# Invariant Relations Involving the Additive Bandwidth

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ABSTRACT. Both the bandwidth and additive bandwidth of a graph supply information about the storage requirements of a representation of the graph. In particular, the bandwidth measures how far 1's must be from the main diagonal of the graph's adjacency matrix while the additive bandwidth yields the same information with respect to the main contradiagonal. Thus storage can be significantly reduced from that required by the full adjacency matrix if at least one of the two types of bandwidths is small, which is most likely to occur for sparse matrices. Alternatively, one could store a representation of the complement of the graph if one of its two bandwidths is small. We relate the additive bandwidth to other graphical invariants and then concentrate on Nordhaus-Gaddum type results to show there are graphs for which both the bandwidth and the additive bandwidth of both the graph and its complement are large. In other words, some graphs require near maximum storage.

## 1 Introduction

Sparse graphs often can be represented efficiently by retaining only a portion of the graphs' adjacency matrices, for example, only the diagonals, or contradiagonals (diagonals composed of elements $a_{ij}$ where $i+j$ is a constant), containing non zero entries. The bandwidth and additive bandwidth are graphical invariants which specify the minimum number of diagonals and contradiagonals needed. Let $G=(V,E)$ be a graph on $p=|V|$ vertices and $e=|E|$ edges. A *numbering* $f$ is a bijection $f\colon V \to \{1,2,\ldots,p\}$. The *bandwidth of G under numbering f* is $B(G,f)=\max\{|f(u)-f(v)|\colon uv \in E\}$; the *bandwidth* of $G$ is $B(G)=\min\{B(G,f)\colon f \text{ is a numbering}\}$; and an

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$f$ which achieves $B(G)$ is called a *bandwidth numbering*. The value of $B(G)$ can range from 1 to $p-1$. An excellent survey of many of the properties of bandwidth can be found in Chinn, Chátalová, Dewdney, and Gibbs [3]. The adjacency matrix of $G$ when the vertices are numbered according to a bandwidth numbering has all the 1's within $B$ diagonals of the main diagonal, meaning all the defining information about the graph is stored in the $B$ diagonals which lie above the main diagonal. In a similar manner Bascuñán, Ruiz, and Slater [1] defined the *additive bandwidth of $G$ under numbering $f$* as $B^+(G,f) = \max\{|f(u)+f(v)-(p+1)|: uv \in E(G)\}$, and the *additive bandwidth* of $G$ as $B^+(G) = \min\{B^+(G,f): f \text{ is a numbering}\}$. The quantity $p+1$ is called the *target* and the *edge sum of edge $e = uv$ under numbering $f$* is $f(u)+f(v)$. A numbering which achieves $B^+$ is called an *additive bandwidth numbering*. An additive bandwidth numbering places all the 1's of the adjacency matrix within $B^+$ contradiagonals of the main contradiagonal (composed of elements $a_{ij}$ where $i+j=p+1$), and thus all the defining information about the graph is stored in that portion of those contradiagonals which lie above the main diagonal. The value of $B^+(G)$ can range from 0 to $p-2$. We shall need the facts that $B^+(C_p) = 1$ [2] and $B^+(K_p) = p-2$, where $C_p$ is the cycle and $K_p$ is the complete graph on $p$ vertices. Depending on the graph, either of $B(G)$ or $B^+(G)$ can be smaller. Results about additive bandwidth can be found in [1,2,5,6,8].

Section 2 relates the value of $B^+$ to other graphical invariants. Sections 3 and 4 yield Nordhaus-Gaddum type results, one for $B^+$ in Section 3 and a mixed theorem involving both $B$ and $B^+$ in Section 4. In Section 5 it is noted that there are graphs for which all of $B(G)$, $B^+(G)$, $B(\overline{G})$, and $B^+(\overline{G})$ are large, where $\overline{G}$ is the complement of $G$, meaning neither bandwidth nor additive bandwidth is useful for such graphs in reducing storage requirements.

## 2 Relations Between Additive Bandwidth and Other Graphical Invariants

Previous work [1,2] has found some invariant relations involving additive bandwidth, and they are listed here for completeness, where $\Delta(G)$ is the maximum degree of graph $G$ and $\beta_0(G)$ is the vertex independence number.

**Theorem 1.**

(a) $B^+(G) \geq \frac{\Delta(G)-1}{2}$.

(b) $p - 1 - \left\lfloor \frac{\beta_0(G)}{2} \right\rfloor \geq B^+(G) \geq p - 2\beta_0(G)$.

(c) $B^+(G) \geq \frac{B(G)}{2}$.

Here we present a series of additional relationships where $\delta(G)$ is the minimum degree of $G$, $\chi(G)$ is its chromatic number, and $\chi_1(G)$ is its edge chromatic number.

**Proposition 2.** *For any graph $G$, $B^+(G) \geq \delta(G) - 1$ and this bound is sharp.*

**Proof:** In an additive bandwidth numbering, vertex 1 has degree at most $B^+(G) + 1$. Sharpness is shown by cycles and complete graphs. □

**Proposition 3.** *For any graph $G$, $B^+(G) \geq \chi(G) - 2$ and this bound is sharp.*

**Proof:** Let the vertices be labeled by an additive bandwidth numbering. Vertices $1, 2, \ldots, \left\lceil \frac{p-B^+}{2} \right\rceil$ are independent since $(p+1) - \left[\left(\left\lceil \frac{p-B^+}{2} \right\rceil - 1\right) + \left\lceil \frac{p-B^+}{2} \right\rceil\right] > B^+$. Similarly, vertices $p - \left\lceil \frac{p-B^+}{2} \right\rceil + 1, p - \left\lceil \frac{p-B^+}{2} \right\rceil + 2, \ldots, p$ are independent. The graph thus can be colored by employing one color for each of these two independent sets and a separate color for each of the $p - 2\left\lceil \frac{p-B^+}{2} \right\rceil \leq B^+$ remaining vertices. Therefore $\chi(G) \leq B^+(G) + 2$. Complete graphs achieve the bound. □

**Proposition 4.** *For any graph $G = (V, E)$, $B^+(G) \geq \frac{\chi_1(G)-1}{2}$ and this bound is sharp.*

**Proof:** Assume $f$ is an additive bandwidth numbering. Then, for $-B^+(G) \leq i \leq B^+(G)$, the sets $E_i = \{e = uv \in E\colon f(u) + f(v) - (p+1) = i\}$ are a partition of $E$. Any two edges in the same $E_i$ are independent, for otherwise the non adjacent end vertices of two edges with a common endpoint would have the same label. Thus all edges in $E_i$ can be colored the same and the result follows. Equality occurs for odd cycles. □

The following lemma and subsequent theorem relate the additive bandwidths of a graph $G$ and its complement, $\overline{G}$, with the number of edges $e$ of $G$.

**Lemma 5.** *For any graph $G$ with $p$ vertices, $e \geq \left\lfloor \frac{(p-B^+(\overline{G})-1)^2}{2} \right\rfloor$.*

**Proof:** Let $f$ be an additive bandwidth labeling of $\overline{G}$, $A$ be the adjancy matrix of $G$ induced by $f$, and consider $i$ such that $1 \leq i \leq k = \left\lfloor \frac{p-B^+(\overline{G})-1}{2} \right\rfloor$. Then each of row $i$ and column $p-i+1$ of $A$ contains at least $p - B^+(\overline{G}) - 2i$ ones (corresponding to edges of $G$) which are not counted for any other $i$. Thus, $e \geq 2\sum_{i=1}^{k}(p - B^+(\overline{G}) - 2i) = 2\left[k(p - B^+(\overline{G})) - 2\frac{k(k+1)}{2}\right] = 2k(p - B^+(\overline{G}) - k - 1)$. The result follows by substitution for $k$ and straightforward manipulation. □

**Theorem 6.** *For any graph $G$, $B^+(G) \geq p - 1 - \sqrt{p(p-1) - 2e + 1}$.*

**Proof:** In Lemma 5 interchange the roles of $G$ and $\overline{G}$ and solve for $B^+(G)$ to obtain $B^+(G) \geq p - 1 - \sqrt{2e(\overline{G}) + 1}$ where $e(\overline{G})$ is the number of edges in $\overline{G}$. Now replace $e(\overline{G})$ by $\frac{p(p-1)}{2} - e$. □

## 3 A Nordhaus-Gaddum Result for Additive Bandwidth

In this section we prove the following Nordhaus-Gaddum result for the additive bandwidth.

**Theorem 7.** *For any graph $G$, $p - 4 \leq B^+(G) + B^+(\overline{G}) \leq 2p - 3 - c\log p$ where $c$ is a positive constant independent of $p$. Moreover, the lower bound is sharp and the upper bound is best possible to within the choice of $c$.*

The theorem is proven by the next four lemmas. We first attack the upper bound with an approach motivated by the corresponding work for bandwidth given by Chinn, Chung, Erdös, and Graham [4].

**Lemma 8.** *For any graph $G$, $B^+(G) + B^+(\overline{G}) \leq 2p - 3 - c\log p$.*

**Proof:** It is well known (see Lovász [7, p. 84] among others) that the Ramsey number $r(k, k) < 4^k$. It follows that any 2-coloring of the complete graph $K_p$ contains a monochromatic $K_k$ with $k \geq c\log p$ where $c$ is a positive constant independent of $p$. Equivalently, for any graph $G$ of order $p$, either $G$ or $\overline{G}$ contains a $K_k$ with $k \geq c\log p$. Without loss of generality assume it is $G$. Then $\beta_0(\overline{G}) \geq c\log p$. Using Theorem 1(b), $B(\overline{G}) \leq p - 1 - c\log p$. Since $B^+(G) \leq p - 2$, $B^+(G) + B^+(\overline{G}) \leq 2p - 3 - c\log p$. □

The next lemma shows that the previous result is best possible by using the graph employed by Chinn, Chung, Erdös, and Graham [4] to show a corresponding statement for bandwidth. The fact that it is the same graph will be important in Section 5.

**Lemma 9.** *There is an infinite family of graphs such that each member $G$ has the property that $B^+(G) + B^+(\overline{G}) \geq 2p - 3 - c\log p$.*

**Proof:** It is known [4] that the edges of $K_p$ can be 2-colored in such a way that the largest monochromatic $K_{x,x}$ has $x < c_1 \log p$ for some positive constant $c_1$ independent of $p$. For given $p$, let $G$ be the subgraph of $K_p$ which contains the edges of one of the colors. For any additive bandwidth numbering of $\overline{G}$, let $y = \lceil c_1 \log p \rceil$ and consider the set of vertices labeled $1, 2, \ldots, y$ and the set labeled $y + 1, y + 2, \ldots, 2y$. Since $K_{y,y}$ is not a subgraph of $G$, at least one edge between a vertex with a label in the first set and a vertex with a label in the second set must lie in $\overline{G}$ and it will have an edge sum of at most $3y$. It follows that $B^+(\overline{G}) \geq p + 1 - 3y > p - 1.5 - 3\lceil c_1 \log p \rceil$. An identical argument starting with an additive bandwidth numbering of $G$ leads to the same lower bound for $B^+(G)$. The result follows. □

We now turn our attention to the lower bound of Theorem 7. It will be convenient to identify the contradiagonal containing adjacency matrix elements $a_{i,j}$ for which $i+j=s$ by the symbol $d_s$.

**Lemma 10.** *For any graph $G$, $B^+(G)+B^+(\overline{G}) \geq p-4$.*

**Proof:** We may assume $B^+(G) \leq p-5$; otherwise the lemma is trivially true. We also may restrict attention to $B^+(G) \geq 2$. To see this, suppose $B^+(G) \leq 1$ and the vertices of $G$ are labeled according to an additive bandwidth numbering. Then, with this same numbering, the adjacency matrix for $\overline{G}$ has ones in every position except possibly in the $2B^+(G)+1$ central contradiagonals and on the main diagonal. This means $\delta(\overline{G}) \geq p-1-[2B^+(G)+1]$. From Proposition 2, we have $B^+(\overline{G}) \geq \delta(\overline{G})-1 \geq p-2B^+(G)-3$. Thus $B^+(G)+B^+(\overline{G}) \geq p-B^+(G)-3 \geq p-4$.

Let $t=p-B^+(G)-1$. In the adjacency matrix for $\overline{G}$ discussed in the previous paragraph, all off diagonal elements of contradiagonals $d_3, d_4, \ldots, d_{t+1}$ and $d_{2p-t+1}, d_{2p-t+2}, \ldots, d_{2p-1}$ are ones. Let $H$ be the subgraph of $K_p$ whose edges are precisely those indicated by the ones in these contradiagonals. Then $H$ is a subgraph of $\overline{G}$ and clearly $B^+(H) \leq B^+(\overline{G})$.

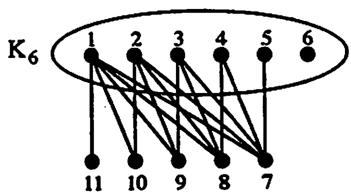
The contradiagonals $d_3, d_4, \ldots, d_{t+1}$ show that vertex $i$ is adjacent to vertex $j$ if $i+j \leq t+1$. Hence vertices $1, 2, \ldots, \lceil \frac{t+1}{2} \rceil$ induce a complete subgraph, $M$, of $H$. Similarly, $H$ also contains a complete subgraph, $N$, on vertices $p-\lceil \frac{t+1}{2} \rceil+1, p-\lceil \frac{t+1}{2} \rceil+2, \ldots, p$. Furthermore, $M$ and $N$ are vertex disjoint since $\lceil \frac{t+1}{2} \rceil = \left\lceil \frac{p-B^+(G)-1+1}{2} \right\rceil \leq \lceil \frac{p-2}{2} \rceil < \frac{p}{2}$ and $M$ and $N$ have the same order.

Consider an additive bandwidth numbering of $H$. Each of $M$ and $N$ requires $\lceil \frac{t+1}{2} \rceil$ of the labels, leaving $p-2\lceil \frac{t+1}{2} \rceil$ labels for other vertices. Of these latter vertices, either at most $\lfloor \frac{p}{2} \rfloor - \lceil \frac{t+1}{2} \rceil$ have labels at most $\lfloor \frac{p}{2} \rfloor$, or at most $\lfloor \frac{p}{2} \rfloor - \lceil \frac{t+1}{2} \rceil$ have labels at least $\lceil \frac{p}{2} \rceil + 1$. Without loss of generality we assume the former. Consider the labels $1, 2, \ldots, \lfloor \frac{p}{2} \rfloor - \lceil \frac{t+1}{2} \rceil, \lfloor \frac{p}{2} \rfloor - \lceil \frac{t+1}{2} \rceil+1, \lfloor \frac{p}{2} \rfloor - \lceil \frac{t+1}{2} \rceil+2$, and $\lfloor \frac{p}{2} \rfloor - \lceil \frac{t+1}{2} \rceil+3$. Since $\lceil \frac{t+1}{2} \rceil = \left\lceil \frac{p-B^+(G)-1+1}{2} \right\rceil \geq \left\lceil \frac{p-(p-5)}{2} \right\rceil = 3$, all of these labels are $\lfloor \frac{p}{2} \rfloor$ or less. At least three of these labels must be assigned to vertices of $M$ or $N$, and of these at least two must be in one of $M$ or $N$. Thus there is an edge sum of at most $(\lfloor \frac{p}{2} \rfloor - \lceil \frac{t+1}{2} \rceil + 2) + (\lfloor \frac{p}{2} \rfloor - \lceil \frac{t+1}{2} \rceil + 3) = 2\lfloor \frac{p}{2} \rfloor - 2\left\lceil \frac{p-B^+(G)}{2} \right\rceil + 5$. Hence $B(\overline{G}) \geq B^+(H) \geq (p+1) - \left(2\lfloor \frac{p}{2} \rfloor - 2\left\lceil \frac{p-B^+(G)}{2} \right\rceil + 5\right) \geq p-4-p+p-B^+(G)$ implying $B^+(G)+B^+(\overline{G}) \geq p-4$. □

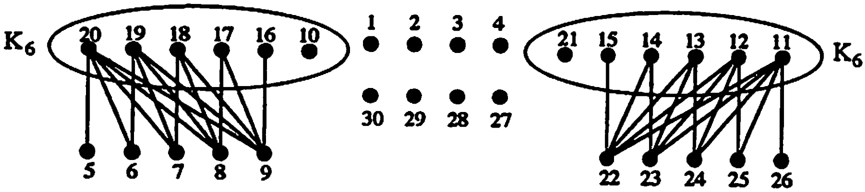
The proof of Theorem 7 is completed by the sharpness result of the next lemma.

**Lemma 11.** *There is an infinite family of graphs such that each member $G$ has the property that $B^+(G)+B^+(\overline{G}) = p-4$.*

**Proof:** We restrict attention to $p$ and $B^+(G)$ even, and values of $B^+(G)$ in the range $\frac{p}{2}-1 \le B^+(G) \le p-8$ ($p$ must be at least 14 for this to make sense). Consider the graph $H$ define in the proof of Lemma 10 and observe $t = p - B^+(G) - 1 \le \frac{p}{2}$ for the given range of $B^+(G)$. Thus, $H$ is composed of $p-2t$ isolated vertices and two isomorphic non trivial components, one of which, $\hat{M}$, is induced by vertices labeled $1, 2, \ldots, t$ and the other, $\hat{N}$, by vertices labeled $p-t+1, p-t+2, \ldots, p$. By symmetry, $\hat{M}$ and $\hat{N}$ are isomorphic. In $\hat{M}$, vertices labeled $1, 2, \ldots, \frac{t+1}{2}$ induce a complete subgraph as discussed earlier, and each vertex labeled $\frac{t+1}{2}+i$, $1 \le i \le \frac{t-1}{2}$, is adjacent only to the vertices labeled $1, 2, \frac{t-1}{2}+1-i$. The following figure illustrates $\hat{M}$ when $t = 11$ and $p \ge 22$. The numbers indicate the vertex labels associated with the additive bandwidth numbering of $G$.




The next step is to define a numbering of $H$. The $p-2t$ isolated vertices are labeled with $1, 2, \ldots, \frac{p}{2}-t, \frac{p}{2}+t+1, \frac{p}{2}+t+2, \ldots, p$. The subgraph $\hat{M}$ is labeled as follows. The vertices not in the $K_{\frac{t+1}{2}}$ are labeled in order of increasing degree by $\frac{p}{2}-t+1, \frac{p}{2}-t+2, \ldots, \frac{p}{2}-\frac{t+1}{2}$. The unicliqual vertex of $K_{\frac{t+1}{2}}$ is labeled with $\frac{p}{2}-\frac{t-1}{2}$. The remaining vertices of $K_{\frac{t+1}{2}}$ are labeled in order of decreasing degree by $\frac{p}{2}+\frac{t-1}{2}, \frac{p}{2}+\frac{t-1}{2}-1, \ldots, \frac{p}{2}+1$. Finally a vertex $v$ of $\hat{N}$ is labeled with $p+1$ minus the label of the vertex of $\hat{M}$ which corresponds to $v$ in the isomorphism between $\hat{M}$ and $\hat{N}$. Thus the differences between $p+1$ and edge sums for edges in $\hat{N}$ will be the same as such differences for edges in $\hat{M}$. The following figure illustrates this labeling when $p = 30$ and $t = 11$ ($B^+(G) = 18$).




In view of the earlier comment, we investigate only edge sums in $\hat{M}$. To achieve our goal, all must lie in the interval $[p+1-(p-4-B^+(G)), p+1+(p-4-B^+(G))] = [B^+(G)+5, 2p-B^+(G)-3]$. For edges in the

$K_{\frac{t+1}{2}}$, the largest edge sum is $\frac{p}{2} + \frac{t-1}{2} + \frac{p}{2} + \frac{t-1}{2} - 1 = 2p - B^+(G) - 3$. The smallest is $\frac{p}{2} - \frac{t-1}{2} + \frac{p}{2} + 1 = \frac{p+4+B^+(G)}{2}$ which is at least $B^+(G) + 5$ when $B^+(G) \leq p - 6$. The smallest edge sum involving a vertex not in $K_{\frac{t+1}{2}}$ is $\frac{p}{2} - t + 1 + \frac{p}{2} + \frac{t-1}{2} = \frac{p+2+B^+(G)}{2}$ which is at least $B^+(G)+5$ when $B^+(G) \leq p - 8$. The largest such edge sum is $\frac{p}{2} - \frac{t+1}{2} + \frac{p}{2} + \frac{t-1}{2} = p - 1$ which is at most $2p - B^+(G) - 3$ when $B^+(G) \leq p - 2$. All of the stated restrictions are met by our initial assumptions. □

## 4 A Nordhaus-Gaddum Type Result involving Bandwidth and Additive Bandwidth

Theorem 7 shows the sharp bound $B^+(G) + B^+(\overline{G}) \geq p - 4$. Similarly, Chinn, Chung, Erdös, and Graham [4] prove $B(G) + B(\overline{G}) \geq p - 2$, which also is sharp. Thus the following theorem, in which both bandwidth and additive bandwidth appear, is somewhat surprising.

**Theorem 12.** *For any graph $G$, $B^+(G) + B(\overline{G}) \geq \frac{2(p-2)}{3}$. Moreover, this bound is sharp for an infinite family of graphs.*

**Proof:** In any additive bandwidth numbering of $G$, the vertex labeled 1 is not adjacent to the vertex labeled $j$, for $2 \leq j \leq p - B^+(G) - 1$, meaning $\Delta(\overline{G}) \geq p - B^+(G) - 2$. Since $B(\overline{G}) \geq \frac{\Delta(\overline{G})}{2}$ (see [3]), $B(\overline{G}) \geq \frac{p-B^+(G)-2}{2}$. Now consider a bandwidth numbering of $\overline{G}$. Examining the resultant adjacency matrix shows that the degree in $G$ of the vertex labeled 1 is at least $p-B(\overline{G})-1$. Using Theorem 1(a), we have $B^+(G) \geq \frac{p-B(\overline{G})-2}{2}$. Thus, $B^+(G) + B(\overline{G}) \geq \frac{p-B(\overline{G})-2}{2} + \frac{p-B^+(G)-2}{2}$, or $3[B^+(G) + B(\overline{G})] \geq 2(p-2)$.

To see sharpness, let $G$ be a graph with $3m - 1$ vertices, additive bandwidth $B^+(G) = m - 1$, and having the maximum possible number of edges. Thus vertices labeled $i$ and $j$ in an additive bandwidth numbering are adjacent if and only if $|i + j - (p + 1)| = |i + j - 3m| \leq m - 1$. Identify a vertex by its label under this numbering. Define a numbering $f$ for $\overline{G}$ by

$$f(i) = \begin{cases} m+1-i & \text{if } 1 \leq i \leq m \\ i & \text{if } m+1 \leq i \leq 2m-1 \\ 5m-1-i & \text{if } 2m \leq i \leq 3m-1 \end{cases}.$$

Now, $B(\overline{G}) \geq \frac{2(p-2)}{3} - B^+(G) = 2(m-1)-(m-1) = m-1$. We complete the proof by showing equality occurs, that is, by showing $|f(i) - f(j)| \leq m - 1$ whenever $f(i)$ and $f(j)$ are adjacent. Suppose, for some $i$ and $j$, $f(i)$ and $f(j)$ are adjacent but $|f(i) - f(j)| \geq m$. Without loss of generality we may assume $f(i) > f(j)$, so $f(i) \geq f(j) + m$.

**Case 1:** $1 \le f(j) \le m$, meaning $1 \le j \le m$, $f(j) = m+1-j$, and $f(i) \ge m+1$. Suppose, first, that $m+1 \le f(i) \le 2m-1$ so $f(i) = i$. Then $f(i) - f(j) = i + j - m - 1 \ge m$ or, equivalently, $i + j - 3m \ge -(m-1)$. Since $i \le 2m-1$ and $j \le m$, we have $|i+j-3m| \le m-1$. This implies $i$ and $j$ are adjacent in $G$ and, hence, are not adjacent in $\overline{G}$, that is $f(i)$ and $f(j)$ are not adjacent in $\overline{G}$, a contradiction. Alternatively, we may have $2m \le f(i) \le 3m-1$, so $2m \le i \le 3m-1$. Since $1 \le j \le m$, we have $2m+1 \le i+j \le 4m-1$ or $|i+j-3m| \le m-1$ and, again, $f(i)$ is not adjacent to $f(j)$ in $\overline{G}$.

**Case 2:** $m+1 \le f(j) \le 2m-1$. Thus $f(j) = j$, $f(i) = 5m-1-i$, and $f(i) - f(j) = 5m-1-i-j \ge m$ or, equivalently, $i+j-3m \le m-1$. Since $m+1 \le j$ and $2m \le i$, we have $1 = 2m+m+1-3m \le i+j-3m$ and $|i+j-3m| \le m-1$. Once again, $f(i)$ is not adjacent to $f(j)$ in $\overline{G}$. □

## 5 Concluding Remarks

When this study began, we hoped to show that, for any graph $G$, at least one of $B^+(G)$, $B^+(\overline{G})$, $B(G)$ or $B(\overline{G})$ would be small so efficient storage of $G$ could occur. Unfortunately, this is not the case, as can be seen for the graph $G$ described in the proof to Lemma 9. For this graph, $B^+(G)$ and $B^+(\overline{G})$ both exceed $p - c\log p$ for some positive constant $c$. In [4], $B(G)$ and $B(\overline{G})$ are shown to have similar values. Nevertheless, for many graphs at least one of the four values is small. Further research to determine when this occurs seems worthwhile.

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