Counting the Number of Star-Factors in Graphs

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ABSTRACT. A star-factor of a graph G is a spanning subgraph of G such that each component of which is a star. In this paper, we study the structure of the graphs with a unique star-factor and obtain an upper bound on the number of edges such graphs can have. We also investigate the number of star-factors in a regular graph.

1 Introduction

For a fixed integer k, let $S(k) = \{K_{1,i} \mid 1 \leq i \leq k\}$. An *$S(k)$-factor* (also called a *star-factor*) of a graph G is a spanning subgraph of G, each component of which is isomorphic to a member of $S(k)$. (Note that an $S(1)$-factor is simply a 1-factor.) An $S(k)$-factor is *proper* if one of its components is isomorphic to $K_{1,k}$. The complete bipartite graph $K_{1,i}$ is called an *i-star* (or simply a *star*).

In 1947, Tutte [8] gave a criterion for a graph to have a 1-factor (that is, an $S(1)$-factor). This criterion was then used by many others to study properties of graph with 1-factors. In particular, Lovász [7] showed that a graph with a unique 1-factor cannot have large minimum degree, and Hetyei (see [7]) proved that the largest number of edges in a graph G of order $2m$ with a unique perfect matching is m^2. Lovász [7] and Zaks [9] obtained a lower bound on the number of 1-factors in n-connected graphs.

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In this paper, we focus on $S(k)$-factors with $k \geq 2$. A characterization of $S(k)$-factor for $k \geq 2$ was given by Las Vergnas [6], Hell and Kirkpatrick [5] and Amahashi and Kano [2] independently. In section 2, we study the structure of the graphs which have a unique $S(k)$-factor, $k \geq 2$, and obtain an upper bound on the number of edges such graphs can have, and also constructing an extremal graph with a unique $S(k)$-factor which attains that bound. In section 3, we show that any r-regular graph of order n has at least n distinct $S(k)$-factors $(1 \leq k \leq r)$.

For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S, $I(G-S)$ denote the set of isolated vertices in $G - S$, and $i(G - S) = |I(G - S)|$. Other notations and terminology will follow [3].

2 Graphs with a unique $S(k)$-factor

In order to study the structure of those graphs with a unique $S(k)$-factor, we need to introduce another notation.

In the star $K_{1,i}$, $i \geq 2$, we call the vertex of degree i the *centre*, and the vertices of degree 1 the *leaves*. For $K_{1,1}$ we arbitrarily prescribe one vertex to be the centre and the other the leaf.

Let F be an $S(k)$-factor of G, and suppose that F has m_i components which are isomorphic to $K_{1,i}$, $1 \leq i \leq k$; implying that $\sum m_i(1+i) = |V(G)|$. We denote the centres of these m_i stars by $x(i,1), x(i,2), \ldots, x(i,m_i)$, $1 \leq i \leq k$, and the leaves of the star with centre $x(i,j)$ by $y_1(i,j), y_2(i,j), \ldots, y_i(i,j)$. So the components of F can be described by $\{S(i,j) = \{x(i,j); y_1(i,j), \ldots, y_i(i,j)\} \colon 1 \leq i \leq k, 1 \leq j \leq m_i\}$. For convenience we write $x(1,j) = x_j$ and $y_1(1,j) = y_j$. Finally, we let S_c denote the set of all centres, that is, $S_c = \{x(i,j) \colon 1 \leq i \leq k, 1 \leq j \leq m_i\}$.

For $k = 1$, an $S(1)$-factor is simply a 1-factor. Hetyei (see [7]) determined that the largest number of edges in the graph G of order $2m$ with a unique 1-factor is m^2. Hence, we may now restrict our attention to the case $k \geq 2$.

Lemma 2.1. *If G has a unique $S(k)$-factor F $(k \geq 2)$, then there exists a set S, $S \subseteq V(G)$, so that $I(G - S) = V(G) - S$, and the number of components in F is $|S|$.*

Proof: We will show that the centres of the stars $K_{1,1}$'s in the $S(k)$-factor can be chosen so that S_c satisfies the requirement. First we choose the centres of the $K_{1,1}$'s arbitrarily and let S be the resulting set S_c. Since F is unique, the only possible edges in $G[V - S]$ are those joining leaves of stars $K_{1,1}$. Suppose we have such an edge, say $y_j y_t$. Then x_j and x_t have no other neighbors in $\{y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{t-1}, y_{t+1}, \ldots, y_{m_1}\}$ or we get a path of length 5 and hence two $K_{1,2}$'s instead of the three $K_{1,1}$'s. It is clear that $x_j x_t \notin E(G)$. Moreover if $k \geq 3$, then $x_j y_t \notin E(G)$ and $x_t y_j \notin E(G)$ since otherwise the edges $x_j y_j$, $x_t y_t$ are replaced by a $K_{1,3}$.

Finally $x_jx_t \notin E(G)$. Exchange the centre-leaf roles of x_jy_j and x_ty_t, and let $S' = (S - \{x_j, x_t\}) \cup \{y_j, y_t\}$. Then $|E(G[V - S'])| < |E(G[V - S])|$. Now we may proceed inductively to complete the proof. □

From now on, we assume that S_c satisfies $I(G - S_c) = V(G) - S_c$. The following lemma is easily proven.

Lemma 2.2. *If a graph G has a unique $S(k)$-factor F. Then the only vertices that the leaves of any component $K_{1,i}$ ($2 \le i \le k$) of F can be adjacent to are their own centres and the centres of k-stars.*

We next show that for $k \ge 2$, a graph with a proper unique $S(k)$-factor (that is, the graph has a unique $S(k)$-factor, and that unique $S(k)$-factor is proper) has at least k vertices of degree one. Note that this result does not hold for $k = 1$.

Theorem 2.3. *If G has a unique proper $S(k)$-factor F ($k \ge 2$), then the leaves of one of k-stars in F are vertices of degree one in G.*

Proof: The leaves of k-stars cannot be adjacent to any other vertices except centres of these k-stars. So if the $S(k)$-factor has only one k-star, we are done. Otherwise, we assume that for each k-star there is an edge from one of its leaves to the centre of another k-star. Construct a digraph H with $V(H) = \{a_i \mid 1 \le i \le m_k\}$ and $(a_i, a_j) \in A(H)$ if there is an edge from a leaf of $S(k, i)$ to the centre of $S(k, j)$. If H has a directed cycle of length at least two, we exchange edges between the k-stars on this cycle to get another $S(k)$-factor. So we assume that H is acyclic. Then H has a vertex with outdegree 0 meaning that there are no edges out of the leaves of the corresponding k-star and so the leaves of this k-star are the vertices of degree one in G. □

Corollary 2.4. *If G has a proper $S(k)$-factor ($k \ge 2$) and $\delta(G) \ge 2$, then G has at least two $S(k)$-factors.*

Proof: It follows from Theorem 2.3. □

Remark 2.5. At this point it is helpful to provide the reader with a description of what we now know of graphs with a unique $S(k)$-factor F, as shown in Figure 1 (the centres are at the top). We describe the other edges which may lie in the graph.

From the leaves of the k-stars we have edges to centres of k-stars so that the digraph H of Theorem 2.3 is acyclic. Leaves of i-stars, $2 \le i \le k$, can only be adjacent to centres of k-stars. Any two centres can be adjacent, but if the centre of a 1-star in F is adjacent to another centre then its leaf can not be adjacent to the same centre unless it is the centre of a k- or $(k-1)$-star. By Lemma 2.1 no leaves are adjacent.

In order to obtain an upper bound on the number of edges in a graph G with a unique $S(k)$-factor F ($k \ge 2$), we associate G and F with two graphs G_1 and F_1 defined as follows.

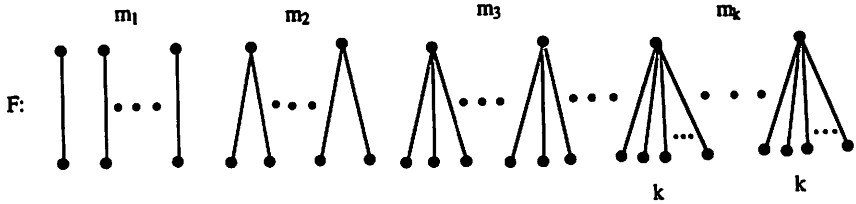

Figure 1

Without loss of generality, suppose that $m_k \neq 0$ and let $S(k,1)$ be the k-star whose leaves are vertices of degree 1 in G (as described in Theorem 2.3).

Let $V(G_1) = V(G)$, where the edges of G_1 are those of G except that if both $x(i_1,j)y_s(i_2,r) \in E(G)$ and $x(i_1,j)x(i_2,r) \notin E(G)$, then $x(i_1,j)x(i_2,r) \in E(G_1)$ and $x(i_1,j)y_s(i_2,r) \notin E(G_1)$. So $|E(G)| = |E(G_1)|$.

Define F_1 as follows:

$$V(F_1)=V(F)=V(G)$$
$$E(F_1)=\{x(i,s)x(j,r) \mid x(i,s)\neq x(j,r), 1 \le i,j \le k, 1 \le s \le m_i, 1 \le r \le m_j\}$$
$$\cup \{x(k,s)y_r(i,h) \mid s = 1,2,\dots,m_k;\ \text{if } i=k,\ \text{then } s+1 \le h \le m_k,$$
$$1 \le r \le k; \text{and if } 1 \le i < k,\ \text{then } 1 \le h \le m_i, 1 \le r \le i\} \cup E(F).$$

That is, F_1 contains all edges of the $S(k)$-factor F, a complete subgraph on the vertex-set S_c, and contains edges from leaves to centres of k-stars. It is easy to see that F is the unique $S(k)$-factor of the new graph F_1.

Lemma 2.6. *For a given graph G with a proper unique $S(k)$-factor F ($k \ge 2$), we define G_1 and F_1 as mentioned above. Then $|E(G_1)| \le |E(F_1)| + \epsilon$ where if $k = 2$ and $m_1 = 2$ or 3, or if $m_{k-1} = 1$ and $m_1 \ge 1$, then $\epsilon = 1$, and in all other cases $\epsilon = 0$.*

Proof: We prove the lemma by constructing a one-to-one mapping f from $E(G_1)$ or $E(G_1) - \{e\}$, $e \in E(G_1)$ (as appropriate), into $E(F_1)$.

The mapping f acts as the identity on (1) the edges of the $S(k)$-factor $F = \cup S(i,j)$; (2) the edges $x(k,s)y_r(i,h) \in E(G_1)$; and (3) the edges $x(i,s)x(j,r) \in E(G_1)$.

By Lemma 2.2 all that remains is to define the action of f on the edges $y_s x(i,j) \in E(G_1)$, $1 \le i \le k-1$. (Recall that $x(1,j) = x_j$ and $y_1(1,j) = y_j$.) If $y_s x(i,j) \in E(G_1)$, then, by the construction of G_1, $x_s x(i,j) \in E(G_1)$ and so both $y_s x(i,j)$ and $x_s x(i,j)$ are edges of G. If $2 \le i \le k-2$ we then obtain another $S(k)$-factor in G. Hence $y_s x(i,j) \in E(G_1)$ implies that $i \in \{1, k-1, k\}$. We have already defined $f(y_s x(k,j))$ so only two cases remain. Consider first $y_s x(k-1,j) \in E(G_1)$, $k-1 \neq 1$ so $k \ge 3$.

If $m_{k-1} = 1$, let $I = \{s \mid y_s x(k-1,1) \in E(G_1)\}$. It is easy to see that if $s, r \in I$, $s \neq r$, then $x_s x_r \notin E(G_1)$. Provided that $|I| \geq 3$, we can extend the one-to-one map f by mapping $\{y_s x(k-1,1) \mid s \in I\}$ into $\{x_s x_r \mid s, r \in I, s \neq r\}$. If $0 < |I| < 3$ such an extension is only possible from $\{y_s x(k-1,1) \mid s \in I - \{j\}, j \in I\}$ into $\{x_r x_s \mid r, s \in I, r \neq s\}$ and we have $\epsilon = 1$. Consider the case of $m_{k-1} \geq 2$ next. Since $y_s x(k-1,j) \in E(G_1)$ implies that $y_s x(k-1,j)$, $x_s x(k-1,j) \in E(G)$ and F is unique, it follows that $y_s x(k-1,t)$, $x_s x(k-1,t) \notin E(G)$ where $t \neq j$. So we put $f(y_s x(k-1,j)) = x_s x(k-1,t)$.

Finally we consider the edges $y_i x_j \in E(G_1)$, $i \neq j$. Clearly $m_1 \geq 2$ (or there are no such edges).

Case 1. $k \geq 3$. Since $y_i x_j \in E(G_1)$ $(i \neq j)$, it follows that $x_i x_j \in E(G_1)$ and both $y_i x_j$, $x_i x_j$ are edges of $E(G)$ and thus we can construct a 3-star instead of two 1-stars, a contradiction.

Case 2. $k = 2$. If $m_1 = 2$ or 3 then either there are no edges of type $y_i x_j$, $i \neq j$, or the subgraphs spanned by the 1-stars are isomorphic to one of the four shown in Figure 2 (a) (b) (c) (d).

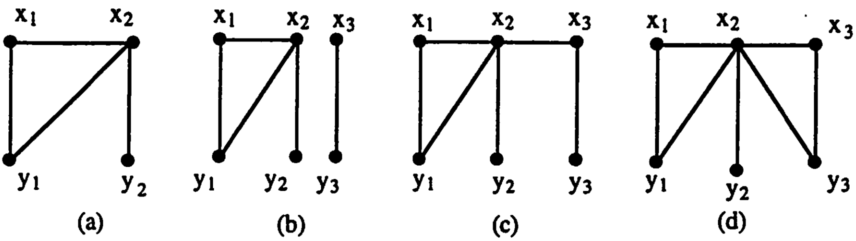

Figure 2

In case (a) the edge $y_1 x_2$ has no image, but cases (b) and (c) the edge $y_1 x_2$ has an image $x_1 x_3$, and in the fourth (Figure 2 (d)) put $f(y_3 x_2) = x_1 x_3$ $(x_1 x_3 \notin E(G_1))$. (Observe that if $m_{k-1} = 1$ and $|I|$ is 2 or 3, then $i, j \in I$ $y_i x_j \notin E(G_1)$ and no conflict can arise.)

What now remains is the case $k = 2$ and $m_1 \geq 4$. Let G_1' be a subgraph of G_1 induced by vertices $x_1, \ldots, x_{m1}$, $y_1, \ldots, y_{m1}$ and F_1' be a subgraph of F_1 induced by vertices $x_1, \ldots, x_{m1}$, $y_1, \ldots, y_{m1}$. If we can show that $|E(G_1')| \leq |E(F_1')|$, then we will be able to define f on these remaining edges and so obtain the described one-to-one mapping.

The proof is by induction on m_1. Calculation shows that the claim is valid when $m_1 = 4$. Suppose now that the claim holds for $m_1 < m$ and consider the case $m_1 = m$. Without loss of generality, we suppose that $y_1 x_2 \in E(G_1)$, implying that $x_1 x_2 \in E(G_1)$, $x_1 y_i \notin E(G_1)$, $2 \leq i \leq m$, $x_1 x_j \notin E(G_1)$ and $y_1 x_j \notin E(G_1)$, $3 \leq j \leq m$. Thus $|E(G_1')| = |E(G_1' - \{x_1, y_1\})| + 3$. But $|E(F_1')| = |E(F_1' - \{x_1, y_1\})| + m + 1$ and by the induction hypothesis $|E(G_1' - \{x_1, y_1\}| \leq |E(F_1' - \{x_1, y_1\})|$ so $|E(G_1')| \leq |(G(F_1')| + 3 - m - 1 \leq |E(F_1')|$ as required.

Thus we have described the required mapping f and the proof is complete. □

With Remark 2.5 and Lemma 2.6, we now are able to describe exactly the graphs with maximum number of edges which have F as a proper unique $S(k)$-factor.

Corollary 2.7. *If a graph G has the subgraph F as a proper unique $S(k)$-factor ($k \geq 2$), then $|E(G)| \leq |E(F_1)| + 1$.*

We next determine the maximum of $|E(F_1)|$ over all $S(k)$-factors F with n vertices. Given n and $k \geq 2$ we denote by $f(n,k)$ the maximum number of edges in a graph of n vertices which has a proper unique $S(k)$-factor. Hence for any graph G of order n which has a proper unique $S(k)$-factor we have $|E(G)| \leq f(n,k)$.

Theorem 2.8. *If a graph G of order n has proper unique $S(k)$-factor ($k \geq 2$), then*

$$f(n,k) = \begin{cases} \frac{n(n+1)}{6} + 1 & \text{if } k = 2 \text{ and } n \equiv 0,2 \pmod 3 \\ \frac{(n-1)(n+2)}{6} & \text{if } k = 2 \text{ and } n \equiv 1 \pmod 3 \\ \frac{(n-1)(n+3)}{8} + 1 & \text{if } k = 3 \text{ and } n \equiv 1 \pmod 4 \\ \frac{(n+1)^2}{8} + 1 & \text{if } k = 3 \text{ and } n \equiv 3 \pmod 4 \\ \frac{n(n+2)}{8} & \text{if } k = 3 \text{ and } n \text{ is even} \\ \frac{(n-k)^2-9}{8} + n & \text{if } k \geq 4 \text{ and } n \not\equiv k \pmod 2 \\ \frac{(n-k)(n-k-2)}{8} + n & \text{if } k \geq 4 \text{ and } n \equiv k \pmod 2 \end{cases}$$

Proof: As mentioned in the beginning of this section we assume $k \geq 2$. Suppose that G has a proper unique $S(k)$-factor F which has m_i components isomorphic to $K_{1,i}$. Then $n = |V(G)| = \sum_{i=1}^{k} m_i(i+1))$. Thus, letting $m = \sum_{i=1}^{k} m_i$, the number of edges in F_1 is given by

$$\begin{aligned} |E(F_1)| &= |E(K_m)| + \sum_{i=1}^{k} im_i + (\sum_{i=1}^{k} im_i - k)) + (\sum_{i=1}^{k} im_i - 2k)) + \dots \\ &\quad + (\sum_{i=1}^{k} im_i - km_k)) \\ &= \frac{1}{2}m(m-1) + (m_k+1)\sum_{i=1}^{k} im_i - (k + 2k + \dots + km_k) \\ &= \frac{1}{2}(m(m-1) + (m_k+1)(2\sum_{i=1}^{k} im_i - km_k)) \end{aligned}$$

Let

$$g(F) = 2|E(F_1)| = m(m-1) + (m_k+1)(2n-2m-km_k), \qquad (2.1)$$

where $m = \sum_{i=1}^{k} m_i$ and $n = \sum_{i=1}^{k} m_i(i+1)$. Setting $m^* = \sum_{i=1}^{k} m_i$, we obtain

$$\begin{aligned} g(F) &= (m^* + m_k)(m^* - m_k - 3) + (m_k+1)(2n - km_k) \\ &= (m^*)^2 - 3m^* - m_k(m_k+3) + (m_k+1)(2n-km_k). \end{aligned}$$

If m_k is fixed, then $g(F)$ is a quadric function about m^* and reaches its maximum value when m^* is maximized. Hence, in order to maximize $g(F)$ one should take as many components as possible in F. Hence $m_3 = m_4 = \cdots = m_{k-1} = 0$ and $m_2 = 0$ or 1. Therefore determining $\frac{1}{2}f(n,k)$ now becomes an integer programming problem as follows:

$$\frac{1}{2}f(n,k) = \max\{(m_1+m_2)^2 - 3(m_1+m_2) - m_k(m_k+3) + (m_k+1)(2n-km_k)\}$$

Subject to $\quad 2m_1 + 3m_2 + (k+1)m_k = n$
$\qquad\qquad\quad m_1, m_2, m_k \geq 0 \qquad$ integers.

We now start to determine the solutions for this problem.

If $n-(k+1)m_k$ is even, then $m_2 = 0$ and $m_1 = \frac{1}{2}(n-(k+1)m_k)$, and if $n-(k+1)m_k$ is odd, then $m_2 = 1$ and $m_1 = \frac{1}{2}(n-3-(k+1)m_k)$. Moreover, in order to obtain the value of m_k which maximizes $g(F)$ we consider following cases.

(1) Suppose that $n-(k+1)m_k$ is even, that is $n = (k+1)m_k + 2m_1$ and $m = m_1 + m_k$.

Suppose that $k+1$ is even (so $k \geq 3$) and $m_k \geq 2$. Let F' be an $S(k)$-factor with $|V(F)| = |V(F')|$, $m'_k = m_k - 1$, $m'_1 = m_1 + \frac{k+1}{2}$ and $m'_j = m_j = 0$, $2 \leq j \leq k-1$. So

$$g(F) = (m_1+m_k)(m_1+m_k-1) + (m_k+1)(km_k+2m_1)$$

and

$$g(F') = (m_1 + m_k + \frac{k-1}{2}(m_1 + m_k - 1 + \frac{k-1}{2} + m_k(km_k + 2m_1 + 1). \qquad (2.2)$$

Thus

$$g(F') - g(F) = (k-3)m_1 + \frac{(k-1)(k-3)}{4} \geq 0. \qquad (2.3)$$

So $g(F') \geq g(F)$ and the maximum number of edges is obtained when we have only one k-star in F.

Suppose that $k+1$ is odd and $m_k \geq 2$. Let F' be an $S(k)$-factor with $m'_k = m_k - 1$ k-stars and therefore one 2-star and $m'_1 = m_1 + \frac{k-2}{2}$ 1-stars. So $m' = m_1 + m_k + \frac{k-2}{2}$ and

$$g(F') = (m_1 + m_k + \frac{k-2}{2})(m_1 + m_k + \frac{k-4}{2}) + m_k(km_k + 2m_1 + 2). \tag{2.4}$$

Thus, from (2.2) and (2.4), we have

$$g(F') - g(F) = (k-4)m_1 + \frac{(k-2)(k-4)}{4} \geq 0 \text{ if } k \geq 4. \tag{2.5}$$

So for $k = 2$ we expect to have as many 2-stars as possible. This case will be considered in more detail later.

(2) Suppose that $n - (k+1)m_k$ is odd, that is $n = (k+1)m_k + 3 + 2m_1$ and $m = m_1 + m_k + 1$.

Suppose that $k+1$ is even and $m_k \geq 2$. Let F' be an $S(k)$-factor with $m'_k = m_k - 1$, $m'_2 = m_2 = 1$, $m'_1 = m_1 + \frac{k+1}{2}$ and $m' = m_1 + m_k + \frac{(k+1)}{2}$. So

$$g(F) = (m_1 + m_k + 1)(m_1 + m_k) + (m_k + 1)(km_k + 2m_1 + 4) \tag{2.6}$$

and

$$g(F') = (m_1 + m_k + \frac{k+1}{2})(m_1 + m_k + \frac{k-1}{2}) + m_k(km_k + 2m_1 + 5).$$

Thus

$$g(F') - g(F) = (k-3)m_1 + \frac{(k+1)(k-1)}{4} > 0 \text{ if } k \geq 5. \tag{2.7}$$

If $k = 3$, then we expect to have many k-stars. This case will be considered in detail later.

Suppose that $k+1$ is odd and $m_k \geq 2$. Let F' be an $S(k)$-factor with $m'_k = m_k - 1$, $m'_2 = 0$, $m'_1 = m_1 + \frac{k+4}{2}$ and $m' = m_1 + m_k + \frac{k+2}{2}$. So

$$g(F') = (m_1 + m_k + \frac{k+2}{2})(m_1 + m_k + \frac{k}{2}) + m_k(km_k + 2m_1 + 4) \tag{2.8}$$

Thus, from (2.6) and (2.8), we have

$$g(F') - g(F) = (k-2)m_1 + \frac{k(k+2)}{4} - 4 > 0 \text{ if } k \geq 4 \tag{2.9}$$

If $k = 2$, then it is better to have more k-stars.

From the above discussion we conclude that, except when (1) $k = 2$; and (2) $k = 3$ and n is odd, if G has a unique proper $S(k)$-factor F and as many edges as possible, we should choose F to have exactly one k-star, at most one 2-star and all other components 1-stars.

So if $k \geq 4$ we easily obtain

$$|E(F_1)| = \begin{cases} \frac{(n-k)^2-9}{8} + n & \text{if } n \not\equiv k \pmod 2 \\ \frac{(n-k)(n-k-2)}{8} + n & \text{if } n \equiv k \pmod 2), \end{cases}$$

If $k = 3$ and n is even, then $m_3 = 1$, $m_2 = 0$ and $m_1 = \frac{n-4}{2}$. Thus we have $|E(F_1)| = \frac{n(n+2)}{8}$.

We now study the exceptional cases.

When $k = 2$, from (2.5) and (2.9), we see that $g(F)$ attains its maximum if m_2 is maximized. So, with fixed n, F_1 has the most edges if the $S(k)$-factor F has as many 2-stars as possible. Hence, if $n \equiv 0 \pmod 3$, then $m_1 = 0$, $m_2 = \frac{n}{3}$ and $|E(F_1)| = \frac{(n+1)}{6}$; if $n \equiv 1 \pmod 3$, then $m_1 = 2$, $m_2 = \frac{n-4}{3}$ and $|E(F_1)| = \frac{(n+2)(n-1)}{6}$; and if $n \equiv 2 \pmod 3$, then $m_1 = 1$, $m_2 = \frac{n-2}{3}$ and $|E(F_1)| = \frac{n(n+1)}{6}$.

When $k = 3$ and n is odd, we see from (2.7) that $g(F)$ is an increasing function of m_3. So, with fixed n, F_1 has the most edges if F has as many 3-star as possible. Hence, if $n \equiv 1 \pmod 4$, then $m_1 = 1$, $m_2 = 1$, $m_3 = \frac{n-5}{4}$ and $|E(F_1)| = \frac{(n-1)(n+3)}{8}$; and if $n \equiv 3 \pmod 4$, then $m_1 = 0$, $m_2 = 1$, $m_3 = \frac{n-3}{4}$ and $|E(F_1)| = \frac{(n+1)^2}{8}$.

Summarizing the above conclusions, we obtain

$$|E(F_1)| = \begin{cases} \frac{n(n+1)}{6} & \text{if } k = 2 \text{ and } n \equiv 0,2 \pmod 3 \\ \frac{(n-1)(n+2)}{6}, & \text{if } k = 2 \text{ and } n \equiv 1 \pmod 3 \\ \frac{(n-1)(n+3)}{8} & \text{if } k = 3 \text{ and } n \equiv 1 \pmod 4 \\ \frac{(n+1)^2}{8} & \text{if } k = 3 \text{ and } n \equiv 3 \pmod 4 \\ \frac{n(n+2)}{8} & \text{if } k = 3 \text{ and } n \text{ is even} \\ \frac{(n-k)^2-9}{8} + n & \text{if } k \geq 4 \text{ and } n \not\equiv k \pmod 2 \\ \frac{(n-k)(n-k-2)}{8} + n & \text{if } k \geq 4 \text{ and } n \equiv k \pmod 2 \end{cases}$$

But, by Lemma 2.6, we have that $f(n,k) = |E(F_1)| + \epsilon$ where if $k = 2$ and $m_1 = 2$ or 3, or if $m_{k-1} = 1$, then $\epsilon = 1$, and otherwise $\epsilon = 0$. From the calculation, this implies that if $k = 2$ and $n \equiv 0, 1 \pmod 3$, or if $k = 3$ and $n \equiv 1$ or $3 \pmod 4$, then $\epsilon = 1$; otherwise $\epsilon = 0$. Therefore, we obtain the desired $f(n,k)$. □

Corollary 2.9. *If a graph H of order n has an $S(k)$-factor and $|E(H)| > f(n,k)$, where $f(n,k)$ is defined as in the Theorem 2.8, then H has at least two $S(k)$-factors.*

3 The number of $S(k)$-factors in an r-regular graph

For graphs H_1 and H_2, the *join* of H_1 and H_2, denoted by $H_1 + H_2$, is obtained from $H_1 \cup H_2$ by joining all vertices in $V(H_1)$ to those in $V(H_2)$. Let $e_G(S_1, S_2)$ denote all edges in G which have one end in S_1 and the other in S_2.

The following characterization will be needed in this section.

Theorem 3.1. (Las Vergnas [6]; Hell and Kirkpatrick [5] and Amahashi and Kano [2]) *For $k \geq 2$, the graph G has an $S(k)$-factor if and only if*

$$i(G - S) \leq k|S| \qquad \textit{for all } S \subseteq V(G).$$

Theorem 3.2. *Let G be a connected r-regular graph ($r \geq 4$) of order n which is not isomorphic to $K_{r,r}$. Then G has at least n star-factors each of which is either a proper $S(r)$-factor or a proper $S(r-1)$-factor.*

Proof: We shall show that either G is a special graph which has at least n proper $S(r)$-factors or $S(r-1)$-factors, or that for every vertex x of G, G has an $S(k)$-factor having exactly one $K_{1,k}$-component whose center is x, for some $k \in \{r, r-1\}$.

Let $x \in V(G)$ and the neighbors of x be denoted by $N_G(x) = \{y_1, y_2, \ldots, y_r\}$. Let $G_x = G[V(G) - \{x\} - N_G(x)]$ and $I(G_x) = \{z_1, z_2, \ldots, z_h\}$ (Recall that $I(G_x)$is the set of isolated vertices in G_x). Obviously, we have $h \leq r-1$. We study the structure of G by considering several cases.

(i) Suppose $|I(G_x)| = 0$. In this case we claim that G_x has an $S(r-1)$-factor or $G \cong K_{r+1,r+1} - F$, where F is a 1-factor in $K_{r+1,r+1}$. If G_x has no $S(r-1)$-factor, then by Theorem 3.1 there exists a set S in $V(G_x)$ so that $i(G_x - S) > (r-1)|S|$. Since $N_G(I(G_x - S)) \subseteq S \cup N_G(x)$, by counting edges between $S \cup N_G(x)$ and $I(G_x - S)$ we have $r|I(G_x - S)| \leq r|S| + r(r-1)$ or $|S| + r - 1 \geq i(G_x - S) > (r-1)|S|$. Simplifying it we get $|S| = 0$ or 1 as $r \geq 4$. But $I(G_x) = \emptyset$, so $S \neq \emptyset$ and thus $|S| = 1$. Let $S = \{s\}$. Then $i(G_x - \{s\}) \leq r$. But $i(G_x - \{s\}) > (r-1)$, and thus $i(G_x - \{s\}) = r$. Moreover, as $r\, i(G_x - \{s\}) = r^2 = e_G(I(G_x - \{s\}, \{s\} \cup N_G(x))$ and G is connected, it follows that $G \cong K_{r+1,r+1} - F$ and the claim is proved.

(ii) If $|I(G_x)| = r-1$, it is easy to see that $G \cong K_{r,r}$, which has been excluded.

(iii) If $0 < |I(G_x)| < r-1$ and $V(G) = I(G_x) \cup \{x\} \cup N_G(x)$, then $G \cong \{x, z_1, \ldots, z_h\} + G[\{y_1, \ldots, y_r\}]$ and we can easily see that G has n $S(r)$-factors.

(iv) Suppose that $0 < |I(G_x)| < r-1$ and $V(G) \neq I(G_x) \cup \{x\} \cup N_G(x)$. Let $G'_x = G[V(G) - \{x, z_1, \ldots, z_h, y_1, \ldots, y_r\}]$. Then $|V(G'_x)| \geq 2$ and $I(G'_x) = \emptyset$. We will show that G'_x has an $S(r-2)$-factor. In fact, if G'_x has no $S(r-2)$-factor, then by Theorem 3.1 there exists a set S' in $V(G'_x)$ so that $i(G'_x - S') > (r-2)|S'|$. Moreover, as $I(G'_x) = \emptyset$, S' is nonempty. Counting edges we have

$$(r-2)|S'| < i(G'_x - S') \leq |S'| + r - h - 1$$

or

$$1 \leq |S'| < (r-h-1)/(r-3) = 1 + \frac{2-h}{r-3}.$$

Since $r \geq 4$ and h is a positive integer, $h = 1$. This implies that $|S'| = 1$. Thus we have $i(G'_x - S') = r - 1$. Now each vertex of $I(G'_x - S')$ is adjacent to the one vertex of S' and to $r-1$ vertices of $\{y_1, y_2, \ldots, y_r\}$. But as x and z_1 are each adjacent to all of $\{y_1, y_2, \ldots, y_r\}$, we have at least $2r + (r-1)^2 = r^2 + 1$ edges incident with $\{y_1, y_2, \ldots, y_r\}$ which is impossible.

Thus we conclude that G must be as described in (i),(ii) and (iv) and we now study these graphs.

If $G \cong K_{r+1,r+1} - F$ or $G \cong \{x, z_1, z_2, \ldots, z_h\} + G[\{y_1, y_2, \ldots, y_r\}]$, it is not hard to find n proper $S(r-1)$-factors in G. In case (i), each vertex u of G is the centre of an r-star which is easily extended to an $S(r)$-factor and this $S(r)$-factor has the only r-star centred at u; thus giving n distinct proper $S(r)$-factors in G. In (iv), each vertex is the centre of the only $(r-1)$-star of the $S(r-1)$-factor. Thus we obtain n proper $S(r-1)$-factors and each of these $S(r-1)$-factors has only one $(r-1)$-star centred at the different vertices. We have the required factors. □

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