$\lvert L4 \cap L9\rvert = 6$	$\lvert L1 \cap L7\rvert = 7$	$\lvert L3 \cap L5\rvert = 8$	$\lvert L2 \cap L5\rvert = 9$
$\lvert L5 \cap L8\rvert = 10$	$\lvert L4 \cap L7\rvert = 11$	$\lvert L3 \cap L11\rvert = 12$	$\lvert L3 \cap L7\rvert = 13$	$\lvert L1 \cap L5\rvert = 14$
$\lvert L2 \cap L7\rvert = 15$	$\lvert L4 \cap L14\rvert = 16$	$\lvert L1 \cap L6\rvert = 17$	$\lvert L3 \cap L8\rvert = 18$	$\lvert L4 \cap L8\rvert = 19$
$\lvert L4 \cap L15\rvert = 20$	$\lvert L6 \cap L9\rvert = 21$	$\lvert L8 \cap L29\rvert = 22$	$\lvert L6 \cap L27\rvert = 23$	$\lvert L3 \cap L10\rvert = 24$
$\lvert L19 \cap L24\rvert = 25$	$\lvert L3 \cap L6\rvert = 26$	$\lvert L2 \cap L8\rvert = 27$	$\lvert L24 \cap L17\rvert = 28$	$\lvert L3 \cap L4\rvert = 29$
$\lvert L2 \cap L4\rvert = 30$	$\lvert L14 \cap L22\rvert = 31$	$\lvert L16 \cap L24\rvert = 32$	$\lvert L2 \cap L6\rvert = 33$	$\lvert L20 \cap L23\rvert = 34$
$\lvert L14 \cap L23\rvert = 35$	$\lvert L15 \cap L16\rvert = 36$	$\lvert L6 \cap L8\rvert = 37$	$\lvert L15 \cap L19\rvert = 38$	$\lvert L8 \cap L10\rvert = 39$
$\lvert L19 \cap L21\rvert = 40$	$\lvert L16 \cap L22\rvert = 41$	$\lvert L22 \cap L25\rvert = 42$	$\lvert L13 \cap L16\rvert = 43$	$\lvert L1 \cap L4\rvert = 44$
$\lvert L31 \cap L26\rvert = 45$	$\lvert L12 \cap L16\rvert = 46$	$\lvert L16 \cap L21\rvert = 47$	$\lvert L23 \cap L21\rvert = 48$	$\lvert L13 \cap L15\rvert = 49$
$\lvert L12 \cap L14\rvert = 50$	$\lvert L7 \cap L10\rvert = 51$	$\lvert L26 \cap L30\rvert = 52$	$\lvert L13 \cap L17\rvert = 53$	$\lvert L1 \cap L2\rvert = 54$
$\lvert L7 \cap L18\rvert = 55$	$\lvert L13 \cap L14\rvert = 56$	$\lvert L11 \cap L12\rvert = 57$	$\lvert L22 \cap L23\rvert = 58$	$\lvert L29 \cap L31\rvert = 59$
$\lvert L30 \cap L31\rvert = 60$	$\lvert L32 \cap L36\rvert = 61$	$\lvert L34 \cap L36\rvert = 62$	$\lvert L1 \cap L3\rvert = 63$	$\lvert L33 \cap L36\rvert = 64$
$\lvert L30 \cap L34\rvert = 65$	$\lvert L32 \cap L35\rvert = 66$	$\lvert L30 \cap L36\rvert = 67$	$\lvert L12 \cap L19\rvert = 68$	$\lvert L29 \cap L34\rvert = 69$
$\lvert L26 \cap L27\rvert = 70$	$\lvert L29 \cap L32\rvert = 71$	$\lvert L2 \cap L3\rvert = 72$	$\lvert L27 \cap L28\rvert = 73$	$\lvert L30 \cap L32\rvert = 74$
$\lvert L26 \cap L28\rvert = 77$				

Appendix F

L1	13658710924	91017683452	27369141085	10874132569	49108526137
	35497612108	64231057891	56924108713	81510249376	72813956410
L2	13658710924	92176834510	27369141085	10874132569	49108526137
	35497612108	64231057891	56924108713	81510249376	71081395642
L3	13658710924	91017683452	27369141085	10874132569	49108526137
	35497612108	64231059871	56924108713	81510247396	72813956410
L4	13658710924	92176103458	27369141085	10874132569	49108526137
	35497816102	64231057891	51092468713	81510249376	76813952410
L5	13658710924	92176103458	27369141085	10874132569	49108526137
	35497618102	64231058791	51092487613	81510249376	76813952410
L6	13658710924	92176103458	27369141085	10874132569	49108526137
	35497816102	64231059871	51092468713	81510247396	76813952410
L7	13657810924	96171023458	27369141085	10874132569	49108651732
	32495681107	54238109671	61092475813	81510247396	75813962410
L8	13657109824	96171023458	27369141085	10874132569	49108651732
	32495876101	54238769110	81092465173	71510248396	65813910247
L9	83657109124	92171063458	27369141085	10874132569	49108651732
	36495278101	54238769110	11092485673	71510248396	65813910247
L10	14523108796	48619310257	56381294710	21891076345	39110587624
	10327894561	81096745132	72436518109	95742631018	67105412983
L11	94523108716	48619310257	56108129473	21810795364	39171084625
	10329876541	81095467132	72436518109	15762431098	67345129810
L12	91054368712	10869735241	56381274109	49816510327	37165429810
	63254191078	85710293164	72439101856	14102876593	21971084635
L13	42513108796	21698310547	56381294710	19841076325	38110795264
	10327984651	81096547132	75432618109	94726531018	67105412983
L14	21543108796	12698310547	56108129473	49811076325	38110795264
	10327984651	81096547132	75432618109	94726531018	67354129810
L15	13651098724	92178103456	27369141085	10874132569	49108576231
	31049765812	84236571910	75924816103	61510249378	56813210947
L16	13279104856	32110897564	21354761098	71059381642	98436510217
	10978563421	47611032985	85106249173	56941287310	64827153109
L17	32179104856	21310897564	13254761098	71059381642	98436510217
	10978563421	47611032985	85106249173	56941287310	64827153109

$|L1 \cap L14| = 2$ $|L14 \cap L15| = 14$ $|L11 \cap L12| = 32$ $|L13 \cap L11| = 50$
$|L1 \cap L9| = 59$ $|L3 \cap L8| = 61$ $|L3 \cap L15| = 62$ $|L6 \cap L15| = 65$
$|L6 \cap L8| = 67$ $|L1 \cap L7| = 68$ $|L4 \cap L15| = 69$ $|L5 \cap L7| = 71$
$|L6 \cap L7| = 74$ $|L1 \cap L6| = 86$ $|L16 \cap L17| = 91$

Appendix G

L1	2135476981110	1327109411658	3211198106574	5711811024963
	4109151187326	7981011613245	6410281751139	9116473510182
	8659321114107	1157624381091	1084365927111	
L2	8325476191110	3217109411658	2131198106574	5711416291083
	4109151187326	7986114132105	6410281751139	1116107358492
	9658321110147	1157921034861	1084365927111	
L3	1325476891110	3217109411658	2131198106574	5711416291083
	4109151187326	7986114132105	6410281751139	8116107351492
	9658321110147	1157921034861	1084365927111	
L4	2135476891110	1327109411658	3211198106574	5711614291083
	4109151187326	7984116132105	6410281751139	8116107351492
	9658321110147	1157921034861	1084365927111	
L5	1325476891110	3217109411658	2186941035711	5761113291084
	4109151187326	7943118162105	6410281751139	8113107651492
	9658321110147	1157921034861	1081146592713	
L6	8325476191110	3217109411658	2111694108573	5764132910811
	4109151187326	7943118162105	6410281751139	1118107653492
	9658321110147	1157921034861	1083116592714	
L7	8325476191110	3412109711658	2111694108573	5267134910811
	4109151187326	7943118162105	6710481251139	1118107653492
	9658321110147	1157921034861	1083116592714	
L8	8325476191110	3917106411258	2186941035711	5761113291084
	4109158116327	7643821511109	6410211198735	1113106587492
	9258311710146	1157921034861	1081147952613	
L9	6325419871110	3917106411258	2110694875311	5761113291084
	4109158116327	1643821051179	9482111073165	8117106531492
	7258311110946	1153927648101	1081147952613	
L10	8325469171110	3617102411958	2111694875103	5764132910811
	4109158116327	6243811051179	9482111073165	1117106538492
	7958311110246	1151092764831	1083117952614	
L11	6325419871110	3617102411958	2111694875103	5764132910811
	4109158116327	1243861051179	9482111073165	8117106531492
	7958311110246	1151092764831	1083117952614	
L12	6325419871110	3617102411958	2111694875103	5761113291084
	4109158116327	1243861051179	9482111073165	8117106531492
	7958311110246	1151092764831	1083479526111	
L13	1325469871110	3617102411958	2111694875103	5764132910811
	4109158116327	6243811051179	9482111073165	8117106531492
	7958311110246	1151092764831	1083117952614	
L14	1132546987101	3617102411958	2136948751110	5761113291084
	4109158116327	6243811051179	9482111073165	8117106531492
	7958311110246	1051192764813	1810479526311	
L15	1325789104116	3211141056789	2138641051197	9564111831072
	4782931115610	1185310671924	5911102764138	6104732119815
	7111015928643	8679115423101	1049681372511	
L16	2135789104116	1321141056789	3218641051197	9564111831072
	4782931115610	1185310671924	5911102764138	6104732119815
	7111015928643	8679115423101	1049681372511	
L17	3615429871011	6937101411258	1326948751110	5761113291084
	4109158116327	2143861051179	9482111073165	8117106531492
	7258311110946	1051192764813	1181047952631	
L18	4651329117108	6119710148352	5936148721110	1765432910811
	3101498116527	2143861051179	9482111073165	1187106532491
	7328511110946	1051192764813	8210117951634	
L19	4651329117108	6297101483511	5936148721110	1761153291084
	3101548116927	2143861051179	9482111073165	1187106532491
	7328911110546	1051192764813	8111047951632	
L20	1065732411918	6492101118357	5936148721110	7264113191085
	3101115896724	2143861051179	4118191073562	1187106532491
	9328711510146	1511927648103	8710549216311	

L21	1697102411358 6754312891011 9536248711110 7462113191085
	1032115896714 2143861051179 4281910113567 1187106532491
	3918711510246 5101191764823 8111054971632
L22	5697124103811 6258311191047 9536248711110 7106411312895
	1321158967104 2143861051179 4118191073562 1047965311218
	3811071154926 8911210761453 1171054928631
L23	3615429871011 6297101411358 1936548721110 5761113291084
	4105138116927 2143861051179 9482111073165 8117106531492
	7328911110546 1051192764813 1181047952631

$\|L1 \cap L21\| = 2$	$\|L2 \cap L22\| = 5$	$\|L16 \cap L22\| = 8$	$\|L1 \cap L15\| = 11$	$\|L2 \cap L21\| = 14$
$\|L11 \cap L16\| = 17$	$\|L1 \cap L16\| = 20$	$\|L3 \cap L15\| = 23$	$\|L3 \cap L20\| = 26$	$\|L5 \cap L20\| = 29$
$\|L6 \cap L19\| = 32$	$\|L5 \cap L19\| = 35$	$\|L1 \cap L9\| = 38$	$\|L8 \cap L20\| = 40$	$\|L4 \cap L23\| = 41$
$\|L2 \cap L17\| = 44$	$\|L3 \cap L17\| = 47$	$\|L5 \cap L23\| = 50$	$\|L4 \cap L13\| = 53$	$\|L2 \cap L9\| = 56$
$\|L3 \cap L9\| = 59$	$\|L5 \cap L14\| = 62$	$\|L5 \cap L11\| = 65$	$\|L5 \cap L12\| = 68$	$\|L12 \cap L19\| = 71$
$\|L1 \cap L5\| = 74$	$\|L2 \cap L8\| = 76$	$\|L14 \cap L18\| = 77$	$\|L14 \cap L19\| = 80$	$\|L6 \cap L8\| = 82$
$\|L11 \cap L23\| = 83$	$\|L17 \cap L18\| = 84$	$\|L10 \cap L17\| = 86$	$\|L1 \cap L2\| = 87$	$\|L5 \cap L8\| = 88$
$\|L13 \cap L17\| = 89$	$\|L11 \cap L17\| = 90$	$\|L3 \cap L7\| = 91$	$\|L4 \cap L6\| = 92$	$\|L12 \cap L17\| = 93$
$\|L2 \cap L7\| = 94$	$\|L14 \cap L23\| = 95$	$\|L9 \cap L14\| = 98$	$\|L10 \cap L14\| = 101$	$\|L9 \cap L11\| = 104$
$\|L9 \cap L12\| = 107$	$\|L10 \cap L12\| = 110$	$\|L6 \cap L7\| = 112$	$\|L12 \cap L13\| = 113$	

Appendix H

L1	13254769811101312	32167451013812119	21396587412131011	56911121013374128
	47615321211139810	74523121111098613	65810211331241197	91071312113812654
	81343111012196572	11812713942610135	10121349811651723	13111012869573241
	12911810137425316			
L2	13254126981110137	32167451013812119	21396587412131011	56911121013374128
	47615321211139810	74523613111098112	13581021731241196	91071312113812654
	81343111012196572	11812713942610135	10121349811651723	61110128139573241
	12911810714253613			
L3	13254121398111076	32167451013812119	21396587412131011	56941210133711128
	47615321211139810	13452310911126817	12581021311364791	91071312113812654
	81343117121109562	11812713642910135	10121311987651423	61110128915732413
	79118101642531312			
L4	13254121398111076	32167451013812119	21396587412131011	56911121013374128
	47615321211139810	13452369111210817	12581021373641191	91071312113812654
	81343117121109562	11812713104296135	10121349811651723	61110128915732413
	79118101642531312			
L5	21354121398111076	13267451013812119	32196587412131011	56911121013374128
	47615321211139810	13452369111210817	12581021373641191	91071312113812654
	81343117121109562	11812713104296135	10121349811651723	61110128915732413
	79118101642531312			
L6	13254121398111076	32167451013812119	21396587412131011	56912121113310748
	47615321211139810	13452310911126817	12581121310367491	91071312113812654
	81343117121109562	11812101367294135	10121379846511123	61110489157321213
	79118101642531312			
L7	21354121398111076	13267451013812119	32196587412131011	56912121113310748
	47615321211139810	13452310911126817	12581121310367491	91071312113812654
	81343117121109562	11812101367294135	10121379846511123	61110489157321213
	79118101642531312			
L8	13254121398111076	32167451013812119	21436587912131011	56312121113109748
	47618325111391210	13452369111210817	12581121310367491	91071351131212684
	81391011712134562	11812913107246135	10121379846511123	61110412918732513
	79118101642531312			
L9	17109121121343658	72431312111109865	10432891156171312	93246587112131011
	12138654937111012	11129546131038127	21118913104573126	13115731048126291
	41061735121129813	39112118762135410	68713101329512114	56131012129841173
	85121127611310439			
L10	17109121121343658	72431312111109865	10452391186171312	93246587112131011
	12133684957111012	11129546131038127	21118913104573126	13118751043126291
	41061735121129813	39112118762135410	68713101329512114	56131012129841173
	85121127611310439			

$|L1 \cap L1| = 2$ $|L1 \cap L9| = 5$ $|L2 \cap L7| = 116$ $|L1 \cap L6| = 122$

$|L2 \cap L6| = 125$ $|L1 \cap L5| = 128$ $|L3 \cap L8| = 131$ $|L3 \cap L4| = 158$